(1) Suppose \( f : \mathbb{R}^d \to \mathbb{R} \) is a \( C^2 \) function. Show that \( f \) is convex if and only if the second derivative matrix \( D^2 f \) is nonnegative definite.

(2) Suppose \( C \) and \( D \) are disjoint convex subsets of \( \mathbb{R}^{d+1} \). Prove there exists a \( d \) dimensional hyperplane \( H_d \) which separates \( C \) from \( D \).

(3) Suppose \( A \) is an \( m \times n \) matrix with adjoint \( A^* \), \( c \in \mathbb{R}^n \), \( b \in \mathbb{R}^m \). Show that
\[
\inf \{ c \cdot x \mid x \in \mathbb{R}^n, \ Ax = b, \ x \geq 0 \} = \sup \{ b \cdot y \mid y \in \mathbb{R}^m, \ A^* y \leq c \} .
\]

(4) Let \( f : \mathbb{R}^d \to \mathbb{R} \cup \{ \infty \} \) be a closed convex function and
\[
C = \{(x,y) \in \mathbb{R}^{d+1} : f(x) < \infty, \ y \geq f(x) \} .
\]
(a) Show that \( C \) is a closed convex set.
(b) Suppose \( (x_0, y_0) \notin C \). Show there exists \( m \in \mathbb{R}^d \) such that
\[
C \subset \{ (x,y) : y > y_0 + m \cdot (x - x_0) \} .
\]

(5) (a) Find the Legendre transforms \( f^* \) of the functions \( f : \mathbb{R} \to \mathbb{R} \) given by \( f(x) = |x|^p/p, \ p > 1 \), and \( f(x) = e^{\vert x \vert} \).
(b) Verify for the functions in (a) that the Legendre transform of \( f^* \) is again \( f \).

(6) For a function \( H : \mathbb{R}^2 \to \mathbb{R} \) which we write as \( H(x,v) \) let \( \hat{H} \) be the Legendre transform of \( H \) in the second variable \( v \) i.e.
\[
\hat{H}(x,z) = \sup_{v \in \mathbb{R}} [vz - H(x,v)] .
\]
Assuming that \( H \) and \( \hat{H} \) are \( C^2 \) functions prove that under the change of variable \( z = H_v \), there are the following relations between partial
derivatives of $H$ and of $\hat{H}$:

$$v = \hat{H}_z, \quad H_x = -\hat{H}_x, \quad H_{vv} = \frac{1}{\hat{H}_{zz}},$$

$$H_{xv} = -\frac{\hat{H}_{xz}}{\hat{H}_{zz}}, \quad H_{xx} = \frac{(\hat{H}_{xz})^2}{\hat{H}_{zz}} - \hat{H}_{xx}.$$