

EQUIVARIANT INTERSECTION THEORY

§2: CHERN CLASSES

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1. DEFINITIONS

Let X be a topological space, and let $E \xrightarrow{\pi} X$ be a (complex) vector bundle of rank r on X . Associated to E , there are *Chern classes* $c_i(E) \in H^{2i}(X)$, for $i \geq 0$, with $c_0(E) = 1$ and $c_i(E) = 0$ for $i > r$. These satisfy the following properties:

- (1) If $f : Y \rightarrow X$ is a continuous map, then $c_i(f^*E) = f^*c_i(E)$.
- (2) If $E' \subset E$ is a subbundle, with quotient bundle $E/E' \cong E''$, then $c_k(E) = \sum_{i+j=k} c_i(E') \cdot c_j(E'')$. (One can write this more succinctly by introducing the *total Chern class* $c(E) = c_0(E) + c_1(E) + \cdots + c_r(E)$. The above relation becomes $c(E) = c(E') \cdot c(E'')$.)
- (3) If L and M are line bundles, $c_1(L \otimes M) = c_1(L) + c_1(M)$.
- (4) If X is a nonsingular algebraic variety (or complex manifold), L is a line bundle, and s is a section whose zero locus $D = Z(s)$ is reduced and irreducible, then $c_1(L) = [D]$.

These classes can be defined in the following way. First, we say what the first Chern class of a line bundle is:

Definition. Let $L \xrightarrow{\pi} X$ be a line bundle, and let $\gamma_L \in H^2(L, L \setminus X)$ be its Thom class. Define $c_1(L) \in H^2(X)$ to be the image of γ_L under the map

$$H^2(L, L \setminus X) \rightarrow H^2(L) \xrightarrow{(\pi^*)^{-1}} H^2(X).$$

That is, $c_1(L)$ is the unique class such that $\pi^*(c_1(L))$ is the restriction of γ_L to $H^2(L)$.

The classes defined this way satisfy property (1) by naturality of the Thom class. The proof of (4) is essentially the observation that the Thom class γ_L pulls back to the refined class of D in $H^2(X, X \setminus L)$.

Exercise 1.1. If $Z(s) = \sum m_i D_i$ (as a divisor), with each D_i reduced and irreducible, show that $c_1(L) = \sum m_i [D_i]$. [Hint: reduce to showing that the map $f^* : \mathbb{Z} = H^2(\mathbb{C}, \mathbb{C} \setminus 0) \rightarrow H^2(\mathbb{C}, \mathbb{C} \setminus 0) = \mathbb{Z}$ is multiplication by m , where f^* is induced by $f : z \mapsto z^m$.]

A consequence of naturality (property (1) is that Chern classes (of line bundles) need only be defined for the universal line bundle $\mathcal{O}(1)$ on \mathbb{P}^n .¹ Indeed, suppose s_1, \dots, s_n are continuous sections of L , which generate L (meaning for each $p \in X$ there is some i such that $s_i(p) \neq 0$). These give rise to a surjection $\mathbb{C}_X \rightarrow L \rightarrow 0$. By the universal property of \mathbb{P}^n , this corresponds to a continuous map $f : X \rightarrow \mathbb{P}^n$ such that $L \cong f^*\mathcal{O}(1)$.

Exercise 1.2. Check that $c_1(\mathcal{O}(1)) = [H] = \zeta$, so that $c_1(L) = f^*\zeta$.

Exercise 1.3. Use the previous exercise, together with the Segre embedding, to prove property (3). (That is, $c_1(L \otimes M) = c_1(L) + c_1(M)$.)

Having defined Chern classes of line bundles, we can define Chern classes of arbitrary vector bundles by considering the associated projective bundle. Let $E \xrightarrow{\pi} X$ be a rank- r vector bundle. The *projective bundle* $\mathbb{P}(E) \xrightarrow{p} X$ is the bundle of lines in the fibers of E . This is a locally trivial fiber bundle, with fiber \mathbb{P}^{r-1} , and on it there is a tautological line bundle $\mathcal{O}(-1) \subset p^*E$.

Set $\zeta = c_1(\mathcal{O}(1)) = -c_1(\mathcal{O}(-1))$.

Claim: The classes $1, \zeta, \zeta^2, \dots, \zeta^{r-1}$ form a basis for $H^*(\mathbb{P}(E))$ over $H^*(X)$.

(This is a special case of a general ‘‘Leray-Hirsch’’-type result, which we will discuss later. The usual method of proving such a statement is to first do it for trivial bundles (using a Künneth formula), and then use a Mayer-Vietoris argument.)

Thus there are unique classes $a_i \in H^{2i}(X)$, for $1 \leq i \leq r$, such that $\zeta^r + p^*(a_1)\zeta^{r-1} + \dots + p^*(a_r) = 0$.

Definition. With notation as above, set $c_i(E) = a_i$.

Property (1) is easily verified using the naturality of the constructions involved in the definition. To prove (2), we will need some lemmas.

Lemma 1.4. *If $E = \bigoplus_{i=1}^r L_i$ is a sum of line bundles L_i , then $c_k(E) = e_k(c_1(L_1), \dots, c_1(L_r))$, where e_k is the k th elementary symmetric polynomial.*

Proof. On $\mathbb{P}(E)$, we have $\mathcal{O}(-1) \hookrightarrow \bigoplus p^*L_i$, and tensoring with $\mathcal{O}(1)$, this becomes $\mathbb{C}_{\mathbb{P}(E)} \hookrightarrow \bigoplus (p^*L_i \otimes \mathcal{O}(1))$. From this, we see that $\bigoplus (p^*L_i \otimes \mathcal{O}(1))$ has a nowhere-vanishing section $s = \bigoplus s_i$, where s_i is a section of $p^*L_i \otimes \mathcal{O}(1)$. Set $U_i = \{x \in \mathbb{P}(E) \mid s_i(x) \neq 0\}$, so $\mathbb{P}(E) = \bigcup U_i$. Now since $Z(s_i) = \mathbb{P}(E) \setminus U_i$, the class $c_1(p^*L_i \otimes \mathcal{O}(1))$ comes from a class in $H^2(\mathbb{P}(E), U_i)$. Therefore $\prod_i c_1(p^*L_i \otimes \mathcal{O}(1))$ comes from a class in $H^2(\mathbb{P}(E), \bigcup U_i)$. Since this group is 0, we have

$$\prod_i c_1(p^*L_i \otimes \mathcal{O}(1)) = \prod_i (p^*c_1(L_i) + \zeta) = 0,$$

¹Here and throughout, $\mathcal{O}(-1) \rightarrow \mathbb{P}^n$ is the tautological subbundle of $\mathbb{C}_{\mathbb{P}^n}^{n+1}$, and $\mathcal{O}(1) = \mathcal{O}(-1)^\vee$ is its dual.

so $\zeta^r + p^*(a_1)\zeta^{r-1} + \cdots + p^*(a_r) = \prod_i(p^*c_1(L_i) + \zeta)$, and the statement follows. \square

2. THE SPLITTING PRINCIPLE

Proposition 1. *Given a vector bundle $E \rightarrow X$, there is a space X' , with $f : X' \rightarrow X$, such that*

- (1) $f^*E \cong L_1 \oplus \cdots \oplus L_r$, where the L_i 's are line bundles, and
- (2) $f^* : H^*X \rightarrow H^*X'$ is injective.

The classes $c_1(L_i) \in H^*X'$ are called the *Chern roots* of E .

Proof. (Sketch.) The standard construction goes as follows. Consider $\mathbb{P}(E) \xrightarrow{p} X$, with the tautological line bundle $L \subset p^*E$. Choose a Hermitian metric on E (here assuming X is paracompact), and let $E' = L^\perp \subset p^*E$, and let $E' = L^\perp$, so $p^*E = L \oplus E'$. Repeat this process with $\mathbb{P}(E') \xrightarrow{p'} \mathbb{P}(E)$; get $(p')^*E' = L'' \oplus E''$, etc. \square

Remark 2.1. In fact, one doesn't need to require that E splits as a direct sum of line bundles on X' to use the splitting principle. It is enough to have a sequence of subbundles

$$0 \subset E_1 \subset \cdots \subset E_r = E,$$

with line bundle factors $L_i = E_i/E_{i-1}$.

Exercise 2.2. In this setup, show by induction that

$$c_k(E) = e_k(c_1(L_1), \dots, c_1(L_r)).$$

Remark 2.3. Although we used the non-algebraic technique of choosing a Hermitian metric, the above constructions can be done within algebraic geometry, and therefore apply to Chow groups. One proceeds as follows:

- (1) Given a subbundle $E' \subset E$, show that there is an *affine bundle* $A \xrightarrow{\pi} X$ – that is, A is a locally trivial bundle with fibers \mathbb{A}^N and affine isomorphisms for transition functions – such that $\pi^*E = \pi^*E' \oplus E''$, and $H^*X \xrightarrow{\pi^*} H^*A$ is an isomorphism. In fact, take A to be the subbundle of $\text{Hom}(E, E')$ consisting locally of those $\varphi : E \rightarrow E'$ such that the composition $E' \subset E \xrightarrow{\varphi} E'$ is the identity.
- (2) Given a vector bundle E on X , show there is a locally trivial bundle $A \xrightarrow{\pi} X$ such that $\pi^*E = L_1 \oplus \cdots \oplus L_r$, and $H^*X \hookrightarrow H^*A$.

The splitting principle allows a proof of the Whitney sum formula:

$$c_k(E) = \sum_{i+j=k} c_i(E')c_j(E''),$$

for $E = E' \oplus E''$ (or for exact sequences $0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$). The idea is to find a space $X' \rightarrow X$ which splits both E' and E'' , and reduce to the case $E = \bigoplus L_i$.

Exercise 2.4. Find and prove a formula for $c_k(E \otimes L)$ in terms of $c_i(E)$ and $c_1(L)$, where L is a line bundle.

Exercise 2.5. If $\text{rk}(E) = r$, show that $c_1(E) = c_1(\bigwedge^r E)$.

Remark 2.6. In principle, one could do the same for $c(E \otimes E')$, $c(\bigwedge^i E)$, and $c(\text{Sym}^i E)$, with E and E' of arbitrary rank. A closed formula is not known, though!

Exercise 2.7. For a any vector bundle E , $c_i(E^\vee) = (-1)^i c_i(E)$.

Remark 2.8. Suppose E is a rank- r vector bundle on X , with continuous sections s_1, \dots, s_m which *generate* E , in the sense that there is a surjective map of bundles $\mathbb{C}^m \rightarrow E$. This data gives rise to a continuous map $f : X \rightarrow \text{Gr}(m - r, \mathbb{C}^m)$, such that $E = f^*Q$, where

$$0 \rightarrow S \rightarrow \mathbb{C}_{\text{Gr}}^m \rightarrow Q \rightarrow 0$$

is the tautological/universal exact sequence on Gr . Therefore $c_i(E) = f^*c_i(Q)$. If one computes $H^*\text{Gr}(m - r, m)$, and computes (or specifies) the Chern classes $c_i(Q)$, one gets an alternative construction of $c_i(E)$. (Cf. [Milnor-Stasheff, §§5-7,14].)

We will need a general fact from algebraic topology:

Theorem 2.9. *Let $E \xrightarrow{\pi} B$ be a locally trivial fiber bundle, with fiber F . (That is, there is an open cover $\{U_\alpha\}$ of B , such that $\pi^{-1}U_\alpha \cong U_\alpha \times F$ via an isomorphism compatible with the projections to U_α .) Let R be a commutative ring with unit, and consider cohomology with coefficients in R . Let $m \geq 0$. Assume there exist a finite number of elements $x_{ij} \in H^i(E)$ for $0 \leq i \leq m$, such that their retrictions to each fiber $F (= \pi^{-1}(p))$ give a basis for $H^i(F)$ over R .*

Then every element in $H^m(E)$ can be uniquely expressed in the form $\sum_{ij} c_{ij} x_{ij}$ with $c_{ij} \in H^{m-i}(B)$.

This is a general form of a ‘‘Leray-Hirsch’’-type theorem. As far as we know, the statement closest to this appearing in the literature is [Hatcher, p. 432]; a similar method can be used to prove the slightly more general statement above.

An example we have already seen is the case of a projective bundle $\mathbb{P}(E) \rightarrow X$, with the classes $1, \zeta, \zeta^2, \dots, \zeta^{r-1}$ giving a basis for $H^*(\mathbb{P}(E))$ over $H^*(X)$.

We will also use the following basic facts relating Chern classes with classes of certain subvarieties:

Lemma 2.10. *Let E be a vector bundle of rank d on a nonsingular variety X , and let $s : X \rightarrow E$ be a section. If s intersects the zero-section transversally, then $[Z(s)] := [s^{-1}(0)] \in H^{2d}X$ is equal to $c_d(E)$.*

Proof. We may use the splitting principle to assume $E = L_1 \oplus \cdots \oplus L_d$, and write $s = s_1 \oplus \cdots \oplus s_d$. Since s is transversal to the zero-section of E , each s_i is transversal to the zero-section of L_i . Therefore $[Z(s_i)] = c_1(L_i) \in H^2 X$, and

$$\begin{aligned} [Z(s)] &= [Z(s_1) \cap \cdots \cap Z(s_d)] \\ &= [Z(s_1)] \cdots [Z(s_d)] \\ &= c_1(L_1) \cdots c_1(L_d) \\ &= c_d(E). \end{aligned}$$

□

Lemma 2.11. *Let $F \subset E$ be a subbundle on a nonsingular variety X , and let $d = \text{rk}(E/F)$. Then the class $[\mathbb{P}(F)] \in H^{2d}(\mathbb{P}(E))$ is equal to $c_d(E/F \otimes \mathcal{O}_{\mathbb{P}(E)}(1))$.*

Proof. On $\mathbb{P}(E)$, we have

$$\begin{array}{ccccccc} & & & \mathcal{O}(-1) & & & \\ & & & \downarrow & \searrow & & \\ 0 & \longrightarrow & F & \longrightarrow & E & \longrightarrow & E/F \longrightarrow 0. \end{array}$$

Now $\mathbb{P}(F) \subset \mathbb{P}(E)$ is the locus where $\mathcal{O}(-1)$ is contained in F ; that is, the zero-locus of the map $\mathcal{O}(-1) \rightarrow E/F$. (This holds scheme-theoretically, and in fact, functorially.) Another way to put this is to consider the section s of $E/F \otimes \mathcal{O}(1)$ coming from $\mathcal{O}(-1) \otimes \mathcal{O}(1) = \mathcal{O} \rightarrow E/F \otimes \mathcal{O}(1)$. One checks that s is transversal to the zero-section, and therefore $[\mathbb{P}(F)] = [Z(s)] = c_d(E/F \otimes \mathcal{O}(1))$. □

3. MORE ON THE SPLITTING PRINCIPLE: BASIC CONSTRUCTIONS

Let E be a rank- r vector bundle on X . There is a *flag bundle* $Fl(E) \rightarrow X$, whose fiber over $x \in X$ is $\{V_1 \subset V_2 \subset \cdots \subset V_r = E_x\}$, with $\dim V_i = i$. (That is, the fiber over x is a flag of subspaces of the fiber E_x .)

This is constructed as follows. Start with $\mathbb{P}(E) \xrightarrow{p_1} X$, with tautological line bundle $L_1 \subset p_1^*E$. Then construct $\mathbb{P}(p_1^*E/L_1) \xrightarrow{p_2} \mathbb{P}(E)$, with tautological line bundle $L_2 \subset p_2^*(p_1^*E/L_1)$. Note that $L_2 \cong U_2/p_2^*L_1$ for some rank-2 subbundle $U_2 \subset p_2^*(p_1^*E)$. Note also that a line in p_1^*E/L_1 is a plane containing L_1 , so the fiber of $\mathbb{P}(p_1^*E/L_1) \rightarrow X$ over x is the set of chains $V_1 \subset V_2 \subset E_x$. Continue in this fashion, projectivizing the quotients of E by tautological line bundles, to obtain $Fl(E)$.

This flag bundle has the following universal property: $Fl(E)$ represents the functor which assigns to each (continuous) map $S \xrightarrow{f} X$ the set of flags $\{E_1 \subset E_2 \subset \cdots \subset E_r = f^*E\}$. That is, given such a map f , there is a

unique map φ making the diagram

$$\begin{array}{ccc} & Fl(E) & \\ & \nearrow \varphi & \downarrow p \\ S & \xrightarrow{f} & X \end{array}$$

commute, and such that $E_i = \varphi^*F_i$, where $F_1 \subset \cdots \subset F_r = p^*E$ is the tautological/universal flag on $Fl(E)$. This universal property follows from the construction of $Fl(E)$, and the universal property of $\mathbb{P}(E)$.

There are other ways to construct flag bundles. For example, one could start by projectivizing the dual bundle $\mathbb{P}(E^\vee) \rightarrow X$; on $\mathbb{P}(E^\vee)$ there is the universal quotient bundle $E \rightarrow L$, with kernel $E_{r-1} \subset E$. At the next step, projectivize E_{r-1}^\vee , obtaining $\mathbb{P}(E_{r-1}^\vee) \rightarrow \mathbb{P}(E^\vee)$, with subbundles $E_{r-2} \subset E_{r-1} \subset E$, and so on.

If L_1, \dots, L_r are the line bundles $F_1, F_2/F_1, \dots, F/F_{r-1}$, in either order, dualized or not (we will often use duals, so read these bundles in reverse order), and $x_i = c_1(L_i)$, then $H^*Fl(E)$ is free over H^*X with basis $\{x_1^{i_1} x_2^{i_2} \cdots x_r^{i_r} \mid i_j \leq r-j\}$. This follows by induction from the case of $\mathbb{P}(E) \rightarrow X$. Thus $H^*Fl(E)$ is free over H^*X of rank $r!$, and $H^*X \hookrightarrow H^*Fl(E)$.

This construction gives a filtration of E with line bundle factors, but not yet a splitting. To find a splitting, assume we have a filtration $E_1 \subset E_2 \subset \cdots \subset E_r = E$ of bundles on X . We construct an affine bundle $X^s \rightarrow X$ such that on X^s , we have L_1, \dots, L_r , with $E_i = L_1 \oplus \cdots \oplus L_i$. Indeed, we already have $L_1 = E_1$ on X . To get L_2 , any other line in E_2 will suffice, and similarly, L_i can be taken to be a line in E_i which is not in E_{i-1} . Thus

$$X^s = (\mathbb{P}(E_2) \setminus \mathbb{P}(E_1)) \times_X (\mathbb{P}(E_3) \setminus \mathbb{P}(E_2)) \times_X \cdots \times_X (\mathbb{P}(E) \setminus \mathbb{P}(E_{r-1})),$$

and the universal line bundle on the factor $(\mathbb{P}(E_i) \setminus \mathbb{P}(E_{i-1}))$ serves as the desired L_i .

Since X^s is an affine bundle, we have $H^*X \xrightarrow{\sim} H^*X^s$. Starting with $E \rightarrow X$, then, we can take $X' = Fl(E)^s \rightarrow Fl(E) \rightarrow X$ as our space over which E splits, with $H^*X \hookrightarrow H^*X'$. Moreover, this construction is universal: if $Y \xrightarrow{f} X$ is another space over which E splits, then the universal property of $Fl(E)$ guarantees a factorization of f through $Fl(E)$, and the universal property of $Fl(E)^s$ says that in fact f factors through $Fl(E)^s$.

4. GRASSMANN VARIETIES AND GRASSMANN BUNDLES

Let V be a (complex) vector space of dimension n . Then the *Grassmannian* $Gr(r, V) = \{r\text{-dimensional subspaces of } V\}$ is a manifold of (complex) dimension $r(n-r)$. It has an open covering described as follows. For each basis $\{e_1, \dots, e_n\}$ of V , one has an open set U consisting of subspaces spanned

by the rows of the $r \times (n - r)$ matrix

$$\left(\begin{array}{ccc|cccc} 1 & & & * & * & \cdots & * \\ & 1 & & * & * & & * \\ & & \ddots & * & \vdots & \ddots & \vdots \\ & & & 1 & * & * & \cdots & * \end{array} \right)$$

The free entries of this matrix identify U with an affine space $\mathbb{A}^{r(n-r)}$. Transition functions between two such opens U and U' are given by right-multiplication by the corresponding change-of-basis matrix relating $\{e_1, \dots, e_n\}$ and $\{e'_1, \dots, e'_n\}$. This already shows that $Gr(r, V)$ is an abstract algebraic variety, but in fact it is a projective variety, via the *Plücker embedding*. For more on this, see [Shafarevich, pp. 42-43] or [Fulton 1997, §9.1].

On $Gr(r, V)$, there is the tautological/universal exact sequence

$$(1) \quad 0 \rightarrow S \rightarrow V \rightarrow Q \rightarrow 0,$$

where S has rank r , $V = V_{Gr}$ is the trivial rank- n bundle, and Q has rank $n - r$.

More generally, given any rank- n vector bundle $E \rightarrow X$, there are *Grassmann bundles* $Gr(r, E) \rightarrow X$, for $0 \leq r \leq n$. Locally, these look like $X \times Gr(r, V)$. The Grassmann bundle $Gr(r, E)$ has a universal/tautological sequence

$$(2) \quad 0 \rightarrow S \rightarrow E_{Gr(r, E)} \rightarrow Q \rightarrow 0,$$

and represents the functor $(Y \xrightarrow{f} X) \mapsto \{\text{rank-}r \text{ subbundles of } f^*E\}$. See [EGAI] for details on the general construction.

In fact, there is a natural inclusion $H^*X \hookrightarrow H^*Gr(r, E)$, and projections $Fl(E) \rightarrow Gr(r, E) \rightarrow X$. Note that $Fl(E)$ can be constructed by taking flags in S and Q (the bundles on $Gr(r, E)$), so that $H^*Fl(E)$ has rank $r!(n-r)!$ over $H^*Gr(r, E)$. Since we know $H^*Fl(E)$ has rank $n!$ over H^*X , this suggests

Claim: $H^*(Gr(r, E))$ is free of rank $\binom{n}{r}$.

We will return to this later.

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