CHAPTER I I - STOCHASTIC CONTROL THEORY

JOSEPH G. CONLON

1. Introduction

We wish to generalize the discrete control problem (1.3), (1.4) of Chapter I to a stochastic situation. The simplest way to do this is to modify (1.3) to

\( x(i + 1) = g^{\text{rand}}(x(i), u(i), i) \),

with initial condition \( x(i_1) = x \) at time \( i_1 \). Here \( g^{\text{rand}}(x(i), u(i), i) \) is a random variable with known probability distribution determined by the values of \( (x(i), u(i), i) \).

Thus we have a set of random variables \( g^{\text{rand}}(x, u, i) \) for all possible values of the variables \( x \in \mathbb{R}^n, u \in \mathbb{R}^m, i \in \mathbb{Z} \). The process determined by (1.1) is clearly Markovian. For the cost function we take the expected value of the classical cost function,

\[ C[x, u(\cdot)] = E \left[ \sum_{i = i_1}^{i_2} h(x(i), u(i), i) \mid x(i_1) = x \right] .\]

The goal then is to minimize the cost function (1.2) over all possible choices of the control parameters \( u(\cdot) \). To proceed in analogy with the classical case, we need to make an important assumption here. That is we can observe the position of the particle at all times \( i, i_1 \leq i \leq i_2 \), so the optimal controller \( u(x, i) \) is a function of position and time. Note that in the classical case we need only to know the initial position \( x \) of the particle at time \( i_1 \). Having made this assumption, we define the function \( f_{i_1, i_2}(\cdot) \) by

\[ f_{i_1, i_2}(x) = \min_{u(\cdot)} C[x, u(\cdot)] , \]

where the minimum in (1.3) is taken over all functions \( u(x, i) \in \mathbb{R}^m \) with \( x \in \mathbb{R}^n, i \in \mathbb{Z} \). As in (1.9) of Chapter I we have that

\[ f_{i_1, i_2}(x) = \min_u \{ h(x, u, i_1) + E[f_{i_1+1, i_2}(g^{\text{rand}}(x, u, i_1)) \mid x(i_1) = x] \} . \]

Just as in the classical case we can solve the recurrence equation (1.4) with given terminal condition

\[ f_{i_2, i_2}(x) = \min_u h(x, u, i_2) . \]

Thus we solve (1.4) backwards in time with the terminal condition (1.5), and it is clear that the solution is unique.

We generalize the preceding to the case of continuous time. Thus we assume the dynamics

\[ \frac{dx(t)}{dt} = g^{\text{rand}}(x(t), u(x(t), t), t) , \]
so the controller $u(\cdot)$ can be an arbitrary function of position $x$ of the particle and time $t$. The expected cost over the time period $[t_1, t_2]$ for the particle having initial position $x(t_1) = x$ is given by the formula

$$C[x, u(\cdot)] = E \left[ \int_{t_1}^{t_2} h(x(t), u(x(t), t), t) \mid x(t_1) = x \right].$$

(1.7)

We define now $f_{t_2}(t, x)$ for $t < t_2$, $x \in \mathbb{R}^n$, to be the minimum expected cost over all settings of the control variables $u(\cdot)$ as a function of position and time, for the system with initial condition $x(t) = x$. In analogy to (1.4) we obtain the equation

$$f_{t_2}(t, x) = \min_{u} \{ \Delta t \ h(x, u, t) + E \left[ f_{t_2}(t + \Delta t, x + \Delta t \ g^{\text{rand}}(x, u(x, t), t)) \right] \} + O(\Delta t^2).$$

(1.8)

We rewrite (1.8) as

$$f_{t_2}(t, x) - f_{t_2}(t + \Delta t, x) = \frac{\min_{u} \left\{ h(x, u, t) + E \left[ f_{t_2}(t + \Delta t, x + \Delta t \ g^{\text{rand}}(x, u, t)) - f_{t_2}(t + \Delta t, x) \right] \right\}}{\Delta t}.$$  

(1.9)

It is evident that the limit of the LHS of (1.9) as $\Delta t \to 0$ is $\partial f_{t_2}(t, x)/\partial t$. In order to evaluate the expectation of the RHS as $\Delta t \to 0$, we do a Taylor expansion,

$$E \left[ f_{t_2}(t + \Delta t, x + \Delta t \ g^{\text{rand}}(x, u, t)) \right] = f_{t_2}(t + \Delta t, x) + \Delta t \ \nabla_x f_{t_2}(t + \Delta t, x) \cdot E \left[ g^{\text{rand}}(x, u, t) \right] + \frac{(\Delta t)^2}{2} \sum_{i,j=1}^{n} \frac{1}{2} \frac{\partial^2 f_{t_2}(t + \Delta t, x)}{\partial x_i \partial x_j} E \left[ g^{\text{rand}}_i(x, u, t) g^{\text{rand}}_j(x, u, t) \right] + \cdots.$$  

(1.10)

To compute the limits as $\Delta t \to 0$ in (1.9), (1.10) we must be able to compute the moments of the random variable $g^{\text{rand}}(x, u, t)$. If we assume that $g^{\text{rand}}(x, u, t)$ comes from a diffusion then we have that

$$E \left[ g^{\text{rand}}(x, u, t) \right] = g(x, u, t)$$  

(1.11)

for some function $g(x, u, t)$, since we made the assumption that the pdf of $g^{\text{rand}}(x, u, t)$ depends only on $(x, u, t)$. The second moment for a diffusion process is given by the formula

$$E \left[ g^{\text{rand}}_i(x, u, t) g^{\text{rand}}_j(x, u, t) \right] = \frac{a_{i,j}(x, u, t)}{\Delta t},$$  

(1.12)

for some $n \times n$ symmetric non-negative definite matrix $a(x, u, t) = [a_{i,j}(x, u, t)]$. The higher moments in (1.10) are negligible as $\Delta t \to 0$. Assuming that (1.11), (1.12) hold as $\Delta t \to 0$, it follows from (1.9) that $f_{t_2}(t, x)$ is a solution to the second order PDE

$$- \frac{\partial f_{t_2}(t, x)}{\partial t} = \min_{u} \left\{ h(x, u, t) + \nabla_x f_{t_2}(t, x) \cdot g(x, u, t) + \frac{1}{2} \sum_{i,j=1}^{n} a_{i,j}(x, u, t) \frac{\partial^2 f_{t_2}(t, x)}{\partial x_i \partial x_j} \right\}.$$  

(1.13)

As with the Hamilton-Jacobi equation (1.30) of Chapter I, the PDE (1.13) is to be solved backwards in time for $t < t_2$, with terminal data

$$f_{t_2}(t_2, x) = 0.$$  

(1.14)
The equation (1.13) is known as the Bellman equation for the control problem. Since we get the Hamilton-Jacobi equation (1.30) of Chapter I by setting the matrix \( a(x, u, t) \equiv 0 \), we see that the Bellman equation is a second order generalization of the Hamilton-Jacobi equation. Roughly speaking, stochastic control theory bears the same relationship to classical control theory as quantum mechanics does to classical mechanics. In particular, if one takes the logarithm of the solution to the Schrödinger equation of quantum mechanics, it satisfies an equation similar to (1.13), with the first order partial terms of the equation yielding the Hamilton-Jacobi equation for the corresponding classical mechanics problem.

We need to be more clear about the meaning of the stochastic differential equation (1.6) and the identities (1.11), (1.12). We should think of (1.6) as a random perturbation of the deterministic equation (1.27) from Chapter I. Thus we write (1.6) as

\[
\frac{dx(t)}{dt} = g(x, u, t) + \sigma(x, u, t)W(t),
\]

where \( W(t), t \in \mathbb{R} \), is \( n \) dimensional white noise. Hence \( W(t) = [W_1(t), W_2(t), ..., W_n(t)] \in \mathbb{R}^n \), and each of the \( W_j(\cdot), j = 1, ..., n \), are independent copies of 1 dimensional white noise. The \( n \times n \) matrix \( \sigma(x, u, t) \) is the volatility matrix for the equation. The function \( g(x, u, t) \) is the same as the function in (1.11). The matrix \( a(x, u, t) \) of (1.12) is related to \( \sigma(x, u, t) \) in (1.15) by the identity

\[
a(x, u, t) = \sigma(x, u, t)^* \sigma(x, u, t).
\]

To understand this we consider the 1 dimensional stochastic equation

\[
\frac{dx(t)}{dt} = b(x, t) + \sigma(x, t)W(t),
\]

where \( W(t), t \in \mathbb{R} \), is 1 dimensional white noise. Formally \( W(\cdot) \) is the derivative of Brownian motion. Thus if we define the integral

\[
B(t) = \int_0^t W(s)ds, \quad t \geq 0,
\]

then \( B(t), t > 0 \), is Brownian motion. Now the increments in Brownian motion are independent Gaussian variables. In particular for any \( \Delta t > 0 \), and \( n = 1, 2, ..., \) one has

\[
B(n\Delta t) = \sum_{j=1}^n [B(j\Delta t) - B((j-1)\Delta t)] = \sqrt{\Delta t} \sum_{j=1}^n Y_j,
\]

where the \( Y_j, j = 1, 2, ..., \) are independent identically distributed (i.i.d.) standard normal variables with probability density function (pdf) given by the formula

\[
\frac{1}{\sqrt{2\pi}}e^{-x^2/2}, \quad -\infty < x < \infty.
\]

By letting \( \Delta t \to 0 \) in (1.19) we can see that the paths \( B(t), t \geq 0 \), for Brownian motion are continuous with probability 1. They are however also differentiable nowhere with probability 1. The highest regularity one has for Brownian paths is Hölder continuity with exponent \( \alpha \) where \( \alpha < 1/2 \). That means

\[
|B(t + \Delta t, \omega) - B(t, \omega)| \leq C(\omega)|\Delta t|^\alpha, \quad 0 \leq t \leq 1, \quad 0 < \Delta t < 1,
\]
where the constant $C(\omega)$ depends only on the random path $\omega \in \Omega$, with $\Omega$ being the probability space on which Brownian motion “lives”. We need therefore to be careful about how we interpret the integral (1.18). The function $W(t)$, $t \in \mathbb{R}$, is a distribution, and so we should replace (1.18) by the equation

$$W(\cdot)[\phi(\cdot)] = \int_{-\infty}^{\infty} W(t)\phi(t) \, dt = -\int_{-\infty}^{\infty} B(t)\frac{d\phi(t)}{dt} \, dt,$$

which defines $W(\cdot)$ as a functional on $C^\infty$ functions of compact support $\phi : \mathbb{R} \to \mathbb{R}$. Note that by continuity of Brownian motion, the last expression on the RHS of (1.22) is well defined with probability 1. The formula (1.22) evidently corresponds to integration by parts, if we formally set

$$\int_{t_1}^{t_2} W(t) \, dt = \int_{t_1}^{t_2} W(t) \, dt,$$

with mean $= b$ and variance $= \sigma^2$.

Thus using (1.19), we write for $x$ (1.24)

$$W(x)[\phi(x)] = \int_{-\infty}^{\infty} W(t)\phi(t) \, dt = -\int_{-\infty}^{\infty} B(t)\frac{d\phi(t)}{dt} \, dt,$$

The random variable on the RHS of (1.24) conditioned on $x((j-1)\Delta t)$ is Gaussian with

$$\text{mean} = b(x((j-1)\Delta t), (j-1)\Delta t)\Delta t, \quad \text{variance} = \sigma^2(x((j-1)\Delta t), (j-1)\Delta t)^2 \Delta t.$$

Hence the solution of (1.17) evolves over small time steps by Gaussian variables with mean and variance determined by the current position of the particle. If we compare the formulas (1.11), (1.12) to (1.25), we see that they are giving the dimensional formulas for the mean and variance of the Gaussian variables as the solution of (1.15) evolves over small time intervals.

It is fairly straightforward to prove that continuous solutions $x(t, \omega)$ of (1.17) with given initial condition exist and are unique with probability 1, provided we assume that the functions $b(x,t)$, $\sigma(x,t)$ are uniformly Lipschitz in the $x$ variable i.e.

$$|b(x,t) - b(x',t)| \leq C|x - x'|, \quad |\sigma(x,t) - \sigma(x',t)| \leq C|x - x'|, \quad x, x' \in \mathbb{R},$$

$$b(x,t)^2 + \sigma(x,t)^2 \leq C[1 + x^2], \quad x \in \mathbb{R},$$

where the constant $C$ is independent of $t \in \mathbb{R}$.

To see this let us consider first the deterministic case $\sigma(\cdot, \cdot) \equiv 0$. Then to get a solution of (1.17) with given initial condition $x(0) = x_0$, we define an integral operator $K$ on continuous functions $x(t)$, $0 \leq t \leq T$, by

$$Kx(t) = x_0 + \int_0^t b(x(s), s) \, ds, \quad 0 \leq t \leq T.$$

Let $E_T$ be the Banach space of continuous functions $x : [0, T] \to \mathbb{R}$ with the $L^\infty$ norm $\|x(\cdot)\|_\infty = \sup\{|x(t)| : 0 \leq t \leq T\}$. Now (1.27) and the Lipschitz condition (1.26) imply that for two function $x_1(\cdot)$, $x_2(\cdot) \in E_T$, one has

$$|Kx_1(t) - Kx_2(t)| \leq C \int_0^t |x_1(s) - x_2(s)| \, ds, \quad 0 \leq t \leq T.$$
We conclude from (1.28) that
\[(1.29) \quad \|Kx_1(\cdot) - Kx_2(\cdot)\|_\infty \leq CT \|x_1(\cdot) - x_2(\cdot)\|_\infty .\]

Now let \(S_T(x_0) \subset E_T\) be the set of functions \(x(\cdot) \in E_T\) which satisfy \(x(0) = x_0\). Then \(S_T(x_0)\) is a metric space, and from (1.29) it follows that \(K\) is a contraction mapping on \(S_T(x_0)\) provided \(CT < 1\). The contraction mapping theorem asserts that \(K\) has a unique fixed point \(x(\cdot)\) in \(S_T(x_0)\) i.e.
\[(1.30) \quad Kx(\cdot) = x(\cdot), \quad \text{which implies} \quad x(t) = x_0 + \int_0^t b(x(s), s) \, ds, \quad 0 \leq t \leq T .\]

Evidently (1.30) shows that (1.17) has a unique continuous solution with initial condition \(x_0\) up to time \(T\) provided \(CT < 1\). To solve the equation beyond time \(T\) we simply repeat the procedure with initial condition \(x(T)\) at time \(T\), and so solve the equation up to time \(2T\) etc.

We can more or less apply this argument directly to the stochastic case. The integral operator \(K\) becomes
\[(1.31) \quad Kx(t) = x_0 + \int_0^t b(x(s), s) \, ds + \int_0^t \sigma(x(s), s) \, W(s) ds ,\]
where \(B(\cdot)\) is Brownian motion. To define the second integral on the RHS of (1.31), we denote by \((\Omega, \mathcal{F}, P)\) the probability space on which Brownian motion \(B(t), t \geq 0,\) “lives”, and by \(\mathcal{F}_t\) the \(\sigma\)-field generated by the variables \(B(s), 0 \leq s \leq t\). The Banach space \(E_T\) is now the set of all measurable functions \(x : [0, T] \times \Omega \to \mathbb{R}\) such that:
(a) For \(\omega \in \Omega\) the function \(x(s, \omega), 0 \leq s \leq T\), from \([0, T] \to \mathbb{R}\) is continuous with probability 1 in \(\Omega\).
(b) For \(t \geq 0\), the \(\sigma\)-field generated by the variable \(x(t, \cdot) : \Omega \to \mathbb{R}\) is in \(\mathcal{F}_t\).

The norm on \(E_T\) is then given by the formula
\[(1.32) \quad \|x(\cdot, \cdot)\|^2 = E \left[ \sup_{0 \leq t \leq T} x(s, \cdot)^2 \right] .\]

If (a) and (b) hold then one can define the stochastic integral
\[(1.33) \quad M_t = \int_0^t \sigma(x(s), s) \, dB(s) ,\]
and \(M_t, t \geq 0,\) is a Martingale. This means that \(M_t\) is \(\mathcal{F}_t\) measurable and
\[(1.34) \quad E[|M_t|] < \infty, \quad E[ M_t | \mathcal{F}_{t'} ] = M_{t'}, \quad 0 \leq t' \leq t \leq T .\]

Furthermore one can easily see that
\[(1.35) \quad E[ M_t^2 ] = E \left[ \int_0^t \sigma(x(s), s)^2 \, ds \right], \quad 0 \leq t \leq T .\]

Now the maximal inequality implies that there is a universal constant \(C\) such that
\[(1.36) \quad E \left[ \sup_{0 \leq t \leq T} M_t \right]^2 \leq CE \left[ \int_0^T \sigma(x(s), s)^2 \, ds \right] .\]
The inequality (1.36) together with (1.26) imply that the operator $K$ of (1.31) is bounded on $E_T$ and
\begin{equation}
\|K x_1(\cdot, \cdot) - K x_2(\cdot, \cdot)\| \leq C_1 [T + \sqrt{T}] \|x_1(\cdot, \cdot) - x_2(\cdot, \cdot)\|, \quad x_j(\cdot, \cdot) \in E_T, \ j = 1, 2,
\end{equation}
for some constant $C_1$ independent of $T$. We still need to show that the function $K x(\cdot, \cdot)$ satisfies (a) above. This follows from a maximal inequality like (1.36), just as the proof of continuity with probability 1 of Brownian motion.

We return to proceeding on a heuristic level and consider the stochastic analogue of the minimum time problems in §3 of Chapter I. These led to a time independent Hamilton-Jacobi equation. Here we assume that the functions $g$ and of (1.6) and $h$ of (1.7) depend only on $(x, u)$ with no explicit dependence on time $t$. In the classical case we considered all paths $x(t)$ such that $x(0) = x$, and $x(\tau) = x_0$ for some fixed target $x_0$, with the goal of minimizing
\begin{equation}
\int_0^\tau h(x(t), u(t)) \, dt \quad \text{given } x(0) = x, \ x(\tau) = x_0.
\end{equation}

In the time minimization problem we set $h \equiv 1$. In the stochastic case we cannot set a point $x_0$ as our target since the probability of a random path intersecting a point is 0. Instead we need to choose a target for which random paths have a positive probability of hitting. Thus we consider a domain $\mathcal{D} \subset \mathbb{R}^n$ as our configuration space, with the boundary $\partial \mathcal{D}$ as the target. If we actually want a point $x_0$ as a target we can take $\mathcal{D} = \mathbb{R}^n - B(x_0, r)$, where $B(x_0, r)$ is a ball centered at $x_0$ with radius $r << 1$ (see Figure 1). For the classical problem if $f(x)$ is the minimizer for (1.38) then $f(x)$ satisfies the Hamilton-Jacobi equation
\begin{equation}
H(x, \nabla f(x)) = 0, \quad x \in \mathcal{D},
\end{equation}
with the boundary condition
\begin{equation}
f(x) = 0, \quad x \in \partial \mathcal{D}.
\end{equation}
The Hamiltonian in (1.39) is given by equation (3.3) of Chapter I, assuming the dynamics is given by (3.1).

For the stochastic analogue we consider time independent dynamics for (1.15),
\begin{equation}
\frac{dx(t)}{dt} = g(x, u) + \sigma(x, u) W(t),
\end{equation}
with minimum expected cost function
\begin{equation}
f(x) = \min \left\{ E \left[ \int_0^\tau h(x(t), u(x(t), t)) \, dt \mid x(0) = x \right] \right\}, \quad x \in \mathcal{D}.
\end{equation}
The random variable $\tau$ is the first exit time of the path $x(\cdot)$ from the domain $\mathcal{D}$. Arguing as we did in the derivation of (1.13), we see that the Bellman equation for the problem is the second order time independent PDE
\begin{equation}
0 = \min_u \left\{ h(x, u) + \nabla_x f(x) \cdot g(x, u) + \frac{1}{2} \sum_{i,j=1}^n a_{i,j}(x, u) \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \right\},
\end{equation}
where the $n \times n$ matrix $a(x, u)$ of (1.43) is related to $\sigma(x, u)$ in (1.41) by the identity
\begin{equation}
a(x, u) = \sigma(x, u)^* \sigma(x, u).
\end{equation}
We need to solve (1.43) for $x \in \mathcal{D}$, subject to the boundary condition (1.40). Thus the problem of finding the optimal function $f(x)$, $x \in \mathcal{D}$, becomes one of solving the
A natural question to ask concerning the relation between stochastic control and classical control, is if classical control can be obtained as the zero noise limit of stochastic control. Let us consider the Bellman equation (1.13) corresponding to the stochastic dynamics (1.45)

\[ dx(t) \over dt = g(x, u, t) + \sqrt{\varepsilon} \, W(t) , \]

where \( \varepsilon \in \mathbb{R} \) measures the noise strength. This corresponds to setting \( a(x, u, t) = \varepsilon I_n \) in (1.13), with \( I_n \) being the identity \( n \times n \) matrix. Hence the PDE becomes

(1.46)

\[ -\partial f^{\varepsilon}_t (t, x) \over \partial t = H(x, \nabla_x f^{\varepsilon}_t (t, x), t) + \frac{\varepsilon}{2} \Delta f^{\varepsilon}_t (t, x) , \]

where \( H(x, p, t) \) is the Hamiltonian for the corresponding classical control problem,

(1.47)

\[ H(x, p, t) = \min_u \{ h(x, u, t) + p \cdot g(x, u, t) \} , \]

and \( \Delta \) is the Laplacian

(1.48)

\[ \Delta = \sum_{j=1}^{n} \partial^2 \over \partial x_j^2 . \]

Consider now the Hamilton-Jacobi equation

(1.49)

\[ -\partial f^{\varepsilon}_t (t, x) \over \partial t = H(x, \nabla_x f^{\varepsilon}_t (t, x), t) , \]

for the classical control problem. We solve (1.46) for \( t < t_2 \) with zero terminal data (1.14) and also (1.49) for \( t < t_2 \) with the same terminal data (1.14). We can reasonably expect that

(1.50)

\[ \lim_{\varepsilon \to 0} f^{\varepsilon}_t (t, x) = f_t (t, x) , \quad x \in \mathbb{R}^n, \quad t < t_2 , \]

but this is not so easy to prove in general. We shall prove (1.50) in certain cases where we can more or less explicitly solve the Bellman equation.

2. LINEAR-QUADRATIC (LQ) PROBLEMS

We recall the classical LQ problem studied in Chapter I i.e.

(2.1)

\[ \frac{dx(t)}{dt} = u, \quad h(x, u) = x^2 + qu^2, \quad f(t, x) = \min_u \int_t^T h(x(s), u(s)) \, ds . \]

The Hamiltonian for the problem is given by the formula

(2.2)

\[ H(x, p) = \min_u \{ x^2 + qu^2 + pu \} = x^2 - p^2 / 4q . \]

Hence \( f(t, x) \) satisfies the Hamilton-Jacobi equation,

(2.3)

\[ -\partial f \over \partial t = x^2 - \frac{1}{4q} \left[ \partial f \over \partial x \right]^2 , \quad t < T; \quad f(T, x) = 0, \quad x \in \mathbb{R} . \]

We can solve (2.3) in the usual manner by writing the Hamiltonian equations of motion

(2.4)

\[ \frac{dx}{ds} = \frac{\partial H}{\partial p} = -\frac{p}{2q}, \quad \frac{dp}{ds} = -\frac{\partial H}{\partial x} = -2x. \]
The optimal trajectory from \((t, x)\) to \((T, \cdot)\) is obtained by solving the boundary value problem

\[ \frac{d^2x}{ds^2} = \frac{x}{q}, \quad t < s < T; \quad x(t) = x, \quad \frac{dx(T)}{dt} = 0. \]

The solution to (2.5) is given by the formula

\[ x(s) = x \frac{\cosh[(T - s)/\sqrt{q}]}{\cosh[(T - t)/\sqrt{q}]}, \quad t \leq s \leq T. \]

The corresponding value of the optimal controller is

\[ u(s) = \frac{-p(s)}{2q} = -\frac{x}{\sqrt{q}} \frac{\sinh[(T - s)/\sqrt{q}]}{\cosh[(T - t)/\sqrt{q}]}, \quad t \leq s \leq T. \]

Hence we have that

\[ h(x(s), u(s)) = x^2 \frac{\cosh[2(T - s)/\sqrt{q}]}{\cosh^2[(T - t)/\sqrt{q}]}, \quad t \leq s \leq T. \]

We conclude that

\[ f(t, x) = \int_t^T h(x(s), u(s)) \, ds = \frac{x^2 \sqrt{q} \sinh[2(T - t)/\sqrt{q}]}{2 \cosh^2[(T - t)/\sqrt{q}]} = x^2 \sqrt{q} \tanh[(T - t)/\sqrt{q}]. \]

Next we consider the corresponding stochastic control problem

\[ \frac{dx(t)}{dt} = u + \sqrt{\varepsilon} W(t), \quad h(x, u) = x^2 + qu^2, \]

\[ f^\varepsilon(t, x) = \min_u E \left[ \int_t^T h(x(s), u(s)) \, ds \mid x(t) = x \right]. \]

Now \(f^\varepsilon(t, x)\) satisfies the Bellman equation

\[ -\frac{\partial f^\varepsilon}{\partial t} = x^2 - \frac{1}{4q} \left[ \frac{\partial f^\varepsilon}{\partial x} \right]^2 + \frac{\varepsilon}{2} \frac{\partial^2 f^\varepsilon}{\partial x^2}, \quad t < T; \quad f^\varepsilon(T, x) = 0, \quad x \in \mathbb{R}. \]

We can explicitly solve (2.11) by looking for a solution of the form

\[ f^\varepsilon(t, x) = A_\varepsilon(t)x^2 + B_\varepsilon(t)x + C_\varepsilon(t). \]

Substituting (2.12) into the Bellman equation (2.11) yields the equation

\[ -A_\varepsilon'(t)x^2 - B_\varepsilon'(t)x - C_\varepsilon'(t) = x^2 - \frac{1}{4q} [2A_\varepsilon(t)x + B_\varepsilon(t)]^2 + \varepsilon A_\varepsilon(t). \]

We identify the coefficients of \(x^0, x^1, x^2\) on both sides of (2.13) to obtain a system of three equations. Thus on identifying the coefficients of \(x^2\) in (2.13), we obtain the equation and boundary condition

\[ -A_\varepsilon'(t)x^2 = B_\varepsilon'(t)x - C_\varepsilon'(t) = x^2 - \frac{1}{4q} [2A_\varepsilon(t)x + B_\varepsilon(t)]^2 + \varepsilon A_\varepsilon(t). \]

We identify the coefficients of \(x^0, x^1, x^2\) on both sides of (2.13) to obtain a system of three equations. Thus on identifying the coefficients of \(x^2\) in (2.13), we obtain the equation and boundary condition

\[ -A_\varepsilon'(t) = 1 - A_\varepsilon(t)^2/q, \quad t < T; \quad A_\varepsilon(T) = 0. \]

Hence \(A_\varepsilon(t)\) is explicitly given by the formula

\[ A_\varepsilon(t) = \sqrt{q} \tanh[(T - t)/\sqrt{q}], \]
which is independent of \( \varepsilon > 0 \). On identifying the coefficients of \( x^1 \) in (2.13), we obtain the equation and boundary condition
\[
B'_{\varepsilon}(t) = A_{\varepsilon}(t)B_{\varepsilon}(t)/q , \quad t < T, \quad B_{\varepsilon}(T) = 0.
\]
We conclude that \( B_{\varepsilon}(t) \equiv 0 \). Finally on identifying the coefficients of \( x^0 \) in (2.13), we obtain the equation and boundary condition
\[
-C'_{\varepsilon}(t) = -B_{\varepsilon}(t)^2/4q + \varepsilon A_{\varepsilon}(t) , \quad t < T, \quad C_{\varepsilon}(T) = 0.
\]
We conclude from (2.15) that \( C_{\varepsilon}(t) \) is given by the formula
\[
C_{\varepsilon}(t) = \varepsilon \sqrt{q} \int_t^T \tanh[(T - s)/\sqrt{q}] \, ds = \varepsilon q \log \{ \cosh[(T - t)/\sqrt{q}] \}.
\]
It is easy to see now that for this problem one has
\[
\lim_{\varepsilon \to 0} f_{\varepsilon}(t, x) = f(t, x), \quad t \leq T, \quad x \in \mathbb{R}.
\]
One should also note that it is possible to solve the Hamilton-Jacobi equation (2.3) by the same method we used to solve the Bellman equation (2.11).

3. Burgers’ Equation

We consider again a classical control problem with linear dynamics, but now with a non-quadratic cost function. Let \( f_0 : \mathbb{R} \to \mathbb{R} \) be a given function. Then the control problem is given by
\[
\frac{dx(t)}{dt} = u, \quad f(t, x) = \min_u \left[ f_0(x(T)) + \int_t^T qu(s)^2 \, ds \mid x(t) = x \right].
\]
Evidently the Hamiltonian \( H(x, p) \) is given by the formula
\[
H(x, p) = \min_u [qu^2 + pu] = -p^2/4q,
\]
whence \( f(t, x) \) is a solution to the Hamilton-Jacobi equation
\[
-\frac{\partial f}{\partial t} = -\frac{1}{4q} \left[ \frac{\partial f}{\partial x} \right]^2 , \quad t < T; \quad f(T, x) = f_0(x), \quad x \in \mathbb{R}.
\]
Note that
\[
\min \left[ \int_t^T \left\{ \frac{dx(s)}{ds} \right\}^2 \, ds \right] = \frac{(x - y)^2}{(T - t)},
\]
where the minimum is taken over all paths \( x(s), t \leq s \leq T, \) with \( x(t) = x, \ x(T) = y \). This follows from the Schwarz inequality i.e.
\[
|x(T) - x(t)| = \left| \int_t^T \frac{dx(s)}{ds} \, ds \right| \leq (T - t)^{1/2} \left[ \int_t^T \left\{ \frac{dx(s)}{ds} \right\}^2 \, ds \right]^{1/2}.
\]
The function \( f(t, x) \) of (3.1) is therefore given by the formula
\[
f(t, x) = \min_{y \in \mathbb{R}} \left[ f_0(y) + \frac{q(x - y)^2}{(T - t)} \right].
\]
We can rewrite (3.6) as
\[
f(t, x) - \frac{qx^2}{(T - t)} = \min_{y \in \mathbb{R}} \left[ f_0(y) + \frac{qy^2}{(T - t)} - \frac{2qxy}{(T - t)} \right].
\]
Figure 2. In Figure 2 are portrayed the graph of \( S \) from the 3 parameter family of parabolas \( S \). One can see how this may happen even when the function \( f \) with possible discontinuities in the first derivative which are always jump downs. The parabola lies above the graph of \( f \) in value as one moves from left to right. Hence the function \( f(t, \cdot) \) is continuous with possible discontinuities in the first derivative which are always jump downs. One can see how this may happen even when the function \( f_0(\cdot) \) is smooth (see Figure 2). In Figure 2 are portrayed the graph of \( S = -f_0(y) \), \( y \in \mathbb{R} \), and a graph from the 3 parameter family of parabolas \( S = q(x - y)^2/(T - t) + C \), \( y \in \mathbb{R} \), where \( C, x \in \mathbb{R}, t < T \), are the parameters. We have chosen \( C \) sufficiently large so that the parabola lies above the graph of \( S = -f_0(y) \), \( y \in \mathbb{R} \). Let \( y = \xi(x, t) \) be the point for which the vertical distance between the two graphs is minimized. Thus if we let \( C \) decrease in value until the parabola touches the graph \( S = -f_0(y) \), \( y \in \mathbb{R} \), then the touching point is \( y = \xi(x, t) \). Since there is a common slope at \( \xi(x, t) \), we can conclude that

\[
\frac{\partial f(t, x)}{\partial x} = f_0'(\xi(x, t)) .
\]

To see this note from (3.6) that \( \xi(x, t) \) satisfies the equation

\[
2q[x - \xi(x, t)] - f_0'(\xi(x, t)) = 0.
\]

We differentiate the formula

\[
f(x, t) = \frac{q[x - \xi(x, t)]^2}{(T - t)} + f_0(\xi(x, t))
\]

with respect to \( x \) to obtain

\[
\frac{\partial f(t, x)}{\partial x} = \frac{2q[x - \xi(x, t)]}{(T - t)} - \frac{\partial \xi(x, t)}{\partial x} \left[ \frac{2q[x - \xi(x, t)]}{(T - t)} - f_0'(\xi(x, t)) \right]
= \frac{2q[x - \xi(x, t)]}{(T - t)} = f_0'(\xi(x, t)) .
\]

It is clear that in Figure 2 when we decrease \( C \) until the parabola touches the graph \( S = -f_0(y) \), \( y \in \mathbb{R} \), it is possible to have double contact unless the function \( f_0(\cdot) \) is convex. In the case of double contact there is a discontinuity in the derivative \( \partial f(t, x)/\partial x \). Evidently the contact set can be arbitrarily complicated, which raises the issue of what we mean by claiming that the function \( f(t, x) \) satisfies the PDE (3.3). One way of resolving this issue is to regard the solution set of (3.3) as the zero noise limit of the solution set of the corresponding stochastic problem. Unlike for the classical problem, there is no difficulty with sufficient smoothness of solutions to the Bellman equation for the stochastic problem.

The stochastic problem is given by

\[
\frac{dx(t)}{dt} = u + \sqrt{\varepsilon} W(t), \quad f^\varepsilon(t, x) = \min_u E \left[ f_0(x(T)) + \int_t^T q\varepsilon(s)^2 ds \mid x(t) = x \right].
\]
The Bellman equation for (3.12) is then

\[ (3.13) \quad -\frac{\partial f^\varepsilon}{\partial t} = -\frac{1}{4q} \left( \frac{\partial f^\varepsilon}{\partial x} \right)^2 + \frac{\varepsilon}{2} \frac{\partial^2 f^\varepsilon}{\partial x^2}, \quad t < T; \quad f^\varepsilon(T, x) = f_0(x), \quad x \in \mathbb{R}. \]

Rather remarkably we can linearize (3.13) by means of the Hopf-Cole transformation. Thus we set

\[ (3.14) \quad u(t, x) = \exp[-\alpha f^\varepsilon(t, x)] \]

for some parameter \( \alpha \in \mathbb{R} \) to be determined, whence

\[ (3.15) \quad \frac{\partial u}{\partial t} = -\alpha u \frac{\partial f^\varepsilon}{\partial t}, \quad \frac{\partial u}{\partial x} = -\alpha u \frac{\partial f^\varepsilon}{\partial x}, \quad \frac{\partial^2 u}{\partial x^2} = -\alpha u \left( \frac{\partial^2 f^\varepsilon}{\partial x^2} - \alpha \left( \frac{\partial f^\varepsilon}{\partial x} \right)^2 \right). \]

On taking \( \alpha = 1/2q\varepsilon \) we see from (3.13), (3.15) that \( u(t, x) \) satisfies the heat equation

\[ (3.16) \quad \frac{\partial u}{\partial t} + \varepsilon \frac{\partial^2 u}{\partial x^2} = 0, \quad t < T; \quad u(T, x) = \exp\left[-\frac{1}{2q\varepsilon} f_0(x)\right]. \]

Now the solution of the heat equation is explicitly known and is given by the formula

\[ (3.17) \quad u(t, x) = \frac{1}{\sqrt{2\pi\varepsilon(T-t)}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-y)^2}{2\varepsilon(T-t)}\right] u(T, y) \, dy, \quad t < T. \]

It follows then from (3.14), (3.16) that

\[ (3.18) \quad f^\varepsilon(t, x) = -2q\varepsilon \log \left( \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2q\varepsilon} \left( f_0(y) + \frac{g(x-y)^2}{(T-t)} \right) \right] \, dy \right) \]

\[ + 2q\varepsilon \log \left( \sqrt{2\pi\varepsilon(T-t)} \right). \]

Since \( \lim_{\eta \to 0} \eta \log \eta = 0 \) the limit \( \lim_{\varepsilon \to 0} f^\varepsilon(t, x) \) is given by the limit of the first term on the RHS of (3.18). This can be evaluated by the method of steepest descent, also known as Laplace’s theorem.

Thus let \( g : \mathbb{R} \to \mathbb{R} \) be a continuous function which has the property that \( |g(y)| \geq C|y|^\beta \) for all sufficiently large \( y \in \mathbb{R} \), for some positive constants \( C, \beta \). Then Laplace’s theorem states that

\[ (3.19) \quad \lim_{\eta \to 0} \eta \log \left( \int_{-\infty}^{\infty} \exp\left[-\frac{g(y)}{\eta} \right] \, dy \right) = -\inf_{y \in \mathbb{R}} g(y). \]

To prove (3.19) observe that there exists \( y_0 \in \mathbb{R} \) such that \( \inf_{y \in \mathbb{R}} g(y) = g(y_0) \). Since \( g(\cdot) \) is continuous at \( y_0 \), there is for any \( \delta > 0 \) an \( \varepsilon(\delta) > 0 \) such that \( g(y) \leq g(y_0) + \delta \) for \( |y - y_0| \leq \varepsilon(\delta) \). Hence

\[ (3.20) \quad \eta \log \left( \int_{-\infty}^{\infty} \exp\left[-\frac{g(y)}{\eta} \right] \, dy \right) \geq \eta \log \left( 2\varepsilon(\delta) \exp\left[-\frac{g(y_0) + \delta}{\eta} \right] \right) = \eta \log \left( 2\varepsilon(\delta) \right) - g(y_0) - \delta. \]

If we let \( \eta \to 0 \) in (3.20) and then \( \delta \to 0 \), we see that the LHS of (3.19) is bounded below by \(-g(y_0)\). To get an upper bound we observe that
(3.21) \( \eta \log \left\{ \int_{-\infty}^{\infty} \exp \left[ -\frac{g(y)}{\eta} \right] dy \right\} = \\
- g(y_0) + \eta \log \left\{ \int_{-\infty}^{\infty} \exp \left[ -\frac{g(y) - g(y_0)}{\eta} \right] dy \right\} . \)

By our assumption on the growth of the function \( g(\cdot) \) we see that

(3.22) \( \int_{-\infty}^{\infty} \exp \left[ -\frac{g(y) - g(y_0)}{\eta} \right] dy \leq K, \quad 0 < \eta < 1, \)

for some constant \( K \), whence the second term on the RHS of (3.21) converges to 0 as \( \eta \to 0 \).

We compare the \( \varepsilon \to 0 \) limit of (3.18) to the formula (3.6) for \( f(t,x) \), whence (3.19) implies that \( \lim_{\varepsilon \to 0} f^\varepsilon(t,x) = f(t,x) \), so we have shown again that the zero noise limit of the cost function for the stochastic control problem is the cost function for the classical control problem. Note that for this example it is not so easy to see how the derivative \( \partial f^\varepsilon(t,x) / \partial x \) is related as \( \varepsilon \to 0 \) to \( \partial f(t,x) / \partial x \). Observe that the optimal controller \( u^\varepsilon(t,x) \) for the stochastic control problem, and the optimal controller \( u(t,x) \) for the classical control problem are given by the formulas

(3.23) \( u^\varepsilon(t,x) = -\frac{1}{2q} \frac{\partial f^\varepsilon(t,x)}{\partial x} , \quad u(t,x) = -\frac{1}{2q} \frac{\partial f(t,x)}{\partial x} . \)

Hence we expect that \( \lim_{\varepsilon \to 0} u^\varepsilon(t,x) = u(t,x) \), but this result cannot hold pointwise in \((t,x)\) since \( u(t,x) \) is discontinuous in general. The correct interpretation of the limit is a delicate matter, and involves the notion of weak solution of the Hamilton-Jacobi equation (3.3).