EQUIVARIANT INTERSECTION THEORY
§1: TOPOLOGICAL INTERSECTION THEORY

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1. A brief review of singular (co)homology

Let X be any space. One has the *singular homology modules* $H_*X = \bigoplus H_iX$, and the *relative singular homology modules* $H_*(X,Y)$ for subspaces $Y \subseteq X$.

When X is an oriented manifold, Lefschetz and others worked to make H_*X into a ring.

When homology was incorporated into the modern framework of algebraic topology, this became easier. Just as homology modules are defined by $H_iX = H_i(C_*X)$ (where C_*X is the complex of singular chains), one can define cohomology modules by $H^iX = H_i(\mathrm{Hom}(C_*X, \mathbb{Z}))$, and similarly for relative cohomology modules. This is a purely formal construction, sacrificing the geometric intuition that comes with homology. For instance, can you give an explicit example of a singular n-cochain? Of course, there are the constant cochains, but to give a nontrivial example, one must assign a value to every singular n-simplex – and there are a lot of these.

These cohomology modules have the advantage that it is easy to define a ring structure, though. One has the *cup product*, denoted "$\cup$" or "$\cdot$," a map

$$H^iX \otimes H^jX \to H^{i+j}X$$

defined on simplices $\sigma \in H_{i+j}X$ by the following (unintuitive) formula:

$$(c \cup d)(\sigma) = c(\text{front } i\text{-face of } \sigma)\ d(\text{back } j\text{-face of } \sigma).$$

This makes $H^*X = \bigoplus H^iX$ into a skew-commutative graded ring: For $c \in H^iX$ and $d \in H^jX$, one has $c \cup d = (-1)^{ij} d \cup c$. (Note: The sign conventions in the definition of the cup product vary throughout the literature. For example, ours agrees with that of [Spanier], but is the opposite of that of [Milnor-Stasheff].)

H_*X becomes a (left-)module over H^*X via the *cap product*

$$\cap : H^iX \otimes H_jX \to H_{j-i}X,$$

defined by $c \cap \sigma = c(\text{back } i\text{-face of } \sigma)\ (\text{front } (j-i)\text{-face of } \sigma)$.

Associated to a continuous map $f : X \to Y$, one has natural *pushforward* and *pullback* maps on homology and cohomology, respectively: $f_* : H_*X \to$

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H_*Y and $f^* : H^*Y \to H^*X$. These are related by the *projection formula*, sometimes called the "naturality of the cap product":

$$f_*(f^*c \cap \sigma) = c \cap f_*\sigma.$$

If X is triangulated, one also has the *simplicial homology* $H_*^{simp}X$, and a canonical isomorphism $H_*^{simp}X \xrightarrow{\sim} H_*X$. (This shows $H_*^{simp}X$ is independent of the choice of triangulation, a question which was open for some time.)

If M is a compact, connected, oriented n-manifold, then it has a *fundamental class* $[M] \in H_nM \cong \mathbb{Z}$, and there is a canonical isomorphism (the "Poincaré isomorphism") $H^iM \xrightarrow{\sim} H_{n-i}M$, given by $c \mapsto c \cap [M]$. This isomorphism, then, makes $H_*M \cong H^*M$ into a ring.

This is sometimes called "Poincaré duality." In fact, there are several notions of Poincaré duality on a manifold, all related. Suppose $\{z_\alpha\}$ is a homogeneous basis for H_*M. The "dual" basis for H^*M is one of the following:

1. Take classes $y_\alpha \in H^*M$, with $y_\alpha \cap [M] = z_\alpha$. These are the "same" classes, if H^*M is identified with H_*M via the Poincaré isomorphism. ("Dual" is something of a misnomer here, but these classes often turn out to be a better choice.)
2. Take classes $x_\alpha \in H^*M$, such that under the map $H_*M \to H_0M = \mathbb{Z}$, the class $x_\alpha \cap z_\beta$ maps to $\delta_{\alpha\beta}$. (This is more properly called duality, but one needs to specify the entire basis $\{z_\alpha\}$ to find one x_α.)

To add to the potential confusion, there is a pairing

$$\int_M : H^*M \times H^*M \to \mathbb{Z}$$

such that $\int_M x_\alpha \cdot y_\beta = \delta_{\alpha\beta}$. When field coefficients are used, this (as well as the situation in (2)) is a usual vector space duality pairing.

The fundamental class is characterized by the fact that it maps to a generator of $H_n(M, M \smallsetminus \{x\}) \cong H_n(U, U \smallsetminus \{x\}) \cong \mathbb{Z}$ for all $x \in M$, where U is a ball around x. (One begins to see something sheafy in this. Indeed, one can define the *orientation sheaf*, whose stalks correspond to choices of local orientations; the fundamental class is then a global section.)

Assume M is a compact complex manifold, so it has a canonical orientation coming from the complex structure. In fact, M is a nonsingular (projective) algebraic variety. Let $V \subseteq M$ be a closed algebraic subset of dimension k. (Note: When discussing algebraic varieties, dimension will mean *algebraic* dimension; the topological (real) dimension is twice this for a complex variety.)

Claim:

$$H_iV = \begin{cases} 0 & \text{if } i > 2k; \\ \bigoplus \mathbb{Z} & \text{if } i = 2k, \end{cases} \tag{1}$$

where the sum in the second case gives one copy of $\mathbb{Z}$ for each k-dimensional irreducible component of V.

We will return to the proof of this later. In the meantime, note that if V is irreducible, we obtain a class $[V] \in H_{2k}V = \mathbb{Z}$, and pushing forward by the inclusion of V in M, a class $[V] \in H_{2k}M$. Given two subvarieties V and W, of respective codimensions d and e, we therefore get a class $[V] \cdot [W] \in H^{2d+2e}(M)$.

Question: What is this class?

In fact there is a canonical isomorphism

$$H_i V \cong H^{2n-i}(M, M \smallsetminus V), \tag{2}$$

where n is the (algebraic) dimension of M; this isomorphism is often called the "Alexander-Lefschetz" isomorphism. (See [Spanier, p. 296].)

If V and W are as above, then, we have

$$\begin{aligned} [V] &\in H_{top}V \cong H^{2n-2d}(M, M \smallsetminus V) \\ [W] &\in H_{top}W \cong H^{2n-2e}(M, M \smallsetminus W), \end{aligned} \tag{3}$$

so

$$\begin{aligned} [V] \cdot [W] \in \quad & H^{4n-2d-2e}(M, M \smallsetminus (V \cap W)) \\ & \cong H_{2d+2e-2n}(V \cap W). \end{aligned} \tag{4}$$

If V and W meet properly – that is, if $\dim(V \cap W) = d + e - n$ – then

$$H_{2d+2e-2n}(V \cap W) = \bigoplus \mathbb{Z},$$

with one copy of $\mathbb{Z}$ for each irreducible component of $V \cap W$. This assigns *intersection numbers* to each component, so we have an *intersection cycle* $V \cdot W$. (In fact, these intersection numbers agree with those defined algebraically in intersection theory.)

There are several approaches one might use to prove the claim (1). For example, the following have been used:

1. Triangulate V, and use simplicial theory. Then one needs to worry about independence of choice of triangulation (and appeal to singular theory).
2. Resolve singularities of V by a map $\tilde{V} \to V$. Then one has a manifold $\tilde{V}$ and a pushforward $[\tilde{V}] \mapsto [V]$. Again, it is not obvious that this gives something independent of the choice of resolution. (Historically, this was perhaps the first rigorous method.)

2. Borel-Moore homology

A better way, at least for our purposes, is to use *Borel-Moore homology*, which we will denote by $\overline{H}_i X$. There are several ways to define these groups; for example, via

- sheaf theory (as was done originally in [Borel-Haefliger]);

- locally finite chains;
- a one-point compactification X^+, for good spaces X. (Use $H_i(X^+, X^+ \smallsetminus X)$.)

We will use a definition which comes equipped with many nice properties, and which works for any space which can be embedded as a closed subspace of an oriented smooth manifold M. For such a space X, we define

$$\overline{H}_i X := H^{\dim M - i}(M, M \smallsetminus X). \tag{5}$$

Proposition 2.1. *This definition is independent of the choice of embedding. In fact, given closed embeddings of X into manifolds M and M', there is a canonical isomorphism $H^{\dim M - i}(M, M \smallsetminus X) \cong H^{\dim M' - i}(M', M' \smallsetminus X)$. Moreover, if X is embedded into a third manifold M'', these isomorphisms form a commuting triangle:*

$$\begin{tikzcd}
H^{\dim M-i}(M, M \smallsetminus X) \arrow[rr, "\sim"] \arrow[dr, "\sim"'] & & H^{\dim M'-i}(M', M' \smallsetminus X) \arrow[dl, "\sim"] \\
& H^{\dim M''-i}(M'', M'' \smallsetminus X). &
\end{tikzcd}$$

To prove this, we need three key facts:

1. For $A \subset U \subset X$, with A closed in X and U open, there are natural *excision isomorphisms* $H^i(X, U) \cong H^i(X \smallsetminus A, U \smallsetminus A)$.
2. For an oriented rank-r real vector bundle $E \xrightarrow{\pi} X$, and A any subspace of X, there is the *Thom isomorphism*
$$H^i(X, X \smallsetminus A) \cong H^{i+r}(E, E \smallsetminus A).$$
In fact, there is a class $\eta \in H^r(E, E \smallsetminus X)$, called the *Thom class*, characterized by the fact that it restricts to the chosen generator of $H^r(\pi^{-1}(p), \pi^{-1}(p) \smallsetminus \{p\})$ for all $p \in X$. The above isomorphism is given by $c \mapsto \pi^*(c) \cup \eta$. (Note: we always identify X with its embedding by the zero section in a vector bundle.)
3. For a smooth closed submanifold M of a smooth manifold M', there is a neighborhood U of M in M' such that the pair (U, M) is diffeomorphic to (N, M), where N is the normal bundle of the embedding $M \subset M'$.

Proof. (See [Fulton 1997, pp. 216-217].) □

We continue our introduction to Borel-Moore homology with a discussion of functoriality. Note that these groups are not homotopy invariant! (For example, our definition says that $\overline{H}_n \mathbb{R}^n = \mathbb{Z}$.) Nor are they functorial with respect to arbitrary continuous maps. However, there is functoriality in two important situations.

Proposition 2.2 (Covariance for proper maps)**.** *If $f : X \to Y$ is continuous and* proper *(i.e., the inverse image of a compact set is compact), then there* *are maps*

$$f_* : \overline{H}_i X \to \overline{H}_i Y.$$

Proof. (Sketch.) Suppose X is embedded as a closed subspace of $\mathbb{R}^n$, and Y is embedded as a closed subspace of $\mathbb{R}^m$. Since f is proper, we can find a map $\varphi : X \to I^n$ such that

$$(f, \varphi) : X \to Y \times I^n \subset \mathbb{R}^m \times \mathbb{R}^n$$

is a closed embedding. (Here $I^n = [a, b]^n$, with $a < 0 < b$.)[1] Then

$$\begin{aligned}
\overline{H}_i X \quad &= \quad H^{m+n-i}(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^m \times \mathbb{R}^n \smallsetminus X) \\
&\xrightarrow[\text{(restrict)}]{} H^{n+m-i}(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^m \times \mathbb{R}^n \smallsetminus (Y \times I^n)) \\
&\xrightarrow[\text{(homotopy)}]{\sim} H^{m+n-i}(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^m \times \mathbb{R}^n \smallsetminus (Y \times \{0\})) \\
&\xrightarrow[\text{(Thom)}]{\sim} H^{m-i}(\mathbb{R}^m, Y) \\
&= \quad \overline{H}_i Y.
\end{aligned}$$

□

Exercise 2.3. Check independence of choices, and naturality: $(g \circ f)_* = g_* \circ f_*$ for

$$X \xrightarrow{f} Y \xrightarrow{g} Z.$$

We also have *contravariance for open inclusions.* Let $U \subset X$ be an open subspace, embed X in an n-manifold M, and let $Y = X \smallsetminus U$. We get restriction maps $\overline{H}_i X \to \overline{H}_i U$ from the long exact cohomology sequence of the triad $(M, M \smallsetminus Y, M \smallsetminus X)$. Indeed, U is a closed subspace of the manifold $M \smallsetminus Y$, so the map is

$$\overline{H}_i X = H^{n-i}(M, M \smallsetminus X) \to H^{n-i}(M \smallsetminus Y, (M \smallsetminus Y) \smallsetminus U) = \overline{H}_i U.$$

Exercise 2.4. Check independence of choice, and naturality for a sequence of open inclusions $U' \subset U \subset X$.

In the case where M, X, and Y are complex varieties, we will use this restriction map to find isomorphisms on top-dimensional homology. Also, this allows restriction of homology classes to small (classical!) open sets.

Example 2.5. $X = \mathbb{C}^n$. (Or $X =$ ball in $\mathbb{C}^n$.) This is a manifold, so

$$\overline{H}_i X = H^{2n-i}(X, \emptyset) = \begin{cases} \mathbb{Z} & \text{if } i = 2n; \\ 0 & \text{otherwise.} \end{cases}$$

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[1] Take φ to be the composition of $X \hookrightarrow \mathbb{R}^n$ with a homeomorphism $\mathbb{R}^n \to (a, b)^n$, followed by the inclusion $(a, b)^n \subset I^n$. When f is proper, this choice of φ makes (f, φ) a closed embedding. (Note that $X \hookrightarrow Y \times (a, b)^n$ is always a closed embedding; use properness to show that any sequence in X has a limit point in $Y \times (a, b)^n$.)

These two operations – pushforward and restriction – are compatible. If $f : X \to Y$ is proper and $U \subset Y$ is open, then the diagram

$$\begin{tikzcd}
\overline{H}_i X \arrow[r] \arrow[d] & \overline{H}_i(f^{-1}U) \arrow[d] \\
\overline{H}_i Y \arrow[r] & \overline{H}_i U
\end{tikzcd}$$

commutes. (Exercise)

Proposition 2.6. *For $Y \subset X$ closed, with $U = X \smallsetminus Y$, there is a natural long exact sequence*

$$\cdots \to \overline{H}_i Y \to \overline{H}_i X \to \overline{H}_i U \to \overline{H}_{i-1} Y \to \cdots \to \overline{H}_0 U \to 0. \tag{6}$$

This sequence will allow inductions on dimension, when the homologies of two of the three spaces are known.

Proof. When X is embedded as a closed subspace of a manifold M, this is just the long exact cohomology sequence of the triad $(M, M \smallsetminus Y, M \smallsetminus X)$. $\square$

3. Classes of subvarieties

Now we can prove the claim from Section 1:

Claim: Let V be a k-dimensional quasi-projective algebraic variety. Then $\overline{H}_i V = 0$ for $i > 2k$, and $\overline{H}_{2k} = \bigoplus \mathbb{Z}$, with one copy of $\mathbb{Z}$ for each k-dimensional irreducible component of V.

Proof. In general, if X is a disjoint union of spaces X_j, then $\overline{H}_i X = \bigoplus_j \overline{H}_i X_j$. (Exercise.) Now when V is nonsingular, this observation reduces the claim to the case where V is irreducible. Indeed, in a nonsingular variety, connected components coincide with irreducible components. But then V is a connected manifold, so $\overline{H}_i V = H^{2k-i} V$; this is 0 for $i > 2k$ and $\mathbb{Z}$ for $i = 2k$.

For a general (possibly singular and reducible) k-dimensional complex variety V, we can reduce to the nonsingular case using the long exact sequence of Borel-Moore homology. Let $W \subset V$ be the closed set consisting of the singular locus of V, together with all irreducible components of dimension $< k$; thus W is a variety of dimension $< k$. By induction on dimension, then, we may assume the claim holds for W. In particular, $\overline{H}_i W = 0$ for $i > 2k - 2$. Now apply the long exact sequence for $W \subset V$: we have

$$0 = \overline{H}_{2k} W \to \overline{H}_{2k} V \to \overline{H}_{2k}(V \smallsetminus W) \to \overline{H}_{2k-1} W = 0. \tag{7}$$

Thus $\overline{H}_{2k}(V) \cong \overline{H}_{2k}(V \smallsetminus W)$, and since the latter is a k-dimensional nonsingular variety, the claim follows. (The same argument also shows vanishing for $i > 2k$.) $\square$

Remark 3.1. An essential ingredient in the above proof is the fact that the singular locus of a complex variety is a subvariety of *complex* codimension at least 1, and hence of *real* codimension at least 2. Things are somewhat more subtle in the real world, and one must impose additional hypotheses for a similar claim to hold on real varieties.

We therefore have a *fundamental class* $[V] \in \overline{H}_{2k}V \cong \mathbb{Z}$, for any irreducible k-dimensional variety V. If V is a closed subvariety of a nonsingular variety X, we also get a fundamental class corresponding to V in $H^{2d}(X)$, where d is the codimension of V. This comes from

$$\overline{H}_{2k}V = H^{2d}(X, X \smallsetminus V) \to H^{2d}(X).$$

The element $\eta_V \in H^{2d}(X, X \smallsetminus V)$ corresponding to $[V] \in \overline{H}_{2k}V$ is called the *refined class* of V in X.

Remark 3.2. One should expect a class representing V in $H^{2d}(X)$ to be "supported on V," and hence to come from $H^{2d}(X, X \smallsetminus V)$. We need a canonical representative, though, and the fundamental class of Borel-Moore homology gives us one.

We now turn to the behavior of these classes under morphisms.

Proposition 3.3. *Let X and Y be nonsingular varieties of respective dimensions n and m, and let $f : X \to Y$ be a propoer morphism. Let $V \subset X$ be a closed subvariety, and let $W = f(V) \subset Y$. Then*

$$\begin{tikzcd}
f_* \colon & H^i(X) \arrow[r] \arrow[d, equal] & H^{2m+2n-i}(Y) \arrow[d, equal] \\
& H_{2n-i}(X) \arrow[r] & H_{2n-i}(Y)
\end{tikzcd}$$

maps

$$[V] \mapsto \begin{cases} 0 & \textit{if} \quad \dim W < \dim V; \\ d[W] & \textit{if} \quad \dim W = \dim V, \end{cases}$$

where d is the degree of V over W. (By definition, this is the degree of the field extension $[\mathbb{C}(V) : \mathbb{C}(W)]$.)

Proof. The first case ($\dim W < \dim V$) is clear from the claim (1). In the second case, there is an open set $U \subset W$ such that $f^{-1}U \cap V \to U$ is a d-sheeted covering. Taking U to be sufficiently small, we may assume it is a ball, and we have $f^{-1}U \cap V = U_1 \coprod U_2 \coprod \cdots \coprod U_d$, with each U_i mapping isomorphically to U.

Then we have a commuative diagram

$$\begin{tikzcd}
\bigoplus_{i=1}^{d} \overline{H}_{2k}(U_i) \arrow[r, phantom, "="] \arrow[dr] & \overline{H}_{2k}(f^{-1}U \cap V) \arrow[d] & \overline{H}_{2k}V \arrow[l] \arrow[r] \arrow[d] & \overline{H}_{2k}X \arrow[d] \\
& \overline{H}_{2k}U & \overline{H}_{2k}W \arrow[l, "\sim"'] \arrow[r] & \overline{H}_{2k}Y,
\end{tikzcd}$$

and the proposition follows. $\square$

The next proposition is about "compatibility of pullbacks."

Proposition 3.4. *Let* $f : X \to Y$ *be a morphism of nonsingular varieties, with* $V \subset Y$ *irreducible of codimension* d. *Assume there is a classical open neighborhood* $U \subset Y$ *where* $V \cap U = V^o \subset U$ *is connected, nonsingular, and defined by (holomorphic) equations* $h_1, \ldots, h_d$, *such that* $W \cap f^{-1}U = W^o \subset f^{-1}U$ *is also connected, nonsingular, and defined by the equations* $h_1 \circ f, \ldots, h_d \circ f$. *Then* $f^*\eta_V = \eta_W$.

Proof. We have

$$\mathbb{Z} = H^{2d}(Y, Y \smallsetminus V) \xrightarrow{f^*} H^{2d}(X, X \smallsetminus W) = \mathbb{Z},$$

so $\eta_V \mapsto c \cdot \eta_W$ for some $c \in \mathbb{Z}$. The assumptions on U and $f^{-1}U$ guarantee that the restriction maps are isomorphisms:

$$\begin{array}{rcl} H^{2d}(Y, Y \smallsetminus V) & \xrightarrow{\sim} & H^{2d}(U, U \smallsetminus V^o) \\ H^{2d}(X, X \smallsetminus W) & \xrightarrow{\sim} & H^{2d}(f^{-1}U, f^{-1}U \smallsetminus W^o). \end{array}$$

Thus we may replace Y with U, and reduce to the situation where Y is a vector bundle over V – in fact, a trivial bundle – and similarly X is a vector bundle over W. The claim is that the Thom class in $H^{2d}(Y, Y \smallsetminus V)$ pulls back to the Thom class in $H^{2d}(X, X \smallsetminus V)$. But the hypotheses mean that X is the pullback of Y, when X and Y are considered as vector bundles over W and V. The assertion thus reduces to naturality of the Thom class of vector bundles. $\square$

Thus for irreducible closed subvarieties V and W of a nonsingular variety X, with $\dim V = k$, $\dim W = l$, and $\dim X = n$, we obtain an intersection class in $\overline{H}_{2k+2l-2n}(V \cap W) = H^{2k+2l}(X, X \smallsetminus (V \cap W))$, corresponding to $[V] \cdot [W]$. Each $(k+l-n)$-dimensional irreducible component Z of $V \cap W$ gives an intersection number $i(Z, V \cap W, X)$, which is the projection of $[V] \cdot [W]$ onto the factor of $\overline{H}_{2k+2l-2n}(V \cap W) = \bigoplus \mathbb{Z}$ corresponding to Z.

Exercise 3.5. If there is an open set of Z on which V and W meet transversally, then $i(Z, V \cap W, X) = 1$. (See [Fulton 1997, p. 222].)

Remark 3.6. Geometric considerations ("reduction to the diagonal") can be used to show that the intersection numbers $i(Z, V \cap W, X)$ are nonnegative whenever X is a manifold; hence any product $[V] \cdot [W]$ can be expressed as a *nonnegative* sum of classes $[Z]$. Is this a formal property of the theory, or does it depend on the geometry? For example, if X is a $\mathbb{Q}$-homology manifold (i.e., a variety having the $\mathbb{Q}$-homology of a manifold), is it true that the corresponding intersection numbers $i(Z, V \cap W, X)$ are positive rational numbers?

Remark 3.7. When X is a manifold, the ring structure on H^*X comes from the diagonal embedding. More specifically, one has the diagonal map $\delta : X \to X \times X$, and for classes $\alpha, \beta \in H^*X$,

$$\alpha \cup \beta = \delta^*(\alpha \times \beta) = \delta^*(p_1^*(\alpha) \cup p_2^*(\beta)). \tag{8}$$

For subvarieties V, W, we get $[V] \cdot [W] = \delta^*[V \times W]$. This technique of "reduction to the diagonal" works because δ is a *regular embedding.*

(Q: who first thought of this?)

Proposition 3.8. *If $X = X_s \supset X_{s-1} \supset \cdots \supset X_0 = \emptyset$ are closed algebraic subsets, and $X_i \smallsetminus X_{i-1} = \coprod_j U_{ij}$ with $U_{ij} \cong \mathbb{C}^{n(i,j)}$, then the classes $[\overline{U}_i j]$ form a $\mathbb{Z}$-linear basis for $\overline{H}_*X$.*

Proof. Use induction on i, and assume the proposition holds for X_{i-1}. Associated to the inclusion $X_{i-1} \subset X_i$, we have an exact sequence

$$\to \overline{H}_k(X_{i-1}) \to \overline{H}_k(X_i) \to \bigoplus_j \overline{H}_k(U_{ij}) \to \overline{H}_{k-1}(X_{i-1}) \to . \tag{9}$$

When k is odd, $\overline{H}_k(X_{i-1}) = 0$ by induction, and $\overline{H}_k(U_{ij}) = \overline{H}_k(\mathbb{C}^{n(i,j)}) = 0$ by calculation. Therefore $\overline{H}_k X_i = 0$, as well, and we have short exact sequences

$$0 \to \overline{H}_{2k}(X_{i-1}) \to \overline{H}_{2k}(X_i) \to \bigoplus_j \overline{H}_{2k}(U_{ij}) \to 0. \tag{10}$$

Now $[\overline{U}_{ij}]$ maps to $[U_{ij}]$ under the map $\overline{H}_*(X_i) \to \oplus_j \overline{H}_*(U_{ij})$, and the latter classes form a basis, so the former are independent in $\overline{H}_*(X_i)$. The proposition follows. □

(It isn't so easy to prove an analogous statement in singular homology!)

Exercise 3.9. Show that $H^*\mathbb{P}^n = \mathbb{Z}[\zeta]/(\zeta^{n+1})$, where ζ corresponds to $[H]$, the class of a hyperplane.

Exercise 3.10. Compute f^* and f_* for the Segre embedding $\mathbb{P}^m \times \mathbb{P}^n \hookrightarrow \mathbb{P}^{nm+n+m}$.

Exercise 3.11. If a connected group G acts continuously on a space X, show that g acts trivially on H^*X for each $g \in G$. For varieties, show $g \cdot [V] = [gV] = [V]$.

Remark 3.12. In fact, more is true: The operations from intersection theory in algebraic geometry make sense in Borel-Moore homology, and are compatible. See [Fulton 1984, §19].

4. GYSIN MAPS

Suppose Y is a nonsingular subvariety of dimension n in a nonsingular variety X of dimension m, and write $d = m - n$ for the codimension. Let $\iota : Y \hookrightarrow X$ be the inclusion. Then there are *Gysin maps*

$$H^iY \xrightarrow{\iota_*} H^{i+2d}X.$$

These can be constructed as

$$H^iY = \overline{H}_{2n-i}Y \xrightarrow{\iota_*} \overline{H}_{2n-i}X = H^{i+2d}X.$$

More directly, let $U \subset X$ be a tubular neighborhood of Y in X, and let N be the normal bundle of Y. Then the Gysin map ι_* is the composition

$$\begin{aligned} H^iY \cong H^{i+2d}(N, N \smallsetminus Y) &= H^{i+2d}(U, U \smallsetminus Y) \\ &= H^{i+2d}(X, X \smallsetminus Y) \to H^{i+2d}X. \end{aligned}$$

In particular, ι_* factors through $H^{i+2d}(X, X \smallsetminus Y)$.

This Gysin map has the following properties:

(1) For $c \in H^*X$ and $d \in H^*Y$, there is the *projection formula*

$$\iota_*(\iota^*(c) \cdot d) = c \cdot \iota_*(d).$$

(2) It is functorial for compositions of inclusions $Z \subset Y \subset X$.

(3) If $j : Z \hookrightarrow X$, and $Z \cap Y = \emptyset$, then the composition

$$H^iY \xrightarrow{\iota_*} H^{i+2d}X \xrightarrow{j^*} H^{i+2d}Z$$

is the zero map.

(4) The composition

$$H^iY \xrightarrow{\iota_*} H^{i+2d}X \xrightarrow{\iota^*} H^{i+2d}Y$$

is multiplication by $c_d(N)$ (where N is the normal bundle to Y).

(5) If we have a fiber square

$$\begin{tikzcd} Y \arrow[r, hook, "\iota"] \arrow[d, "g"'] & X \arrow[d, "f"'] \\ Y' \arrow[r, hook, "\iota'"] & X', \end{tikzcd}$$

then the corresponding diagram on cohomology commutes, i.e., $f^* \circ \iota'_* = \iota_* \circ g^*$.

Remark 4.1. More generally, there are Gysin pushforward maps induced by certain "nice" maps of varieties. For example, assuming all varieties are pure-dimensional (but not necessarily irreducible), and all maps are proper, there are Gysin maps for the following:

(1) Inclusions of nonsingular varieties $Y \subset X$, as above.

(2) Any (proper) map $Y \to X$, where X is nonsingular. Regardless of whether Y is nonsingular, there is a map $H^*Y \to \overline{H}_*Y$ given by cap product with $[Y]$. Composing this with $\overline{H}_*Y \to \overline{H}_*X = H^*X$ gives the pushforward.

(3) Regular embeddings $Y \hookrightarrow X$. That is (for general X) Y is locally defined by a regular sequence, so the ideal sheaf I/I^2 is locally free. (This is somewhat harder to construct.)

(4) Smooth maps $Y \to X$.

(5) Local complete intersection morphisms $Y \to X$. (This follows from (3) and (4), since any such map may be factored $Y \hookrightarrow X \times P \to X$ as a regular embedding followed by a smooth projection.)

As a special case of (4), suppose $f : Y \to X$ is proper and a locally trivial fiber bundle, whose fiber is a (compact) oriented n-manifold. Then the Gysin map is $f_* : H^iY \to H^{i-n}X$. When X is an oriented $(m-n)$-manifold, this is just the composition

$$H^iY \xrightarrow{\cap[Y]} \overline{H}_{m-i}Y \xrightarrow{f_*} \overline{H}_{m-i}X \cong H^{i-n}X.$$

In general, f factors as

$$Y \hookrightarrow X \times I^p \hookrightarrow X \times \mathbb{R}^p \to X.$$

There is a tubular neighborhood U of Y in $X \times \mathbb{R}^p$; letting N denote the normal bundle, of rank $d = p - n$, we have

$$\begin{aligned} H^iY \cong H^{i+d}(N, N \smallsetminus Y) &= H^{i+d}(U, U \smallsetminus Y) \\ &= H^{i+d}(X \times \mathbb{R}^p, X \times \mathbb{R}^p \smallsetminus Y) \\ &\to H^{i+d}(X \times \mathbb{R}^p, X \times \mathbb{R}^p \smallsetminus (X \times I^p)) \cong H^{i-n}X. \end{aligned}$$

There is an important subtlety in the form of functoriality of Gysin maps. Suppose X and Y are (real) manifolds, with $f : Y \to X$ a proper map. To construct the Gysin maps associated to an embedding $Y \hookrightarrow X$, we need an *oriented* normal bundle; similarly, if $f : Y \to X$ is a fiber bundle with smooth fiber, the fiber must be an *oriented* manifold.

In the latter case, the construction depends on a factorization of f as the composition $Y \hookrightarrow X \times \mathbb{R}^n \to X$. Let N be the normal bundle to the embedding $Y \subset X \times \mathbb{R}^n$ – this is called the *stable normal bundle.* (Here "stable" is meant in the following sense: two bundles are "stably equivalent" if they become isomorphic after adding a trivial bundle to each.)

Suppose we have $Z \xrightarrow{g} Y \xrightarrow{f} X$, with f, g proper (so $g \circ f$ is proper, too); assume f and g are fiber bundles with oriented stable normal bundles. Then $g \circ f$ has a unique compatible orientation, and with this orientation, we have $(g \circ f)_* = g_* \circ f_*$.

In general, though, given orientations for f, g, and $g \circ f$ may or may not be compatible. If the orientations are not compatible, we will have $(g \circ f)_* = -g_* \circ f_*$.

A similar issue arises in functoriality for embeddings, $Z \subset Y \subset X$. There is an exact sequence

$$0 \to N_{Z/Y} \to N_{Z/X} \to N_{Y/X} \to 0.$$

A general sequence of oriented bundles, $0 \to E \to F \to G \to 0$, is either compatible with the orientations or not – this means the canonical isomorphism $\Lambda^{top}E \otimes \Lambda^{top}G \xrightarrow{\sim} \Lambda^{top}F$ either preserves or reverses signs.

Example 4.2. Let $p : X \to S$ be a proper fiber bundle with fiber F, an n-dimensional oriented manifold. Consider the fiber square

$$\begin{tikzcd}
X \times_S X \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "p"] \\
X \arrow[r, "p"] & X.
\end{tikzcd}$$

An orientation for F determines one for $p : X \to S$, as well as for p_1 and p_2, since these have fiber F. It also determines an orientation for $q : X \times_S X \to S$ (using the standard orientation of the fiber $F \times F$). Now $q = p \circ p_1$, with compatible orientations, so $q_* = p_* \circ (p_1)_*$. On the other hand, $q = p \circ p_2$ has incompatible orientations if n is odd: $q_* = (-1)^n p_* \circ (p_2)_*$. (This is because an orientation of a vector space V induces a standard orientation of $V \oplus V$, which is reversed by the involution $(v, w) \mapsto (w, v)$ if $n = \dim V$ is odd.)

Exercise 4.3. Let $f : X \to Y$ be one of the above types of maps with Gysin pushforwards (an embedding of codimension n, or a smooth map of relative dimension n). Then the projection formula has the following form:

$$f_*(x \cdot f^*(y)) = (-1)^{n \cdot \deg(x)} f_*(x) \cdot y.$$

5. More on Poincaré duality

For our purposes, the right setting in which to express Poincaré duality is the following. Let $p : X \to S$ be a fiber bundle with smooth oriented fiber F, as above. Set $\langle x, y\rangle = p_*(x \cdot y) \in H^*S$. If H^*F is free over the ground ring R, and $H^*X \to H^*F$ is surjective, then H^*X is free over H^*S. (When dealing spaces with homology in odd degrees, one must be careful with conventions about left- and right-modules.)

If $x_1, \ldots, x_m \in H^*X$ are homogeneous elements such that $\bar{x}_1, \ldots, \bar{x}_m$ form an R-basis for H^*F, they give a basis for H^*X over H^*S. (Left or right.)

Lemma 5.1. *Given the above situation, there are unique elements* $y_1, \ldots, y_m \in H^*X$ *such that* $\langle x_i, y_j\rangle = \delta_{ij}$, *the set* $\{y_j\}$ *forms a basis for* H^*X *over* H^*S, *and if* $y = \sum a_i y_i$, *we have* $a_i = \pm\langle x_i, y\rangle$.

Exercise 5.2. Compare $x_i \cdot x_j = \sum_k c_{ij}^k x_k$ with $\delta_*(y_k) = \sum_{i,j} a_{ij}^k y_i \times y_j$, where $\delta : X \to X \times_S X$ is the diagonal map, $y_i \times y_j = p_1^*(y_i) \cdot p_2^*(y_j)$, and $a_{ij}^k, c_{ij}^k \in H^*S$. In fact, show that $c_{ij}^k = \pm a_{ij}^k$, and determine the sign.

(Of course, when all degrees are even – as they will be for applications in algebraic geometry – there is no sign.) The geometric meaning of this exercise is that the structure constants of the cohomology ring determine the Gysin map δ_*, and vice versa.

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