Math 626 Problem Set VI

(1) Let \( U : \mathbb{R} \to \mathbb{R} \) be a concave function. Prove Jensen’s inequality that for any random variable \( X \) there is the inequality,

\[
E[U(X)] \leq U(E[X]) \, .
\]

The inequality shows that for a risk averse utility function the expected utility decreases after playing a fair game.

(2) Consider Merton’s model for optimal portfolio selection where the price \( S \) of a stock evolves by geometric Brownian motion,

\[
\frac{dS}{dt} = S[\mu + \sigma W(t)] \, .
\]

The consumption rate is \( u_2 > 0 \) and a proportion \( u_1, -\infty < u_1 < \infty, \) of the portfolio is kept in stock, the remainder in cash at zero interest rate. Let \( H(v,t) \) be the maximum expected utility at time \( t < T \) when the value of the portfolio is \( v, \)

\[
H(v,t) = \max_{u_1,u_2} E_v \left[ \int_t^T e^{-\rho s}[u_2(s)]\gamma ds \right] ,
\]

where \( 0 < \gamma < 1, \rho > 0. \) Find an explicit formula for \( H(v,t). \)

(3) Consider again the Merton problem in (2) above and suppose that \( u_1 \) is restricted to the interval \( 0 \leq u_1 \leq 1. \) Find an explicit formula for \( H(v,t) \) in the case when \( \sigma^2(1-\gamma) < \mu \) and verify that it is indeed the maximum expected utility.

(4) Let \( g_\beta(x,t) \) be defined by

\[
g_\beta(x,t) = \beta P_x(\tau > t) + E[x + \mu t + \sigma B(t); \tau > t] ,
\]
where $B(t)$ is Brownian motion and $\tau$ is the first time that $x + \mu t + \sigma B(t)$ becomes 0.

(a) Let $p(x,t) = P_x(\tau < t)$ be the solution to the equation,

$$
\frac{\partial p}{\partial t} = \mu \frac{\partial p}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2 p}{\partial x^2}, \quad x, t > 0,
$$

with initial condition $p(x,0) = 0$, $x > 0$, and boundary condition $p(0,t) = 1$, $t > 0$. Show that

$$
g_\beta(x,t) = \beta [1 - p(x,t)] + x + \mu t - \mu \int_0^t p(x,s) ds .
$$

(b) Show using the formula in (a) that $g_\beta(x,t)$ is a concave increasing function of $x > 0$ provided $\beta > 0$.

\textbf{(5)} Let $g_\beta$ be the function in (4) above. Show that for any $u_2 \leq \mu/\rho$, if $\beta \leq \mu/\rho - u_2$ then there is the inequality,

$$
e^{-\rho t} g_\beta(x,t) \leq \mu/\rho + (x - u_2), \quad x \geq u_2 .
$$

\textbf{(6)} Let $H^*(z,t)$, $z < 0, t < T$ be the solution to the equation,

$$
H^*_t + z^2 H^*_{zz} + f^*(t,z) = 0 ,
$$

with terminal condition $H^*(z,T) = 0$. Suppose that $f^*(t,z)$ is a convex function of $z$ for all $t < T$. Prove that $H^*(z,t)$ is also a convex function of $z$ for all $t < T$. 