EQUIVARIANT INTERSECTION THEORY
§3: G-BUNDLES

WILLIAM FULTON
NOTES BY DAVE ANDERSON

1. Principal G-bundles

Let G be a topological group. In our main examples, G will be a Lie group.

Definition. A *principal G-bundle* is a bundle $E \xrightarrow{\pi} X$, together with a (left-)action of G on E, which is locally trivial in the following sense: there is an open covering $\{U_\alpha\}$ of X, such that

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow[\varphi_\alpha]{\sim} & U_\alpha \times G \\ & \searrow \qquad \swarrow & \\ & X & \end{array}$$

commutes, and the isomorphisms φ_α are compatible with the actions of G. (G acts on the product $U_\alpha \times G$ by left-multiplication.)

We will also assume that the cover $\{U_\alpha\}$ is "numerable," i.e., there is a locally finite partition of unity subordinate to $\{U_\alpha\}$. A principal G-bundle is called numerable if the base has a numerable trivializing open cover. (See [Husemoller] for generalities on fiber bundles.)

Definition. A Hausdorff space X is *paracompact* if every open cover has a locally finite refinement.

In fact, X is paracompact if and only if every open cover has a locally finite partition of unity. Most common topological spaces are paracompact: for example, all manifolds, CW-complexes, and compact spaces are paracompact. (See [Miyazaki].)

The main application of this definition is the following:

Proposition 1.1. *There is a* universal *(numerable, principal) G-bundle*

$$EG \to BG,$$

with the following property: For any (numerable, principal) G-bundle $E \to X$, there is a continuous map $f : X \to BG$ such that $E \to X$ is the pullback

Date: October 8, 2005.

of $EG \to BG$. That is,

$$\begin{array}{ccc} E & \xrightarrow{\sim} & X \times_{BG} EG \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

as G-bundles. In fact, f is unique up to homotopy, so

$$\{G\text{-}bundles\ on\ X\}/_{\cong} = [X, BG].$$

In fact, EG is contractible, and if $E \to B$ is any (numerable, principal) G-bundle with E contractible, then it is universal. For proofs of these facts, see [Husemoller, Ch. 4, Sec. 10].

We will frequently use filtrations by closed subspaces

$$\begin{array}{ccccccc} EG_m & \subset EG_{m+1} \subset & \cdots & \subset & EG \\ \downarrow & \downarrow & & & \downarrow \\ BG_m & \subset BG_{m+1} \subset & \cdots & \subset & BG, \end{array}$$

where each $EG_m \to BG_m$ is a G-bundle, and the spaces EG_m "get more contractible" as $m \to \infty$. That is, $\pi_i(EG_m) = 0$ for $i < k(m)$, and $k(m) \to \infty$ as $m \to \infty$.

It is a fact that if X is a CW-complex of dimension at most k, and $E \to B$ is a G-bundle with $\pi_i(E) = 0$ for all $i \leq k$, then

$$\{G\text{-bundles on } X\}/_{\cong} = [X, B].$$

(See [Husemoller, Ch. 4].) In this sense, the bundles $EG_m \to BG_m$ serve to "approximate" the universal bundle $EG \to BG$.

Exercise 1.2. (1) On $\mathbb{P}^n$, show that $\mathcal{O}(1)$ is generated by $n+1$ sections, but cannot be generated by fewer sections – even continuously.
(2) On $\mathbb{P}^\infty = \bigcup_m \mathbb{P}^m$, show that $\mathcal{O}(1)$ is not generated by a finite number of sections.
(3) Show $\mathbb{P}^\infty$ is paracompact.

Thus one may need to require that the base be compact, and not just paracompact, to ensure a vector bundle is finitely generated.

With few exceptions, we will adhere to the following general convention: Work with *right* principal G-bundles, and *left* G-spaces. This will be important in keeping track of signs in cohomology.

Remark 1.3. A vector bundle $E \to X$ is not a principal $\mathbb{C}^n$-bundle. However, the set of isomorphism classes vector bundles of rank n on X is in bijection with the set of isomorphism classes of principal $GL_n(\mathbb{C})$-bundles on X, which in turn is naturally isomorphic to $[X, BGL_n(\mathbb{C})]$, the set of homotopy classes of maps from X to $BGL_n(\mathbb{C})$.

In fact, both vector bundles and principal GL_n-bundles will be trivialized by the same open coverings $\{U_\alpha\}$, with the "same" transition functions. In general, given a principal right G-bundle $P \to X$, and a left action of G on a space V, there is a corresponding locally trivial bundle with fiber V: Use

$$P \times_G V := P \times V/\sim,$$

where $\sim$ is defined by $(p{\cdot}g, v) \sim (p, g{\cdot}v)$. For example, $(U \times G) \times_G V = U \times V$.

Applying this to the left action of GL_n on $\mathbb{C}^n$, we get the correspondence $\{$principal G-bundles$\} \to \{$vector bundles$\}$.

In subsequent sections, we will frequently use the following basic facts about G-bundles to prove things about equivariant cohomology.

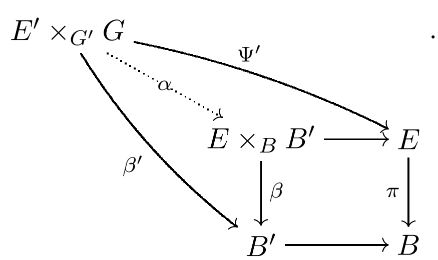
Lemma 1.4. *Let $\pi' : E' \to B'$ be a G'-bundle, and let $\pi : E \to B$ be a G-bundle. Let $\varphi : G' \to G$ be a continuous homomorphism, and let $\Psi : E' \to E$ be equivariant with respect to φ, so there is an induced map $\psi : B' \to B$:*

$$\begin{array}{ccc} E' & \xrightarrow{\Psi} & E \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow{\psi} & B. \end{array}$$

Then the G-bundle induced from $E' \to B'$ and $G' \to G$ (that is, $E' \times_{G'} G \to B'$) is canonically isomorphic to the pullback of $E \to B$ by $\psi : B' \to B$.

In particular, this says that any commutative square of G-bundles is actually a fiber square. (There is a slight clash of notation here: one way of stating the conclusion of the lemma is to write $E' \times_{G'} G = E \times_B B'$, where the LHS is the quotient by the group G', and the RHS is the fiber product over the space B.)

Proof. There are canonical maps $\Psi' : E' \times_{G'} G \to E$ and $\beta' : E' \times_{G'} G \to B'$, induced by the maps $(e', g) \mapsto \Psi(e') \cdot g$ and $(e', g) \mapsto \pi'(e')$, respectively. Therefore there is a diagram



The claim is that α is an isomorphism; we leave this as an exercise. (Both β and β' are G-bundles, so this reduces to showing that a commutative square of G-bundles is a fiber square.) $\square$

Lemma 1.5. *Let $E' \to B'$ and $E \to B$ be (right, principal, numerable) G-bundles, with an equivariant map $E' \to E$, determining a map $f : B' \to B$. Assume that the map $E' \to E$ is a locally trivial fiber bundle, with fiber Y.*

Let $X' \to X$ be an equivariant map of (left) G-spaces, which is also a locally trivial fiber bundle, with fiber Z. Then we have a fiber square

$$\begin{array}{ccc} E' \times X' & \longrightarrow & E \times X \\ \downarrow & & \downarrow \\ E' \times_G X' & \longrightarrow & E \times_G X, \end{array} \tag{1}$$

where the vertical maps are G-bundles, and the horizontal maps are locally trivial fiber bundles, with fiber $Y \times Z$.

In fact, this square comes with projection maps, so there is a diagram

$$\begin{array}{ccccccc} E' & \longleftarrow & E' \times X' & \longrightarrow & E \times X & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ B' & \longleftarrow & E' \times_G X' & \longrightarrow & E \times_G X & \longrightarrow & B. \end{array}$$

Proof. The question is local on B, so we may assume $E \to B$ is trivial, and looks like $B \times G \to B$. Thus $E' \to E$ is the map $(f \times 1) : B' \times G \to B \times G$. Now the diagram (1) looks like

$$\begin{array}{ccc} B' \times G \times X' & \longrightarrow & B \times G \times X \\ {\scriptstyle \pi'}\downarrow & & {\scriptstyle \pi}\downarrow \\ (B' \times G) \times_G X' & \longrightarrow & (B \times G) \times_G X, \end{array}$$

and in fact $(B' \times G) \times_G X' \cong B' \times X'$, and $(B \times G) \times_G X \cong B \times X$. This is not a product diagram – since $\pi(b, g, x) = (b, g \cdot x)$, and similarly for π' – but it is isomorphic to one. Let $\theta : B \times G \times X \to B \times G \times X$ be the homeomorphism given by $(b, g, x) \mapsto (b, g, g^{-1} \cdot x)$, and let $\theta' : B' \times G \times X' \to B' \times G \times X'$ be defined similarly. Then we have

$$\begin{array}{ccccccc} B' \times G \times X' & \xrightarrow[\theta']{\sim} & B' \times G \times X' & \longrightarrow & B \times G \times X & \xleftarrow[\theta]{\sim} & B \times G \times X \\ & {\scriptstyle pr'}\searrow & {\scriptstyle \pi'}\downarrow & & {\scriptstyle \pi}\downarrow & \swarrow{\scriptstyle pr} & \\ & & B' \times X' & \longrightarrow & B \times X, & & \end{array}$$

where pr, pr' are the projections (i.e., $pr(b, g, x) = (b, x)$). Thus our original diagram is isomorphic to

$$\begin{array}{ccc} B' \times G \times X' & \longrightarrow & B \times G \times X \\ {\scriptstyle pr'}\downarrow & & {\scriptstyle pr}\downarrow \\ B' \times X' & \longrightarrow & B \times X. \end{array}$$

Now the statement is clear, given the assumptions on the maps $X' \to X$ and $E' \to E$. □

2. Examples of approximation spaces

We plan to associate to a topological group G a sequence of spaces $EG_m \subset EG$ and $BG_m \subset BG$. Here are some important examples.

Example 2.1. $G = \mathbb{Z}/2\mathbb{Z}$. (An exception, in that most of our groups will be connected.) We take $EG_m = S^m$, with $-1 \in G$ acting by the antipodal map. Thus $BG_m = \mathbb{RP}^m$, and $BG = \mathbb{RP}^\infty$:

$$\begin{array}{ccccccccc} S^m & \subset & S^{m+1} & \subset & \cdots & \subset & S^\infty & = & \bigcup_m S^m \\ \downarrow & & \downarrow & & & & \downarrow & & \\ \mathbb{RP}^m & \subset & \mathbb{RP}^{m+1} & \subset & \cdots & \subset & \mathbb{RP}^\infty & = & \bigcup_m \mathbb{RP}^m. \end{array}$$

In all our examples, the limiting spaces $X_\infty = \bigcup_m X_m$ are given the "induction topology": a set $U \subset X_\infty$ is open if and only if the intersection $U \cap X_m$ is open in X_m for all m.

Example 2.2. $G = S^1$. (Considered as the unit circle in $\mathbb{C}$.) We take $EG_m = S^{2m+1} \subset \mathbb{C}^{m+1}$, with G acting on the right by $(x_0, \ldots, x_m) \cdot \zeta = (\zeta x_0, \ldots, \zeta x_m)$, so $BG_m = \mathbb{CP}^m$:

$$\begin{array}{ccccccc} S^{2m+1} & \subset & \cdots & \subset & S^\infty & = & EG \\ \downarrow & & & & \downarrow & & \\ \mathbb{CP}^m & \subset & \cdots & \subset & \mathbb{CP}^\infty & = & BG. \end{array}$$

Example 2.3. $G = \mathbb{C}^*$. We take $EG_m = \mathbb{C}^{m+1} \smallsetminus \{0\}$, with the same action as in Example 2.2. Thus $BG_m = \mathbb{CP}^m$, as before.

Remark 2.4. Topologists usually just consider the limit space $\mathbb{CP}^\infty$, but as this is not an algebraic variety, algebraic geometers will need the entire sequence. One can speak of a "compatible sequence" of cohomology classes to arrive at cohomology of BG.

Example 2.5. $G = (\mathbb{C}^*)^n$. Here there are two useful possibilities:

(1) $EG_m = (\mathbb{C}^{m+1} \smallsetminus \{0\}) \times \cdots \times (\mathbb{C}^{m+1} \smallsetminus \{0\})$ (n copies), with the action $(v_1, \ldots, v_n) \cdot (\zeta_1, \ldots, \zeta_n) = (\zeta_1 v_1, \ldots, \zeta_n v_n)$. Thus $BG_m = (\mathbb{CP}^m)^n$, and we have

$$\begin{aligned} EG &= \bigcup_m (\mathbb{C}^m \smallsetminus \{0\})^n \\ BG &= (\mathbb{CP}^\infty)^n. \end{aligned}$$

Note that in these direct limits (i.e., unions) of spaces, the indexing is not very important; in particular, the values of m may start at any conveniently large number.

(2) We can also take

$$\begin{aligned} EG_m &= \{(v_1, \ldots, v_n) \mid \{v_i\} \text{ are linearly independent vectors in } \mathbb{C}^m\} \\ &= \{m \times n \text{ matrices of full rank } n\}. \end{aligned}$$

Here we assume $m \geq n$. With this choice of EG_m, the base is

$$\begin{aligned} BG_m &= \{(L_1, \ldots, L_n) \,|\, L_i \subset \mathbb{C}^m \text{ is a line, and } L_1 + \cdots + L_n \text{ is an } n\text{-plane}\} \\ &= \text{the "split Grassmannian" of } n\text{-planes in } \mathbb{C}^m. \end{aligned}$$

Example 2.6. $G = GL_n(\mathbb{C})$. Let

$$EG_m = M^o_{m,n} := \{m \times n \text{ matrices of full rank } n\};$$

here we assume $m \geq n$. Then G acts on the right by matrix multiplication, and we have $BG_m = M^o_{m,n}/G = Gr(n, \mathbb{C}^m)$, via the correspondence assigning to each matrix in $M^o_{m,n}$ the vector space spanned by its columns. Taking unions, $EG = M^o_{\infty,n}$, the space of full rank matrices with arbitrarily many rows (and only finitely many nonzero entries), and $BG = Gr(n, \mathbb{C}^\infty)$.

Example 2.7. We can use the same spaces EG_m as above for any subgroup $G \subset GL_n(\mathbb{C})$. For example, if $G = (\mathbb{C}^*)^n = T \subset GL_n(\mathbb{C})$, then we have exactly the situation of Example 2.5 (2).

When G is the Borel subgroup of upper-triangular matrices in GL_n, we get $BG_m = M^o_{m,n}/G = Fl(1, \ldots, n; \mathbb{C}^m)$, the partial flag variety of chains $(V_1 \subset \cdots V_n \subset \mathbb{C}^m)$, with $\dim V_i = i$. (The map sends an $m \times n$ matrix to the flag $V_\bullet$, where V_i is the span of the first i columns.)

Exercise 2.8. When G is the group of lower-triangular matrices, $BG_m = Fl(1, \ldots, n; \mathbb{C}^m)$, via the map sending a matrix to the flag where V_i is the span of the last i columns.

Example 2.9. For $G = U(n) \subset GL_n(\mathbb{C})$ (the unitary group), we can take $EG_m = \{\text{orthonormal column vectors } v_1, \ldots, v_n \in \mathbb{C}^m\}$, using the standard Hermitian metric on $\mathbb{C}^m$. Then $BG_m = EG_m/G = Gr(n, \mathbb{C}^m)$.

Example 2.10. Let $G = SL_n(\mathbb{C}) \subset GL_n(\mathbb{C})$, and take $EG_m = M^o_{m,n}$ as before. Then in $BG_m = EG_m/G$, two matrices $(v_1 \ v_2 \ \cdots \ v_n)$ and $(v'_1 \ v'_2 \ \cdots \ v'_n)$ are identified exactly when $v_1 \wedge v_2 \wedge \cdots \wedge v_n = v'_1 \wedge v'_2 \wedge \cdots \wedge v'_n$. From this description, one sees that BG_m is the complement of the zero section in the line bundle $\bigwedge^n S \to Gr(n, \mathbb{C}^m)$, where S is the tautological subbundle: $BG_m = \bigwedge^n S \smallsetminus 0$.

Of central importance in this theory is the ring $\Lambda = \Lambda_G = H^*(BG)$. Indeed, this is the ring of all "characteristic classes" for G-bundles – i.e., ways to assign to G-bundle $P \to X$ a cohomology class $c \in H^*X$, functorially in X. (If $f : X' \to X$, then the class $c' \in H^*X'$ assigned to $f^*(P \to X)$ should be f^*c.) For example, when $G = GL_n(\mathbb{C})$, $H^*BG = \mathbb{Z}[c_1, \ldots, c_n]$, so every characteristic class is a polynomial in Chern classes. (Cf. [Milnor-Stasheff, §14].) Let us see what this ring is in some of the above examples.

- $G = C^*$ (Example 2.3). Here $BG = \mathbb{P}^\infty$, so $H^*BG = \mathbb{Z}[x]$, where x is a variable in H^2BG. In fact, x restricts to $c_1(\mathcal{O}(1))$ in each $H^2(\mathbb{P}^m)$.

- $G = (\mathbb{C}^*)^n$ (Example 2.5). Here $BG = \mathbb{P}^\infty \times \cdots \times \mathbb{P}^\infty$, so $H^*BG = \mathbb{Z}[x_1, \ldots, x_n]$, a polynomial ring in variables x_i of degree 2. Under the first description, $BG_m = \mathbb{P}^m \times \cdots \times \mathbb{P}^m$, and x_i restricts to $c_1(\mathcal{O}(1)_i)$, where $\mathcal{O}(1)_i$ is the universal (quotient) bundle on the ith factor.

Exercise 2.11. Work this out using the second choice of approximations EG_m (Example 2.5 (2)).

- $G = GL_n(\mathbb{C})$ (Example 2.6). Now $H^*BG = H^*Gr(n, \mathbb{C}^\infty) = \mathbb{Z}[c_1, \ldots, c_n]$, a polynomial ring with $\deg c_i = 2i$. In fact, c_i restricts to $c_i(S^\vee) = (-1)^i c_i(S)$ in $H^*Gr(n, \mathbb{C}^m)$.
- $G = \{\text{upper-triangular matrices}\} \subset GL_n$ (Example 2.7). Here $BG = Fl(1, \ldots, n; \mathbb{C}^\infty)$, and $H^*BG = \mathbb{Z}[x_1, \ldots, x_n]$, a polynomial ring with generators of degree 2. On each $BG_m = Fl(1, \ldots, n; \mathbb{C}^m)$, there is the tautological flag bundle $F_1 \subset F_2 \subset \cdots \subset F_n \subset \mathbb{C}^m_{BG_m}$, and x_i restricts to $-c_1(F_i/F_{i-1}) = c_1((F_i/F_{i-1})^\vee)$ in H^*BG_m.
- $G = U(n) \subset GL_n(\mathbb{C})$ (Example 2.9). Since BG_m is the same as $(BGL_n(\mathbb{C}))_m$, we see that $H^*BU(n) = H^*BGL_n(\mathbb{C})$.
- $G = SL_n(\mathbb{C})$ (Example 2.10). Here $H^*BG = \mathbb{Z}[c_2, \ldots, c_n]$, a polynomial ring with $\deg c_i = 2i$. In fact, this is naturally identified with $H^*BGL_n(\mathbb{C})/(c_1)$.

3. Cohomology of flag bundles

Logically, one could simply define equivariant cohomology in terms of these sequences of spaces, and ignore the topological considerations requiring EG to be contractible. It is worth making a few remarks about the relationship, however; we will say why these spaces EG_m are "good approximations" to EG. In particular, for most of our examples, we need the following fact:

Claim: The space $M^o_{m,n} \subset M_{m,n}$ is $2(m-n)$-connected, i.e., $\pi_i(M^o_{m,n}) = 0$ for $i \leq 2(m-n)$.

Note that $M_{m,n} = \mathbb{A}^{mn}$, and $M^o_{m,n} = \mathbb{A}^{mn} \smallsetminus Z$, where $Z = Z_{n-1} = \{A \in M_{m,n} \mid \mathrm{rk}(A) \leq n-1\}$. We will show:

Proposition 3.1. *The algebraic set Z is irreducible of dimension $m-n+1$. More generally, $Z_r \subset M_{m,n}$ is irreducible of codimension $(n-r)(m-r)$.*

The claim then follows from the following general fact:

Proposition 3.2. *If $Z \subset \mathbb{C}^N = \mathbb{A}^N$ is a closed algebraic set of codimension d, then $\pi_i(\mathbb{C}^N \smallsetminus Z) = 0$ for $0 < i \leq 2d-2$. Moreover, this bound is sharp: for every such Z, $\pi_{2d-1}(\mathbb{C}^N \smallsetminus Z) \neq 0$.*

[Reference??] We will return to the proof of this later, and discuss Proposition 3.1 now.

Proof. (of Proposition 3.1.) Let V and W be vector spaces of dimensions m and n, respectively. Let $M = \mathrm{Hom}(V, W) \cong M_{m,n}$. On M, there is the tautological map of trivial vector bundles $\varphi_M : V_M \to W_M$. Now $Z_r = \{\varphi \in M \mid \bigwedge^{r+1} \varphi = 0\} \subset M$, and we have

$$Z_0 \subset Z_1 \subset \cdots \subset Z_{\min\{m,n\}} = M.$$

We want to show that Z_r is irreducible of codimension $(n-r)(m-r)$, and $Z_r = \overline{Z_r \smallsetminus Z_{r-1}}$.

Giving a map $V \to W$ is equivalent to specifying a $n-r$-dimensional subspace $K \subset V$ (the kernel), together with an inclusion $V/K \hookrightarrow W$. This is just the usual factorization of a map into a surjection followed by an injection:

$$\begin{array}{ccccc} K & \hookrightarrow & V & \twoheadrightarrow & V/K \\ & \searrow & \downarrow & \swarrow & \\ & & W. & & \end{array}$$

This suggests we look at $Gr = Gr(n-r, V)$, on which we have the universal sequence

$$0 \to S \to V_{Gr} \to Q \to 0.$$

Set $Y = M \times Gr$. On Y, we have

$$\begin{array}{ccc} S_Y & \hookrightarrow & V_Y \\ & \searrow & \downarrow \\ & & W_Y; \end{array}$$

let $\tilde{Z}_r \subset Y$ be the locus of zeroes of the vector bundle map $S_Y \to W_Y$ – that is, $\tilde{Z}_r = \mathcal{Z}(S_Y \to W_Y) = \{(A, K) \mid K \subset \ker(A)\}$. This is locally defined by $(n-r)m$ equations. (Locally, all $K \in Gr(n-r, V)$ can be represented by a $n \times (n-r)$ matrix M_K; the defining equations are the requirement that the $m \times (n-r)$ matrix $A \cdot M_K$ vanish.)

The following two exercises complete the proof.

Exercise 3.3. Show that $\tilde{Z}_r \subset Y$ is a submanifold of codimension $(n-r)m$. (That is, it is smooth of expected dimension.) [Consider the projection map $p_2 : \tilde{Z}_r \to Gr$.]

Exercise 3.4. We have

$$\begin{array}{ccc} \tilde{Z}_r & \hookrightarrow & M \times Gr \\ \downarrow & & \downarrow p_1 \\ Z_r & \hookrightarrow & M. \end{array}$$

Show that $\tilde{Z}_r$ projects surjectively onto Z_r, and that $(p_1)^{-1}(Z_r \smallsetminus Z_{r-1}) \to Z_r \smallsetminus Z_{r-1}$ is an isomorphism.

Thus we conclude that Z_r is irreducible, of the same dimension as $\tilde{Z}_r$, so $\operatorname{codim}(Z_r, M) = \operatorname{codim}(\tilde{Z}_r, Y) - \dim Gr = (n-r)(m-r)$. □

Note that this gives us a canonical resolution of singularities for Z_r, as well as a way of finding the cohomology class of Z_r: we have $[\tilde{Z}_r] \mapsto [Z_r]$.

Evidently, we will need to understand the ring $H^*Gr(n, \mathbb{C}^m)$. We have

Proposition 3.5. *The cohomology ring of the Grassmannian is given by*

$$H^*Gr(n, \mathbb{C}^m) = \mathbb{Z}[x_1, \dots, x_n, y_1, \dots, y_{m-n}]/I,$$

where $\deg x_i = 2i$, $\deg y_j = 2j$, *and* I *has the* m *generators* $\sum_{i+j=k}(-1)^i x_i y_j$, *for* $1 \le k \le m$.

In fact, the isomorphism is given by $x_i \mapsto c_i(S^\vee)$, $y_j \mapsto c_j(Q)$, *where* S *and* Q *are the universal sub- and quotient bundle on* $Gr(n, \mathbb{C}^m)$, *respectively.*

Another presentation for this ring may be given in terms of x*'s only:*

$$H^*Gr(n, \mathbb{C}^m) = \mathbb{Z}[x_1, \dots, x_n]/J,$$

again via $x_i \mapsto c_i(S^\vee)$, *where* J *has* n *generators given as follows. Define elements* $y_1, y_2, \dots$ *formally by*

$$1 + y_1 T + y_2 T^2 + \cdots = (1 - x_1 T + x_2 T - \cdots + (-1)^n x_n T^n)^{-1},$$

i.e., $y_1 = x_1$, $y_2 = x_1^2 - x_2$, *etc. Then* $J = (y_{n-m+1}, \dots, y_m)$.

The first presentation says that the relations forced on Chern classes by the Whitney sum formula are the only relations in $H^*Gr(n, \mathbb{C}^m)$, while the second expresses the fact that $c_j(Q) = 0$ for $j > n - m = \operatorname{rk}(Q)$.

We now compute the cohomology of flag bundles and Grassmann bundles, generalizing the results for flag varieties and Grassmannians.

Proposition 3.6. *Let* $E \to X$ *be a vector bundle of rank* n, *and let* $Fl(E) \to X$ *be the corresponding flag bundle, with tautological flag of subbundles*

$$0 = F_0 \subset F_1 \subset \cdots \subset F_n = E_{Fl}.$$

Then

$$H^*Fl(E) = H^*X[x_1, \dots, x_n]/(e_k(x) - c_k(E^\vee) \mid 1 \le k \le n),$$

with basis $\{x_1^{i_1} \cdots x_n^{i_n} \mid i_j \le n - j\}$. *Here* $e_k(x)$ *is the ith elementary symmetric function in* $x_1, \dots, x_n$, *and the identification is given by* $x_i \mapsto c_1((F_i/F_{i-1})^\vee)$.

Proof. First, consider $\mathbb{P}(E) \to X$, where E is a vector bundle of rank n. Then

$$H^*\mathbb{P}(E) = H^*X[x]/(x^n + c_1(E)x^{n-1} + \cdots + c_n(E)),$$

with basis $\{1, x, \dots, x^{n-1}\}$, via a map $x \mapsto c_1(\mathcal{O}(1)) = \zeta$. Indeed, we know there is a well-defined map from the RHS to the LHS, by our construction of Chern classes; on the other hand, the Leray-Hirsch theorem implies that $\{1, \zeta, \dots, \zeta^{n-1}\}$ forms a basis for $H^*\mathbb{P}(E)$ over H^*X. The construction of $Fl(E)$ as a sequence of projective bundles then shows that the classes $x_1^{i_1}\cdots x_n^{i_n}$ form a basis for $H^*Fl(E)$ over H^*X.

By the Whitney sum formula, there is a map from the RHS to the LHS, since $c(E)=\prod c(F_i/F_{i-1})=\prod(1-x_i)$. To prove the proposition, it suffices to show that the classes $x_1^{i_1}\cdots x_n^{i_n}$ span the RHS.

Exercise 3.7. Complete this last step.

□

Proposition 3.8. *Let $E\to X$ be a rank-n vector bundle on X. On $Gr(r,E)$, there is the tautological sequence*

$$0\to S\to E_{Gr}\to Q\to 0,$$

where S has rank r, and Q has rank $n-r$. Then

$$\begin{aligned} H^*Gr(r,E) &= H^*X[x_1,\dots,x_r,y_1,\dots,y_{n-r}]/I \\ &= H^*X[x_1,\dots,x_r]/J, \end{aligned}$$

via $x_i\mapsto c_i(S^\vee)=(-1)^ic_i(S)$, $y_j\mapsto c_j(Q)$. The ideal I has n generators, coming from the relation $c(E)=c(S)c(Q)$:

$$I=(\sum_{i+j=k}(-1)^ix_iy_j\,|\,1\le k\le n).$$

The ideal J has r generators, coming from the fact that $\operatorname{rk}E/S=n-r$: define elements $z_1,z_2,\dots$ by

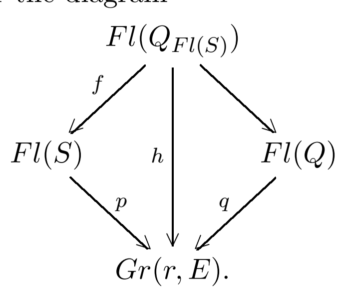
$$1+z_1T+z_2T^2+\cdots=\frac{1+c_1(E)T+\cdots+c_n(E)T^n}{1-x_1T+x_2T^2-\cdots+(-1)^rx_rT^r}.$$

Then

$$J=(z_{n-r+1},\dots,z_n).$$

The z_j's defined in the proposition function as the y_j's – that is, they map to $c_j(Q)$. In fact, the relations imply that $z_j\in J$ for $j>n$, as well. It follows from this second description that $H^*Gr(r,E)$ is free of rank $\binom{n}{r}$ over H^*X.

Proof. (Sketch.) Consider the diagram



We already understand the cohomology of the flag bundles with maps p, q, and f, and therefore also the bundle $h=p\circ f$. This "reduces" the problem to algebra. We will return to fill in the details later, after we have discussed Schubert classes. (In the meantime, note that we do have a map from the RHS to the LHS, as in the flag bundle case.) □

To compute $H^*BSL_n(\mathbb{C})$, we need a topological lemma.

Lemma 3.9. *Let $E \to X$ be a complex vector bundle of rank r (or more generally, an oriented real vector bundle of rank $2r$), and let $\mu \in H^{2r}(E, E \smallsetminus \{0\})$ be the Thom class. Its image in $H^{2r}E$ is the pullback of $c_r(E)$. Then there is a long exact sequence, called the* Gysin sequence *of the vector bundle:*

$$\cdots \to H^iX \xrightarrow{\cdot c_r(E)} H^{i+2r}X \to H^{i+2r}(E \smallsetminus \{0\}) \xrightarrow{\psi} H^{i+1}X \to \cdots .$$

Proof. This is obtained from the long exact sequence of the pair $(E, E \smallsetminus \{0\})$ by using the Thom isomorphism $H^iE \cong H^{i+2r}(E, E \smallsetminus \{0\})$ and the homotopy isomorphism $H^iX \cong H^iE$. In fact, ψ is the composition $H^{i+2r}(E \smallsetminus \{0\}) \to H^{i+2r+1}(E, E \smallsetminus \{0\}) \cong H^{i+1}X$. □

An immediate consequence is the following:

Corollary 3.10. *If multiplication by $c_r(E)$ is injective on H^*X, there are short exact sequences*

$$0 \to H^{i-2r}X \xrightarrow{\cdot c_r(E)} H^iX \to H^i(E \smallsetminus \{0\}) \to 0,$$

so $H^*(E \smallsetminus \{0\}) \cong H^*X/(c_r(E))$.

Note, however, that this never happens when X is finite-dimensional.

Now recall that $(BSL_n(\mathbb{C}))_m = L \smallsetminus \{0\} \to Gr(n, \mathbb{C}^m)$, where L is the line bundle $\bigwedge^n S$. Note that $c_1(L) = c_1(S) = c_1$ (by an exercise from §2), and that c_1 is a non-zerodivisor in the polynomial ring $H^*Gr(n, \mathbb{C}^\infty)$. Applying the corollary to $Gr(n, \mathbb{C}^\infty)$, then, we obtain

Proposition 3.11. *$H^*BSL_n(\mathbb{C}) = \mathbb{Z}[c_1, \ldots, c_n]/(c_1) \cong \mathbb{Z}[c_2, \ldots, c_n]$, where $\deg c_i = 2i$.*

Remark 3.12. Similarly, we have an exact sequence of groups

$$0 \to \mathbb{C}^* \to GL_n(\mathbb{C}) \to PGL_n(\mathbb{C}) \to 0.$$

One might try to compute H^*BG, where $G = PGL_n(\mathbb{C})$, using similar techniques – but in fact, this is a complicated open problem. This BG is known to have odd-dimensional homology, and nonzero torsion in cohomology. At issue is the problem of finding good approximation spaces EG_m for this group. The state of the art seems to be that a presentation for H^*BG is known when n is prime.

We now give a proof of the key fact showing that our approximation spaces do indeed become more contractible.

Proposition 3.13. *Let $Z \subset \mathbb{C}^n$ be a closed algebraic set of codimension d. Then $\pi_i(\mathbb{C}^n \smallsetminus Z) = 0$ for $0 < i \leq 2d - 2$, and $\pi_{2d-1}(\mathbb{C}^n \smallsetminus Z) \neq 0$.*

Proof. (D. Speyer.) Identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. For a smooth (C^∞) map $f : S^i \to \mathbb{C}^n \smallsetminus Z$, let $\mathscr{S} = \{p \in \text{(real) line between } Z \text{ and } f(S^i)\}$. (This is analogous to a secant variety in algebraic geometry.) Consider the number

$$\begin{aligned}\dim_{\mathbb{R}} \mathscr{S} &= \dim_{\mathbb{R}} Z + \dim_{\mathbb{R}} f(S^i) + \dim_{\mathbb{R}} \mathbb{R} \\ &\leq 2n - 2d + i + 1,\end{aligned}$$

since smoothness of f implies $\dim f(S^i) \leq i$. The condition that this number be less than $2n$ is exactly that $i \leq 2d-2$. For such i, then, $\mathscr{S} \subsetneq \mathbb{C}^n$; therefore we can find a point $p \notin \mathscr{S}$. Since p does not lie on any line joining Z and $f(S^i)$, the set of line segments between p and $f(S^i)$ lies in $\mathbb{C}^n \smallsetminus Z$. Use this to extend f to a map of the ball $\tilde{f} : D^{i+1} \to \mathbb{C}^n \smallsetminus Z$, thus showing that f is null-homotopic.

Since every continous map between smooth manifolds is homotopic to a continuous map (see [Bott-Tu, pp. 213-214]), the homotopy groups can be computed using only smooth maps. Thus $\pi_i(\mathbb{C}^n \smallsetminus Z) = 0$ for $i \leq 2d-2$.

On the other hand, it follows from this and the Hurewicz isomorphism theorem that $\pi_{2d-1}(\mathbb{C}^n \smallsetminus Z) = H_{2d-1}(\mathbb{C}^n \smallsetminus Z)$, and $H_{2d-2}(\mathbb{C}^n \smallsetminus Z) = 0$. Now by the universal coefficient theorem and the long exact sequence for the pair $(\mathbb{C}^n, \mathbb{C}^n \smallsetminus Z)$, we have

$$\begin{aligned}H_{2d-1}(\mathbb{C}^n \smallsetminus Z)^\vee &\cong H^{2d-1}(\mathbb{C}^n \smallsetminus Z) \\ &\cong H^{2d}(\mathbb{C}^n, \mathbb{C}^n \smallsetminus Z) \\ &= \overline{H}_{2n-2d} Z,\end{aligned}$$

and we know this top Borel-Moore homology group is nonzero. □

Returning to the presentation of $H^*Gr(r,E)$ over H^*E, consider the following setup. Let $E \to X$ be a rank-n vector bundle, let $s = n - r$, and let $z_i = -c_1(L_i)$ in $H^*Fl(E)$ (where $L_i = F_i/F_{i-1}$).

Lemma 3.14. *$H^*Fl(E)$ has a basis (over H^*X) of classes*

$$e_\lambda \cdot z_1^{i_1} \cdots z_r^{i_r} z_{r+1}^{j_1} \cdots z_n^{j_s},$$

for $i_p \leq r - p$ and $j_q \leq s - q$, and $\lambda = (s \geq \lambda_1 \geq \cdots \geq \lambda_r \geq 0)$ is a partition with at most r parts, and with largest part at most s. Here $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_r}$, and $e_k = e_k(z_1, \ldots, z_r)$ is the kth elementary symmetric polynomial.

Granting this lemma for now, we have

$$\begin{array}{ccc} R[z_1, \ldots, z_n]/K & \xrightarrow{\sim} & H^*Fl(E) \\ \uparrow & & \uparrow{\scriptstyle p^*} \\ R[x_1, \ldots, x_r, y_1, \ldots, y_s]/I & \xrightarrow{\varphi} & H^*Gr(r,E) \\ \uparrow & & \uparrow \\ R & = & H^*X. \end{array}$$

Note that $\varphi(x_k) = c_k(S^\vee)$, and $p^*(c_k(S^\vee)) = e_k(c_1(L_1), \ldots, c_1(L_r))$. Thus the diagram commutes, and we conclude the following:

Corollary 3.15. *The middle row in the above diagram is an isomorphism, and the classes* $\{c_{\lambda_1}(S^\vee) \cdot c_{\lambda_2}(S^\vee) \cdots c_{\lambda_r}(S^\vee) \mid s \geq \lambda_1 \geq \cdots \geq \lambda_r \geq 0\}$ *form a basis for* $H^*Gr(r, E)$ *over* H^*X*.*

For example, with $r = 2$, $n = 4$, a basis is given by $\{1, c_1, c_1^2, c_2, c_1c_2, c_2^2\}$.

Remark 3.16. We have "canonical" maps

$$\begin{array}{ccc} M^o_{m,n} & \hookrightarrow & M^o_{m+1,n} \\ \downarrow & & \downarrow \\ Gr(n, \mathbb{C}^m) & \hookrightarrow & Gr(n, \mathbb{C}^{m+1}), \end{array}$$

where the upper inclusion is given by adding zeroes in the bottom row, and the lower inclusion is induced by the standard inclusion $\mathbb{C}^m \hookrightarrow \mathbb{C}^{m+1}$.

Also, $Gr(n, \mathbb{C}^m) \cong Gr(m - n, \mathbb{C}^m)$, by identifying $(\mathbb{C}^m)^\vee = \mathbb{C}^m$. In fact, $Gr(n, E) \cong Gr(m - n, E^\vee)$, where E is a vector bundle of rank m, by an isomorphism taking the universal sequence

$$0 \to S \to E \to Q \to 0$$

to

$$0 \to Q^\vee \to E^\vee \to S^\vee \to 0.$$

In addition, $Gr(n, E) \cong Gr(E, m - n)$, where the latter is the Grassmannian of rank-$(m - n)$ quotients of E. (This notation is not standard, but suggests the appearance of the tautological bundle to the right of E in the universal sequence.)

References

[Bott-Tu] Bott, R. and L. Tu. *Differential Forms in Algebraic Topology*, Springer, 1995.

[Husemoller] D. Husemoller, *Fibre Bundles*, Springer-Verlag, 1975.

[Milnor-Stasheff] J. W. Milnor and J. D. Stasheff, *Characteristic Classes*, Princeton Univ. Press, 1974.

[Miyazaki] H. Miyazaki, "A note on paracompact spaces," *Tohoku Math. J. (2)*, **4** (1952), 88–92.

[Spanier] E. Spanier, *Algebraic Topology*, McGraw-Hill, 1966.