CHAPTER IV - APPLICATIONS TO MATHEMATICAL FINANCE

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1. Introduction

We consider Merton’s portfolio optimization problem. The portfolio is a combination of cash and stocks. We shall assume the interest rate on cash is 0 and the price $S(t)$ of the stock at time $t$ evolves by geometric Brownian motion

\[
\frac{dS(t)}{dt} = [\mu + \sigma W(t)] S(t),
\]

where $W(\cdot)$ is white noise and $\mu > 0$. If $V(t)$ is the value of the portfolio at time $t$ then

\[
\frac{dV(t)}{dt} = u_1 [\mu + \sigma W(t)] V(t) - u_2,
\]

where $u_1$ is the proportion of the portfolio held in stocks, and $u_2$ is the consumption rate. Evidently it is natural to take $0 \leq u_1 \leq 1$ and $u_2 \geq 0$, but in fact we shall allow $-\infty < u_1 < \infty$. If $u_1 > 1$ then the cash part of the portfolio is negative, so we are borrowing money to buy stocks. If $u_1 < 0$ then the stock part of the portfolio is negative, so we are so called shorting the stock. The goal is to maximize expected consumption over some time period. If $\sigma << 1$ it seems clear since $\mu > 0$ that we should put most of our assets into stocks. Of course there is the possibility that the stock price could fall a lot, so a decision to put most of our assets into stocks is dependent on our attitude to risk. A general principle in finance is risk aversion, which means that one does not agree to play a fair game. In the current situation risk aversion implies that if $\mu = 0$ then we set $u_1 = 0$ so all our assets are in cash. More generally risk aversion is measured by a utility function $U(v)$, where $v > 0$ is the value of our portfolio. Evidently $U(v)$ should be an increasing function of $v$. We also have from risk aversion that

\[
\frac{1}{2} [U(v + \varepsilon) + U(v - \varepsilon)] \leq U(v),
\]

since (1.3) states that my expected utility, after playing a game where I win or lose $\varepsilon$ with equal probability, does not exceed my current utility. Now (1.3) implies that $U''(v) \leq 0$ so a risk averse utility function is concave (see Figure 1). Hence our possible utility functions satisfy

\[
U'(v) \geq 0, \quad U''(v) \leq 0.
\]

Some examples of utility functions are

\[
U(v) = \log(1 + v), \quad U(v) = v^{\gamma}/\gamma \text{ where } 0 < \gamma < 1.
\]
For the Merton problem we measure the utility at time \( t < T \) by a function \( H(v,t) \) defined by

\[
H(v,t) = \max_{u_1 \in \mathbb{R}, u_2 > 0} \{ e^{-\rho t} U(u_2) \Delta t + E[H(V(t + \Delta t), t + \Delta t) | V(t) = v] \}
\]

where \( \rho > 0 \) is a discount factor. Thus \( H(v,t) \) is the expected utility over the time interval \( [t, T] \), and \( H(v,t) \) evidently satisfies the terminal condition \( H(v,T) = 0 \). Since the portfolio will be closed out when its value drops to 0 we also have \( H(0, t) = 0 \), so for boundary conditions we have

\[
H(v,T) = 0 \text{ for } v > 0, \quad H(0,t) = 0 \text{ for } t < T.
\]

We can derive a Bellman equation for \( H(v,t) \) in the usual way. Thus from (1.2), (1.6) we have

\[
H(v,t) = \max_{u_1 \in \mathbb{R}, u_2 > 0} \{ e^{-\rho t} U(u_2) \Delta t + H(v,t) + [u_1 v \mu - u_2] \frac{\partial H(v,t)}{\partial v} \Delta t + \frac{1}{2} u_2^2 \sigma^2 \frac{\partial^2 H(v,t)}{\partial v^2} \Delta t + \frac{\partial H(v,t)}{\partial t} \Delta t \}.\]

The Bellman equation is therefore

\[
\frac{\partial H(v,t)}{\partial t} + \max_{u_1 \in \mathbb{R}} \left\{ u_1 v \mu - \frac{\partial H(v,t)}{\partial v} + \frac{1}{2} u_2^2 \sigma^2 \frac{\partial^2 H(v,t)}{\partial v^2} \right\} + \max_{u_2 > 0} \left\{ e^{-\rho t} U(u_2) - u_2 \frac{\partial H(v,t)}{\partial v} \right\} = 0.
\]

Observe that to have a finite maximum for the function of \( u_1 \in \mathbb{R} \) in (1.9) we need \( H_{v} \leq 0 \) i.e. \( H(v,t) \) is a concave function of \( v > 0 \). In that case the maximum occurs at

\[
u_1 = - \frac{\mu H_v}{\nu \sigma^2 H_{vv}}.
\]

To have a finite maximum for the function of \( u_2 > 0 \) in (1.9) we need \( H_v > 0 \) since \( U(\cdot) \) is an increasing function. Hence the solution \( H(v,t) \) of (1.9) is an increasing concave function of \( v > 0 \) just like the utility function \( U(\cdot) \). In that case the optimal \( u_1 \) in (1.10) is positive, and so our portfolio always contains a proportion of stocks.

Define now the closed convex function \( g : \mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\} \) by

\[
g(x) = -U(x) \text{ if } x \geq 0, \quad g(x) = +\infty \text{ if } x < 0,
\]

and denote by \( g^* : \mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\} \) its Legendre transform. Then it is easy to see that the Bellman equation (1.9) is given by

\[
H_v - \frac{\mu^2 H_v^2}{2 \sigma^2 H_{vv}} + e^{-\rho t} g^*(-e^\rho H_v) = 0, \quad v > 0, \quad t < T.
\]

Note that \( g^*(x^*) = \infty \) for \( x^* > -U'(\infty) \) (see Figure 1). As an example we take the specific utility function given by the second formula of (1.5). In that case \( g^*(\cdot) \) is given by the formula

\[
g^*(x^*) = \frac{1 - \gamma}{\gamma} (\gamma x^*)^{-\gamma/(1-\gamma)} \text{ if } x^* < 0, \quad g^*(x^*) = \infty \text{ if } x^* > 0.
\]

We can solve (1.12) explicitly with the boundary conditions (1.7) by putting in the ansatz

\[
H(v,t) = u(t) v^\gamma.
\]
Then (1.12) becomes
\[ (1.15) \quad \frac{du(t)}{dt} + \frac{\mu^2 \gamma}{2 \sigma^2 (1 - \gamma)} u(t) + \frac{1 - \gamma}{\gamma} \gamma^{-\gamma/(1 - \gamma)} e^{-\rho t/(1 - \gamma)} u(t) - \gamma/(1 - \gamma) = 0. \]
In order to satisfy the first boundary condition of (1.7) we require \( u(T) = 0 \). The second boundary condition of (1.7) holds due to (1.14). If we put
\[ (1.16) \quad w(t) = u(t)^{1/(1 - \gamma)}, \]
then (1.15) implies that \( w(t) \) satisfies the linear equation
\[ (1.17) \quad \frac{dw(t)}{dt} + \frac{\mu^2 \gamma}{2 \sigma^2 (1 - \gamma)^2} w(t) + \gamma^{-1/(1 - \gamma)} e^{-\rho t/(1 - \gamma)} = 0, \]
which we can easily explicitly solve for \( t < T \) with the terminal condition \( w(T) = 0 \).

Observe that for this problem the maximizer \( u_1 \) of (1.10) is given by the formula
\[ (1.18) \quad u_1 = \frac{\mu}{\sigma^2 (1 - \gamma)}. \]
Thus one holds a fixed proportion of the portfolio in stocks. Note that as \( \gamma \to 1 \) or \( \sigma \to 0 \) then \( u_1 \) becomes large, in particular \( u_1 \gg 1 \). Now \( \gamma \to 1 \) means our aversion to risk is small, while \( \sigma \to 0 \) means the risk in investing in stocks is small. In both cases it makes sense to borrow to invest in stocks since the average rate of return on stocks is better than for cash.

We have already stated that solutions of (1.12) only make sense provided \( H(v, t) \) is a concave increasing function of \( v > 0 \). It is therefore natural to see what equation the Legendre transform \( H^*(z, t) \) of the convex function \(-H(v, t)\) satisfies, so \( H^*(z, t) = \sup_{v > 0} [zv + H(v, t)] \). Thus assuming that \( H_v(\infty, t) = 0 \) as in (1.14), we have
\[ (1.19) \quad H^*(z, t) = \sup_{v \geq 0} [zv + H(v, t)] \text{ if } z \leq 0, \quad H^*(z, t) = \infty \text{ if } z > 0. \]
If \( z < 0 \) then
\[ (1.20) \quad z = -H_v, \quad -H_v = 1/H_{zz}^*, \]
whence (1.12) becomes
\[ (1.21) \quad H^*_v + \frac{\mu^2 z^2}{2 \sigma^2} H^*_{zz} + e^{-\rho t} g^*(e^{\rho t} z) = 0, \quad z < 0, \quad t < T. \]
We need now to solve the linear PDE (1.21) with the boundary conditions corresponding to (1.7),
\[ (1.22) \quad H^*(z, T) = 0 \text{ for } z < 0, \quad H^*(0, t) = \infty \text{ for } t < T. \]
Evidently the problem (1.21), (1.22) is much more tractable than (1.12), (1.7).

2. PARTIAL HEDGING OF OPTIONS

We consider a stock whose price evolves randomly by geometric Brownian motion as in (1.1). A European call option with strike price \( K \) and expiration date \( T \) pays the holder of the option \( \max[S(T) - K, 0] \) at time \( T \). Assuming interest rate is \( 0 \), the Black-Scholes theory tells us that the value of this option at time \( t < T \) is \( v_{BS}(S, t) \), where \( S \) is the stock price at time \( t \). It is clear that
\[ (2.1) \quad v_{BS}(S, T) = \max[S - K, 0], \quad S > 0. \]
We compute $v_{BS}(S,t)$ for $t < T$ by solving the PDE
\[ (2.2) \quad \frac{\partial v_{BS}(S,t)}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 v_{BS}(S,t)}{\partial S^2} = 0, \quad S > 0, \ t < T, \]
with terminal condition (2.1). Observe that we have not imposed a boundary condition at $S = 0$. We can determine what this condition is by going to exponential variables $S = e^x$, $-\infty < x < \infty$. If we set $v_{BS}(S,t) = u(x,t)$ then (2.2) implies that $u(x,t)$ is a solution of the constant coefficient PDE
\[ (2.3) \quad \frac{\partial u(x,t)}{\partial t} - \frac{\sigma^2}{2} \frac{\partial^2 u(x,t)}{\partial x^2} = 0, \quad -\infty < x < \infty, \ t < T, \]
and (2.1) implies that $u(\cdot, T)$ satisfies the terminal condition
\[ (2.4) \quad u(x,T) = \max[e^x - K, 0], \quad -\infty < x < \infty. \]
Since $u(x,T) = 0$ for $x < \log K$, we conclude that for $t < T,$
\[ (2.5) \quad \lim_{x \to -\infty} u(x,t) = 0, \quad \text{which implies} \quad \lim_{S \to 0} v_{BS}(S,t) = 0. \]
The basic idea of the B-S theory is that one can reproduce the call option by a dynamic trading strategy. Thus given an amount of cash $v_{BS}(S,t)$ at time $t < T$ when the stock price is $S$, one can design a dynamic trading strategy to reproduce the payoff $\max[S(T) - K, 0]$ at time $T$. To see this we use Ito’s formula

\[ (2.6) \quad v_{BS}(S(T), T) - v_{BS}(S(t), t) = \int_t^T \frac{\partial v_{BS}(S(t'), t')}{\partial t'} dt' + \int_t^T \frac{\partial v_{BS}(S(t'), t')}{\partial S} dS(t') dt' + \int_t^T \frac{\sigma^2 S(t')^2}{2} \frac{\partial^2 v_{BS}(S(t'), t')}{\partial S^2} dt', \]
where the final term on the RHS of (2.6) is Ito’s correction to the usual classical fundamental theorem of calculus. In view of (2.2) we see that (2.6) is the same as
\[ (2.7) \quad v_{BS}(S(T), T) = v_{BS}(S(t), t) + \int_t^T \frac{\partial v_{BS}(S(t'), t')}{\partial S} dS(t'). \]
Equation (2.7) implies that we can reproduce the option payoff $\max[S(T) - K, 0]$ at time $T$ by a portfolio consisting of a position in cash and stocks. Thus at time $t'$ with $t < t' < T$ the portfolio consists of $v_{BS}(S(t'), t')$ in cash and $\partial v_{BS}(S(t'), t')/\partial S$ in stocks. At the initial time $t$, the bank receives $v_{BS}(S(t), t)$ in cash from the purchaser of the option and then takes a position $\partial v_{BS}(S(t), t)/\partial S$ in the stock to offset the risk that the stock price will increase significantly. Note by the maximum principle that
\[ (2.8) \quad 0 < \frac{\partial v_{BS}(S,t)}{\partial S} < 1, \quad S > 0, \ t < T. \]
One should contrast the BS trading strategy with the more naive stop-loss strategy, in which the position in the stock is 1 if $S > K$ and 0 if $S < K$, and observe from (2.1) that the BS strategy becomes stop-loss as $t \to T$. One can therefore view the BS strategy as an averaged stop-loss strategy, where the average is taken around the current stock price with standard deviation $\sim \sigma \sqrt{T-t}$.

Suppose now we are given cash $v$ with $0 < v < v_{BS}(S,t)$ at time $t < T$ when the stock price is $S$, and wish to reproduce the option payoff $\max[S(T) - K, 0]$ at time $T$ as closely as possible. We can no longer implement the BS hedging strategy exactly, but we can seek to minimize the difference between $\max[S(T) - K, 0]$ and
the portfolio value at time $T$. Letting $V(t)$ be the portfolio value at time $t$ we have as in (1.2) that
\begin{equation}
(2.9) \quad \frac{dV(t)}{dt} = u[\mu + \sigma W(t)] \ V(t),
\end{equation}
where $u$ is the proportion of the portfolio in stocks. We choose now a utility function $U(S, v)$ by
\begin{equation}
(2.10) \quad U(S, v) = \frac{1}{p} [h(S)^p - \{\max[h(S) - v, 0]\}^p], \quad \text{where} \ h(S) = \max[S - K, 0].
\end{equation}
Note that the function $v \to U(S, v)$ is an increasing concave function if $p > 1$. We consider the function
\begin{equation}
(2.11) \quad H(S, v, t) = \max_{u \in \mathbb{R}} E[ U(S(T), V(T)) \mid S(t) = S, \ V(t) = v ].
\end{equation}
From (2.10) we see that
\begin{equation}
(2.12) \quad H(S, v, t) \leq \frac{1}{p} E[h(S(T))^p \mid S(t) = S], \quad S > 0, \ t < T.
\end{equation}
Since we can implement the BS strategy if $v \geq v_{BS}(S, t)$, it follows that there is equality in (2.12) for all $v \geq v_{BS}(S, t)$.

We derive the Bellman equation for (2.11). Thus
\begin{equation}
(2.13) \quad H(S, v, t) = \max_{u \in \mathbb{R}} E[ H(S(t + \Delta t), V(t + \Delta t), t + \Delta t) \mid S(t) = S, \ V(t) = v ]
\end{equation}
\begin{equation}
= \max_{u \in \mathbb{R}} \{ H(S, v, t) + [\mu SH_S + \frac{1}{2} \sigma^2 S^2 H_{SS}]\Delta t
\end{equation}
\begin{equation}
+ [\mu u + \frac{1}{2} \sigma^2 u^2 \sigma V v \sigma W_{sv}]\Delta t + u \sigma^2 v^2 H_{sv} \Delta t + H_t \Delta t \}.
\end{equation}
Note that the term in (2.13) involving $H_{sv}$ comes from
\begin{equation}
(2.14) \quad H_{sv} \Delta S \Delta V = H_{sv} \{ \mu V[\mu + \sigma W(t)]\Delta t \}
\end{equation}
\begin{equation}
= H_{sv} Su V \sigma^2 \Delta t, \quad \text{on setting} \ [W(t)\Delta t]^2 = \Delta t.
\end{equation}
From (2.13) it follows that the Bellman equation is given by
\begin{equation}
(2.15) \quad H_t + \mu SH_S + \frac{1}{2} \sigma^2 S^2 H_{SS} - \frac{[\mu H_v + S \sigma^2 H_{sv}]^2}{2 \sigma^2 H_{sv}} = 0, \quad t < T,
\end{equation}
where the maximizing $u \in \mathbb{R}$ is given by the formula
\begin{equation}
(2.16) \quad u = -[\mu H_v + S \sigma^2 H_{sv}]/[\sigma^2 H_{sv}].
\end{equation}

Evidently the equations (1.12) and (2.15) are quite similar. It is not so surprising then that we can also linearize (2.15) by going to the Legendre transform
\begin{equation}
(2.17) \quad H^*(S, z, t) = \sup_{v > 0}[zv + H(S, v, t)].
\end{equation}
In addition to the identities (1.20) we also need the identities
\begin{equation}
(2.18) \quad H_t = \frac{H^*}{H^*}, \quad H_{sv} = \frac{H_{sz}}{H_{zz}}, \quad H_{ss} = \frac{H_{sz}^2}{H_{zz}} - \frac{(H_{sz}^*)^2}{H_{zz}}.
\end{equation}
Hence (2.15) becomes
\begin{equation}
(2.19) \quad H_t^* + \mu SH_S^* + \frac{1}{2} \sigma^2 S^2 \left[ H_{SS}^* - \frac{(H_{sz}^*)^2}{H_{zz}^*} \right] + \frac{H_{zz}^*}{2 \sigma^2} \left[ -\mu z + S \sigma^2 \frac{H_{sz}^*}{H_{zz}^*} \right]^2 = 0.
\end{equation}
Now (2.18) is the same as the linear PDE

\[(2.19) \quad H_t^* + \mu SH_S^* + \frac{1}{2} \sigma^2 S^2 H_{SS}^* + \frac{\mu^2 z^2}{2\sigma^2} H_{zz}^* - \mu SzH_{Sz}^* = 0, \quad t < T.\]

We need to solve (2.19) with the terminal condition

\[(2.20) \quad H^*(S, z, T) = \sup_{v>0} [zv + U(S, v)],\]

where \(U(\cdot, \cdot)\) is given by (2.10). Since \(U(\cdot, \cdot)\) is non-negative we have that

\[(2.21) \quad H^*(S, z, T) = \infty \quad \text{if } z > 0.\]

Going to \(z < 0\) we note (see Figure 2) that

\[(2.22) \quad H^*(S, z, T) = 0 \quad \text{if } z < -U_v(S, 0) = -h(S)^{p-1}.\]

Finally we need to consider \(z\) in the interval \(-h(S)^{p-1} < z < 0\). In that case the unique maximizing \(v = v(z)\) in (2.20) is given by the formula

\[(2.23) \quad z = -[h(S) - v(z)]^{p-1},\]

and

\[(2.24) \quad H^*(S, z, T) = zv(z) + H(S, v(z), T) = -v(z)[h(S) - v(z)]^{p-1} + \frac{1}{p} [h(S) - v(z)]^{p-1}.\]

It follows from (2.23), (2.24) that

\[(2.25) \quad H^*(S, z, T) = \frac{1}{p} h(S)^p + h(S)z + \left(1 - \frac{1}{p}\right) |z|^{p/(p-1)} \quad \text{if } -h(S)^{p-1} < z < 0.\]

We have evaluated in (2.22) and (2.25) the terminal data \(H^*(S, z, T)\) for \(z < 0\). From (2.21) we can add the boundary condition \(H^*(S, 0, t) = \infty\) for \(t < T\) similar to (1.22).

Observe that the graph of \(z \to H^*(S, z, T)\) for \(z < 0\) hits the vertical axis at \(z = 0\) with slope \(h(S) = v_{BS}(S, T)\) (see Figure 2). We show that if \(t < T\) then the graph of \(z \to H^*(S, z, t)\) for \(z < 0\) hits the vertical axis at \(z = 0\) with slope \(v_{BS}(S, t)\). To see this we look for an approximate solution to (2.19) of the form

\[(2.26) \quad H^*(S, z, t) = a(S, t) + b(S, t)z + c(S, t)|z|^{p/(p-1)}, \quad t < T,\]

which gives the leading order behavior of \(H^*(S, z, t)\) for \(z\) small. Evidently the functions in (2.26) have terminal conditions

\[(2.27) \quad a(S, T) = \frac{h(S)^p}{p}, \quad b(S, T) = h(S), \quad c(S, T) = 1 - \frac{1}{p}.\]

If we substitute (2.26) into (2.19) and equate powers of \(z\) we see that the \(z^0\) terms imply that \(a(S, t)\) is a solution to the PDE. It is clear then that we must have

\[(2.28) \quad \frac{\partial a(S, t)}{\partial t} + \mu_S \frac{\partial a(S, t)}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 a(S, t)}{\partial S^2} = 0, \quad t < T.\]

From (2.27), (2.28) we conclude that \(a(S, t)\) is equal to the RHS of (2.12). If we equate \(z^1\) terms then we see that \(b(S, t)\) satisfies the Black-Scholes equation (2.2). In view of (2.27) we conclude that \(b(S, t) = v_{BS}(S, t)\) for \(S > 0, t < T\), whence it follows that

\[(2.29) \quad \lim_{z \to 0} H^*_z(S, z, t) = v_{BS}(S, t), \quad t < T.\]
It is easy now to conclude that the Legendre transform $-H(S,v,t)$ of $H^*(S,z,t)$ has the property $H(S,v,t) = H^*(S,0,t) = a(S,t)$ for $v \geq v_{BS}(S,t)$ as we already had concluded from the BS argument. Hence the value of the portfolio at time $T$ is $V(T) \geq h(S(T))$ with probability 1.

3. Optimal Strategies in Banking

We consider a simple model of how a bank might operate. Let $X(t) > 0$ be the amount of capital the bank can invest in risky assets. We shall assume that

\[(3.1) \quad \frac{dX(t)}{dt} = \mu + \sigma W(t), \quad t > 0,\]

where $W(\cdot)$ is white noise, and assume that $\mu > 0$ as previously. We also assume that bankruptcy occurs at the first time $t = \tau$ for which $X(t) = 0$. The bank wishes to maximize the payout to its investors over the life-time of the bank. Suppose that with a strategy $\pi$, the bank has paid out $L_\pi t$ up to time $t$. Then if the bank begins with an amount of capital $x > 0$, the capital it has at time $t > 0$ is given by the formula

\[(3.2) \quad X(t) = x + \int_0^t [\mu + \sigma W(s)] \, ds - L_\pi t = x + \mu t + \sigma B(t) - L_\pi t,\]

where $B(\cdot)$ is Brownian motion with $B(0) = 0$. If we discount payments by $\rho > 0$, then the total payout over the life-time of the bank is

\[(3.3) \quad \int_0^\tau e^{-\rho t} \, dL_\pi t, \quad \text{where} \quad \tau = \sup\{t > 0 : X(s) > 0 \text{ for } 0 \leq s \leq t\}.\]

We define now the value function $V(x)$ to be the maximum expected discounted payout when the bank’s initial capital is $x > 0$. Thus

\[(3.4) \quad \max_{\pi} E\left[ \int_0^\tau e^{-\rho t} \, dL_\pi t \mid X(0) = x \right],\]

whence $V(0) = 0$.

We wish to find the bank’s optimal strategy. A bank with initial capital $x > 0$ can immediately pay out $h < x$ to investors. Since this strategy is allowable we must have

\[(3.5) \quad V(x-h) + h \leq V(x), \quad \text{implies} \quad V'(x) \geq 1.\]

Another allowable strategy is that the bank pays out nothing over a given time period of length $\Delta t$. Thus

\[(3.6) \quad V(x) \geq e^{-\rho \Delta t} E[V(x + \mu \Delta t + \sigma B(\Delta t))] = [1 - \rho \Delta t] \{V(x) + \mu \Delta t V'(x) + \frac{\sigma^2}{2} \Delta t V''(x)\} + \text{higher order in } \Delta t.\]

Letting $\Delta t \to 0$ in (3.6), we conclude that

\[(3.7) \quad \frac{\sigma^2}{2} V''(x) + \mu V'(x) - \rho V(x) \leq 0, \quad \text{for } x > 0.\]

Observe next that if $V'(x) > 1$, then $V(x - \Delta x) + \Delta x < V(x)$ for small $\Delta x > 0$, in which case the optimal strategy beginning with capital $x$ cannot be to pay out immediately. Hence (3.7) holds with equality when $V'(x) > 1$.\]
We can explicitly solve for $V(x)$ using (3.5), (3.7). Thus we need to find $u_0 > 0$ such that (3.7) holds with equality for $0 < x < u_0$, and $V'(x) = 1$ for $x > u_0$. We also want $V(\cdot)$ to be as smooth as possible, in particular a $C^2$ function. Thus we want $V'(u_0) = 1, V''(u_0) = 0$, from which we conclude using (3.7) that $V(u_0) = \mu/\rho$. We have then that

$$V(x) = \mu/\rho + x - u_0 \quad \text{for } x \geq u_0.$$  

We also have that

$$\frac{\sigma^2}{2} V''(x) + \mu V'(x) - \rho V(x) = 0, \quad \text{for } 0 \leq x \leq u_0, \quad V(0) = 0.$$  

The general solution to (3.9) is

$$V(x) = Ce^{-\mu x/\sigma^2} \sinh(ax/\sigma^2) \quad \text{for } 0 \leq x \leq u_0; \quad a = \sqrt{\mu^2 + 2\rho \sigma^2} > \mu,$$

where $C > 0$ is an arbitrary constant. The function $V(\cdot)$ in (3.10) has exactly one inflection point, which must be the point $u_0$ since we want $V''(u_0) = 0$. Thus we have that

$$u_0 = \frac{\sigma^2}{a} \log \left( \frac{a + \mu}{a - \mu} \right).$$

Since the inflection point for the function (3.10) is unique, we conclude that $V(x)$ is concave for $0 \leq x \leq u_0$. The value of the constant $C$ in (3.11) is chosen so that $V(u_0) = \mu/\rho$, which implies by (3.9) that $V'(u_0) = 1$. It is clear now that the solution defined by (3.8), (3.10), (3.11) is $C^2$ and concavity of $V(\cdot)$ implies that $V'(x) \geq 1$ for all $x \geq 0$. This also shows that $u_0 < \mu/\rho$, a fact that can be independently verified from (3.11). Thus if we set $\xi = 2\rho \sigma^2/\mu^2$ then $u_0 < \mu/\rho$ is equivalent to

$$\log \left( \frac{\sqrt{1 + \xi} + 1}{\sqrt{1 + \xi} - 1} \right) < \frac{2\sqrt{1 + \xi}}{\xi} \quad \text{for } \xi > 0.$$  

We need to show that the function $V(\cdot)$ defined above is indeed the maximizer for (3.4). We do this by obtaining upper and lower bounds for (3.4). To get a lower bound we consider the following strategy: Assuming $0 < \varepsilon < u_0$, then

(a) If $x > u_0$ the bank immediately pay out the dividend $x - (u_0 - \varepsilon)$ and then follows the strategy for $x < u_0$.

(b) If $0 < x < u_0$ the bank lets the capital fluctuate in the risky instrument without paying out any dividends until the amount of capital hits either 0, in which case bankruptcy occurs, or hits $u_0$. If it hits $u_0$ then the bank pays out the dividend $\varepsilon$ and proceeds with the strategy with capital $u_0 - \varepsilon$.

Let $V_\varepsilon(x)$ be the expected amount paid out with this strategy, beginning with capital $x > 0$. Then we have

$$V_\varepsilon(x) = \begin{cases} x - (u_0 - \varepsilon) + V_\varepsilon(u_0 - \varepsilon) & \text{if } x > u_0, \\ E_x[e^{-\rho \tau_1}; \tau_1 < \tau]\{\varepsilon + V_\varepsilon(u_0 - \varepsilon)\} & \text{if } 0 < x \leq u_0. \end{cases}$$

where $\tau_1$ is the first time the diffusion hits $u_0$ and $\tau$ the first time the diffusion hits 0. Observe now that since $V(u_0) = \mu/\rho$ we have that

$$E_x[e^{-\rho \tau_1}; \tau_1 < \tau] = \rho V(x)/\mu \quad \text{if } 0 < x \leq u_0.$$  

Hence from (3.13) it follows that

$$V_\varepsilon(u_0 - \varepsilon) = \rho V(u_0 - \varepsilon)\{\varepsilon + V_\varepsilon(u_0 - \varepsilon)\}/\mu.$$
It follows that
\begin{equation}
V(\varepsilon)(x) = \varepsilon V(x)/[\mu/\rho - V(u_0 - \varepsilon)] \quad \text{if } 0 < x \leq u_0.
\end{equation}
Since \( V(\cdot) \) is concave it follows that
\begin{equation}
[\mu/\rho - V(u_0 - \varepsilon)]/\varepsilon = [V(u_0) - V(u_0 - \varepsilon)]/\varepsilon \geq V'(u_0) = 1.
\end{equation}
Hence \( V(x) < V(x) \) and \( \lim_{\varepsilon \to 0} V(x) = V(x) \) for \( 0 < x < u_0 \). From the first equation of (3.13) we see that the same holds for \( x > u_0 \). We have proved therefore that \( V(\cdot) \) is a lower bound for (3.4). Note however that we have not exhibited a strategy which yields \( V(\cdot) \), but have obtained \( V(\cdot) \) as a limit of \( \varepsilon \) strategies as \( \varepsilon \to 0 \).

Next we need to show that \( V(\cdot) \) is an upper bound on all possible strategies. Let us assume that a strategy consists of a set of stopping times \( 0 < \tau_1 < \tau_2 < \cdots < \tau \). At each time \( \tau_j \) a dividend \( L_{\tau_j}+ - L_{\tau_j}^- \) is paid out. Then if the initial capital is \( x > 0 \), one has
\begin{equation}
V(x) - V(0) = V(X(0)) - e^{-\rho t} V(X(\tau)) = \sum_{j \geq 0} \left[ e^{-\rho \tau_j} V(X(\tau_j+)) - e^{-\rho \tau_{j+1}} V(X(\tau_{j+1})) \right] + \sum_{j \geq 1} \left[ V(X(\tau_j-)) - V(X(\tau_j+)) \right],
\end{equation}
where we have set \( \tau_0 = 0 \). Now by the Ito formula we have
\begin{equation}
e^{-\rho \tau_j} V(X(\tau_j+)) - e^{-\rho \tau_{j+1}} V(X(\tau_{j+1}-)) = - \int_{\tau_j}^{\tau_{j+1}} \frac{d}{dt} \left[ e^{-\rho t} V(X(t)) \right] dt 
= - \int_{\tau_j}^{\tau_{j+1}} e^{-\rho t} \left[-\rho V(X(t)) + V'(X(t)) \mu + \sigma W(t) \right] + \frac{\sigma^2}{2} V''(X(t)) \right] dt,
\end{equation}
Since \( V(\cdot) \) satisfies the differential inequality (3.7), we conclude that
\begin{equation}
E \left[ e^{-\rho \tau_j} V(X(\tau_j+)) - e^{-\rho \tau_{j+1}} V(X(\tau_{j+1}-)) \right] \geq 0.
\end{equation}
We also have that
\begin{equation}
e^{-\rho \tau_j} [V(X(\tau_j-)) - V(X(\tau_j+))] \geq e^{-\rho \tau_j} [V(X(\tau_j+)) + \{L_{\tau_j}+ - L_{\tau_j}^-\} - V(X(\tau_j+))] = e^{-\rho \tau_j} \{L_{\tau_j}+ - L_{\tau_j}^-\},
\end{equation}
where we have used (3.5) i.e \( V'(\cdot) \geq 1 \). We conclude that
\begin{equation}
V(x) \geq E \left[ \sum_{j \geq 1} e^{-\rho \tau_j} \{L_{\tau_j}+ - L_{\tau_j}^-\} \right] = E \left[ \int_0^T e^{-\rho t} dL_t^\pi \right].
\end{equation}
Hence \( V(x) \) is an upper bound on the strategy \( \pi \). The above argument is another example of the \textit{verification theorem} : If we can find a sufficiently smooth solution of the Bellman equation, which in this case is
\begin{equation}
\min\{V' (x) - 1, \rho V(x) - \mu V(x) - \frac{\sigma^2}{2} V''(x)\} = 0, \quad \text{for } x > 0,
\end{equation}
then it is a solution to the optimal control problem.

In the previous model it is clear that the bank will become bankrupt in finite time with probability 1 i.e. \( P(\tau < \infty) = 1 \). This is because when we do well in the risky asset we pay out the profits as dividends. A way of staving off bankruptcy is to allow the bank to raise capital when the funds get low. Suppose then that the bank decides to raise capital at time \( T \). It actually receives the capital at time
\( T + \Delta, \) and it can decide on the amount \( s \) of capital to be raised based on all information up to time \( T + \Delta. \) Furthermore, during the time interval \((T, T + \Delta)\) no dividends are paid out, so \( L_{T+\Delta}^T - L_T^T = 0. \) Finally there is a fixed cost \( K > 0 \) every time the bank raises capital. Now equation (3.2) is replaced by

\[
X(t) = x + \mu t + \sigma B(t) - L_t^T + \sum_j s_j I_{(t_j, T]} ,
\]

where \( I_E \) denotes the indicator function of the set \( E. \) The formula (3.24) corresponds to capital amounts \( s_j \) having been raised during the time intervals \((t_j, t_j + \Delta)\) up to time \( t. \) Instead of (3.3) the total discounted payout is now given by the formula

\[
\int_0^T e^{-\rho t} dL_t^T - \sum_{j} e^{-\rho(t_j + \Delta)}(s_j + K)I_{(t_j, T]} .
\]

Let \( V(x) \) be the maximum of the expectation of (3.25) over all possible strategies \( \pi, \) starting with an amount of capital \( x > 0. \) Evidently \( V(\cdot) \) must satisfy the inequalities (3.5), (3.7). In addition there is a further inequality from the fact that the bank can raise capital. If the current capital is \( x \) and the bank decides to raise more capital, then at time \( \Delta \) it will have an amount \( x + \mu \Delta + \sigma B(\Delta) + s \) of capital, where \( s > 0 \) is the capital raised. If the bank pursues an optimal strategy at time \( \Delta \) then its total expected discounted payout given the evolution of the risky asset in the time interval \((0, \Delta)\) is

\[
V(x) \geq E_x \left[ I_{\{\tau > \Delta\}} e^{-\rho \Delta} \sup_{s > 0} \{ V(x + \mu \Delta + \sigma B(\Delta) + s) - (s + K) \} \right] .
\]

As in the previous example the inequalities (3.5), (3.7), (3.27) hold for all \( x > 0, \) but for each \( x \) one of them must hold with equality, and the boundary condition \( V(0) = 0 \) continues to hold.

We assume that the value function \( V(\cdot) \) is similar to before. In particular there exists \( u_2 > 0 \) such that \( V'(x) = 1 \) for \( x \geq u_2 \) and \( V'(x) > 1 \) for \( 0 \leq x < u_2. \) It follows that \( \sup_{s > 0} \{ V(x + \mu \Delta + \sigma B(\Delta) + s) - (s + K) \} \) occurs for any \( s \) satisfying \( x + \mu \Delta + \sigma B(\Delta) + s \geq u_2 \) whence

\[
\sup_{s > 0} \{ V(x + \mu \Delta + \sigma B(\Delta) + s) - (s + K) \} = [V(u_2) - u_2 - K] + x + \mu \Delta + \sigma B(\Delta) .
\]

If we assume \( V(x) \) is \( C^2 \) at \( x = u_2 \) and that (3.9) also holds at \( x = u_2, \) then \( V(u_2) = \mu/\rho. \) Hence (3.27) yields the inequality

\[
e^{\rho \Delta} V(x) \geq \frac{\mu}{\rho - \mu} - K \cdot P_x(\tau > \Delta) + E(x + \mu \Delta + \sigma B(\Delta); \tau > \Delta) .
\]

We shall show that if the cost \( K \) of raising capital is large enough, then the bank should never raise capital. Thus we need to show that for large \( K \) the function \( V(\cdot) \) defined by (3.8), (3.10) satisfies (3.29) with \( u_2 = u_0. \) To do this we define a function \( g_3(x,t) \) with domain \( \{x > 0, \ t > 0\} \) by

\[
g_3(x,t) = \beta P_x(\tau > t) + E[x + \mu t + \sigma B(t); \tau > t] .
\]
Observe that
\begin{equation}
\lim_{t \to 0} g_{\beta}(x, t) = \beta + x, \quad \lim_{x \to 0} g_{\beta}(x, t) = 0.
\end{equation}
We can also see that $g_{\beta}(x, t)$ satisfies the PDE
\begin{equation}
\frac{\partial g_{\beta}(x, t)}{\partial t} = \mu \frac{\partial g_{\beta}(x, t)}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 g_{\beta}(x, t)}{\partial x^2}.
\end{equation}
Consider now the function $k(x, t)$ defined by
\begin{equation}
k(x, t) = e^{\rho t} V(x) - g_{\beta}(x, t).
\end{equation}
From (3.31) it follows that $k(x, t)$ satisfies the boundary conditions
\begin{equation}
\lim_{t \to 0} k(x, t) = V(x) - [\beta + x], \quad \lim_{x \to 0} k(x, t) = 0.
\end{equation}
We also have from (3.7), (3.32) that $k(x, t)$ satisfies the differential inequality
\begin{equation}
\frac{\partial k(x, t)}{\partial t} \geq \mu \frac{\partial k(x, t)}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 k(x, t)}{\partial x^2}.
\end{equation}
The maximum principle and (3.35) imply that if the function $k(x, 0)$, $x \geq 0$, is non-negative then $k(x, t) \geq 0$ for all $x, t > 0$. We conclude from (3.34) that
\begin{equation}
e^{\rho t} V(x) \geq g_{\beta}(x, t) \quad \text{for } \beta \leq 0, \ x, t > 0.
\end{equation}
Comparing (3.29) to (3.36) with $t = \Delta$, we see that (3.29) holds with $u_2 = u_0$ provided $K \geq \mu/\rho$ $- u_0$. We have therefore shown that if $K \geq \mu/\rho$ $- u_0$ then the bank never raises capital in its optimal strategy.

We assume now that $K < \mu/\rho$ $- u_0$, and will show that if $\Delta > 0$ is sufficiently small, then the bank does raise capital in the optimal strategy. To see this we assume the following properties of the function $g_{\beta}(x, t)$ when $\beta > 0$:
\begin{equation}
\frac{\partial g_{\beta}(x, t)}{\partial x} > 0, \quad \frac{\partial^2 g_{\beta}(x, t)}{\partial x^2} < 0, \quad \lim_{t \to 0} \frac{\partial g_{\beta}(0, t)}{\partial x} = \infty.
\end{equation}
Evidently (3.37) implies that the graphs of $V(\cdot)$ defined by (3.8), (3.10), and $e^{-\rho \Delta} g_{\beta}(\cdot, \Delta)$ for fixed small $\Delta$ and $\beta = \mu/\rho$ $- u_0$ $- K > 0$ are as in Figure 3. Let us denote the function $V(\cdot)$ of (3.8), (3.10) by $V_0(\cdot)$ and $V_s(\cdot)$ for $s > 0$ a translate of $V(\cdot)$ a distance $s$ to the left. We consider now the functions $g_{\beta+s}(\cdot, \Delta)$ and $V_s(\cdot)$. There is a critical value of $s > 0$ for which the two families of graphs touch. We define $u_2 = u_0 - s$ for this critical $s$ and set $u_1 < u_2$ to be the point of double contact of $V_s(\cdot)$ and $g_{\beta+s}(\cdot, \Delta)$. Our candidate for the optimizing $V(\cdot)$ for (3.25) is then given by the formula
\begin{equation}
V(x) = e^{-\rho \Delta} g_{\beta+u_0-u_2}(x, \Delta) = e^{-\rho \Delta} g_{\mu/\rho - u_2-K}(x, \Delta) \quad \text{if } 0 < x < u_1,
\end{equation}
\begin{equation}
V(x) = V_0(x+u_0-u_2) \quad \text{if } x > u_1.
\end{equation}
The function $V(\cdot)$ of (3.38) is $C^1$ by construction, but fails to be $C^2$ at $u_1$. When the strategy now consists of raising capital when the bank funds drop to $u_1$. When the funds get beyond $u_2$ then the bank pays out dividends. When the bank capital lies between $u_1$ and $u_2$, there are no payouts or raising of capital, with capital simply allowed to fluctuate in the risky asset. The bank always raises capital $u_2-u_1$ to get back to the situation where it can pay out dividends again.

We need to show that the inequalities (3.5), (3.7), (3.27) hold for the function (3.38) and equality holds in one of these for each $x > 0$. Evidently equality holds
in (3.5) for \( x > u_2 \), equality in (3.7) for \( u_1 < x < u_2 \), and equality in (3.27) for \( 0 < x < u_1 \).

Finally we consider the limiting case \( \Delta \to 0 \), where we can raise capital instantaneously. In view of the initial condition (3.31) for \( g_\beta(\cdot,t) \), we see that \( \lim_{\Delta \to 0} u_1(\Delta) = 0 \), and the function (3.38) becomes

\[
V(x) = V_0(x + u_0 - u_2), \quad \text{if } x > 0, \quad V(0) = \mu/\rho - u_2 - K.
\]

Thus \( u_2 \) is computed as the unique solution to the equation

\[
V_0(u_0 - u_2) = \mu/\rho - u_2 - K.
\]

In the limiting case \( \Delta \to 0 \) then the bank waits until capital falls to 0 before going out and raising new capital (see Figure 3).

We consider how we might establish the inequalities (3.37) and so write

\[
g_\beta(x,t) = \beta P_\tau(x,t) + w(x,t),
\]

where \( w(x,t) \) is a solution of the PDE (3.32) with the boundary conditions

\[
\lim_{t \to 0} w(x,t) = x, \quad \lim_{x \to 0} w(x,t) = 0.
\]

It follows from the maximum principle or equivalently the representation of \( w(x,t) \) as an expectation value as in (3.30) that \( w(x,t) \geq 0 \) for all \( x > 0, t > 0 \). It is easy to see that \( P_\tau(x,t) \) is an increasing function of \( x > 0 \) and a decreasing function of \( t > 0 \). If we set \( v(x,t) = \partial w(x,t)/\partial x \) then on differentiating the PDE (3.32) we see that \( v(x,t) \) is a solution to the PDE with boundary conditions satisfying

\[
\lim_{t \to 0} v(x,t) = 1, \quad \lim_{x \to 0} v(x,t) \geq 0.
\]

There is some subtlety here because the boundary conditions (3.43) obtained through formal differentiation of (3.42) are only valid since \( \lim_{x \to 0} w(x,0) = 0 \). Otherwise differentiation would generate a delta function singularity at 0 in the initial data because of the inconsistency between the zero Dirichlet condition at \( x = 0 \) in (3.42) which is valid for all \( t > 0 \). The inequalities (3.43) and the maximum principle imply now that \( \partial w(x,t)/\partial x = v(x,t) \geq 0 \) for all \( x,t > 0 \). Since we have already seen that \( P_\tau(x,t) \) is an increasing function of \( x > 0 \), the first inequality of (3.37) holds.

Next let us consider \( u(x,t) = \partial w(x,t)/\partial t \), which we see is also a solution to the PDE (3.32). Furthermore \( u(x,t) \) satisfies the boundary conditions

\[
\lim_{t \to 0} u(x,t) = \mu, \quad \lim_{x \to 0} u(x,t) = 0.
\]

In deriving (3.44) we have used both the boundary conditions (3.42) and the PDE (3.32). Applying the maximum principle again, we see that \( u(x,t) \geq 0 \) for all \( x,t > 0 \). Hence we have that \( \partial w(x,t)/\partial t \geq 0 \), and we have already shown that \( \partial w(x,t)/\partial x \geq 0 \). This situation should be contrasted with the function \( w_1(x,t) = P_\tau(x,t) \) which also satisfies the PDE (3.32). In that case we have seen that \( \partial w_1(x,t)/\partial t \leq 0, \partial w_1(x,t)/\partial x \geq 0 \), from which we can conclude from the PDE (3.32) that \( \partial^2 w_1(x,t)/\partial x^2 \leq 0 \). The second inequality of (3.37) holds therefore for the function \( w_1(x,t) \) but not necessarily for the function \( w(x,t) \). We can argue however that for small \( x,t > 0 \) the function \( w_1(x,t) \) dominates the function \( w(x,t) \) and so we get the graphs as drawn in Figure 3.


**References**


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