EQUIVARIANT INTERSECTION THEORY
§4: EQUIVARIANT COHOMOLOGY

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1. Foundations

Definition. Let X be a left G-space. We define the *equivariant cohomology ring* of X by

$$H_G^*X = H^*(EG \times_G X),$$

where for any right G-space E, the space $E \times_G X$ is defined to be the quotient space $E \times X/(e \cdot g, x) \sim (e, g \cdot x)$. This is a graded ring: $H_G^i(X) = H^i(EG \times_G X)$.

Example 1.1. Let $X = \{p\}$ be a point. Then $H_G^*(\{p\}) = H^*(EG/G) = H^*BG = \Lambda_G$, the ring of characteristic classes for G. Thus the equivariant cohomology of a point is far from trivial.

Claim: $H_G^i X = H^i(EG_m \times_G X)$ for $i \leq k(m)$, with $k(m) \to \infty$ as $m \to \infty$, and this is compatible with products in this range. Also, this is independent of the choice of the spaces EG and EG_m.

Caution: H_G^*X is certainly *not* equal to $\varprojlim H^*(EG_m \times_G X)$ as a ring. For example, if $G = \mathbb{C}^*$ and $X = \{p\}$ is a point, then

$$H^*(EG_m \times_G \{p\}) = H^*(\mathbb{P}^m) = \mathbb{Z}[x]/(x^{m+1}),$$

but $H_G^*(\{p\}) = \mathbb{Z}[x]$ is not the same as $\varprojlim \mathbb{Z}[x]/(x^{m+1}) = \mathbb{Z}[[x]]$.

The key to proving basic facts about equivariant cohomology is the following setup. Let $E \to B$ be a (right) G-bundle, and let X be a (left) G-space. There is a commuting diagram

$$\begin{tikzcd}
E \times X \arrow[r] \arrow[d] & E \arrow[d] \\
E \times_G X \arrow[r] & B.
\end{tikzcd}$$

Lemma 1.2. *This is a fiber square. The vertical maps are locally trivial bundles with fiber G, and the horizontal maps are locally trivial bundles with fiber X.*

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In fact, this is a special case of a lemma from §3, which we recall below as Lemma 1.6.

Some properties follow formally from the definition. If $\varphi : G \to G'$ is a continuous homomorphism of groups, and $f : X \to X'$ is equivariant[*] with respect to φ, we get

$$\begin{tikzcd} EG \arrow[r] \arrow[d] & EG' \arrow[d] \\ BG \arrow[r] & BG', \end{tikzcd}$$

and a map $EG \times X \to EG' \times X'$, which passes to the quotient: $EG \times_G X \to EG' \times_{G'} X'$. Thus we have a ring homomorphism $H^*_{G'}X' \to H^*_G X$. In fact, this is functorial (it respects compositions), so $H^*_{(\cdot)}(\cdot)$ is a contravariant functor of both G and X.

Example 1.3. For $G = G'$, a G-equivariant map $f : X \to X'$ gives rise to a map $H^*_G X' \to H^*_G X$, so H^*_G is contravariant for equivariant maps of G-spaces.

Example 1.4. From the unique map $X \to pt$, we get $\Lambda_G = H^*_G(pt) \to H^*_G X$. When Λ_G is a commutative ring (as it will be in all our examples), and $X \to X'$ is an equivariant map of G-spaces, $H^*_G X' \to H^*_G X$ is a homomorphism of Λ_G-algebras. For $X \to X'$ equivariant with respect to a homomorphism $G \to G'$, we have

$$\begin{tikzcd} \Lambda_{G'} \arrow[r] \arrow[d] & \Lambda_G \arrow[d] \\ H^*_{G'}X' \arrow[r] & H^*_G X. \end{tikzcd}$$

Given a continuous homomorphism $\varphi : G \to G'$, we get a map $B\varphi : BG \to BG'$, which is unique up to homotopy. Indeed, any G-bundle $E \to B$ gives rise to a G' bundle $E \times_G G' \to B$; in particular, there is a G'-bundle $EG \times_G G' \to BG$, and this produces the map $BG \to BG'$. There is also a commuting diagram

$$\begin{tikzcd} EG \arrow[r] \arrow[d] & EG' \arrow[d] \\ BG \arrow[r, "B\varphi"] & BG'. \end{tikzcd}$$

Thus there is an induced map $H^*_{G'}(pt) \to H^*_G(pt)$.

Exercise 1.5. Let $G = G' = \mathbb{C}^*$. Every algebraic homomorphism $\varphi : \mathbb{C}^* \to \mathbb{C}^*$ has the form $\varphi(\zeta) = \zeta^a$ for some $a \in \mathbb{Z}$. This induces

$$\mathbb{Z}[t] = H^* B\mathbb{C}^* \to H^* B\mathbb{C}^* = \mathbb{Z}[t],$$

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[*] That is, G acts on X, G' acts on X', and $f(g \cdot x) = \varphi(g) \cdot f(x)$.

which must be given by $t \mapsto d \cdot t$ for some $d \in \mathbb{Z}$. Show that $d = a$, and find the corresponding maps of approximation spaces $\mathbb{P}^{m-1} \to \mathbb{P}^{m-1}$.

[Hint/Solution: We want to find the maps

$$\begin{tikzcd}\mathbb{C}^m \smallsetminus \{0\} \arrow[r] \arrow[d] & \mathbb{C}^m \smallsetminus \{0\} \arrow[d] \\ \mathbb{P}^{m-1} & \mathbb{P}^{m-1}.\end{tikzcd}$$

First suppose $a \geq 0$. Then the map $\mathbb{C}^m \smallsetminus \{0\} \to \mathbb{C}^m \smallsetminus \{0\}$ defined by $z = (z_1, \dots, z_m) \mapsto (z_1^a, \dots, z_m^a)$ is equivariant with respect to the homomorphism $\mathbb{C}^* \to \mathbb{C}^*$, and induces an algebraic morphism $f : \mathbb{P}^{m-1} \to \mathbb{P}^{m-1}$. This map f has degree a, so $f^*\mathcal{O}(1) = \mathcal{O}(a)$, and therefore $f^*(t) = f^*c_1(\mathcal{O}(-1)) = c_1(\mathcal{O}(-a)) = at$.

Now consider $\zeta \mapsto \zeta^{-a}$, for $a > 0$. We have a map $\mathbb{C}^m \smallsetminus \{0\} \to \mathbb{C}^m \smallsetminus \{0\}$ given by $z = (z_1, \dots, z_m) \mapsto |z|^{-2a}(\overline{z}_1^a, \dots, \overline{z}_m^a)$, which is equivariant, so it induces $f : \mathbb{P}^{m-1} \to \mathbb{P}^{m-1}$. Check that $f^*(\mathcal{O}(1)) = \mathcal{O}(-a)$.]

We will make frequent use of the following general fact, of which Lemma 1.2 is a special case:

Lemma 1.6. *Let $E' \to B'$ and $E \to B$ be (right, principal, numerable) G-bundles, with an equivariant map $E' \to E$, determining a map $f : B' \to B$. Assume that the map $E' \to E$ is a locally trivial fiber bundle, with fiber Y. Let $X' \to X$ be an equivariant map of (left) G-spaces, which is also a locally trivial fiber bundle, with fiber Z. Then we have a fiber square*

$$\begin{tikzcd}E' \times X' \arrow[r] \arrow[d] & E \times X \arrow[d] \\ E' \times_G X' \arrow[r] & E \times_G X,\end{tikzcd} \tag{1}$$

where the vertical maps are G-bundles, and the horizontal maps are locally trivial fiber bundles, with fiber $Y \times Z$.

We now prove independence of choices in our definition of equivariant cohomology.

Proposition 1.7. *H_G^*X is independent of the choice of EG, and $H_G^iX = H^i(EG_m \times_G X)$ for $i \leq k(m)$, if $\pi_i(EG_m) = 0$ for $i \leq k(m)$.*

In fact, this is a special case of a more general fact:

Proposition 1.8. *For any G-bundle $E \to B$, if $\pi_i(E) = 0$ for $0 < i \leq N$ (or, equivalently by the Hurewicz isomorphism theorem, $H^iE = 0$ for $0 < i \leq N$), then there are canonical isomorphisms $H^i(EG \times_G X) \cong H^i(E \times_G X)$ for $i \leq N$.*

Proof. For a locally trivial fiber bundle $E \to B$, with fiber F, there is a natural long exact homotopy sequence (with base points)

$$\to \pi_i(F) \to \pi_i(E) \to \pi_i(B) \to \pi_{i-1}(F) \to \cdots \to \pi_0(B) \to 0.$$

(See [Spanier, p. 377].)

We can apply this to the square of G-bundles

$$\begin{tikzcd}
E \times X \arrow[r] \arrow[d] & EG \times X \arrow[d] \\
E \times_G X \arrow[r] & EG \times_G X.
\end{tikzcd}$$

Note that $\pi_i(E \times X) = \pi_i(EG \times X)$ for $i \leq N$ (they are equal to $\pi_i(X)$), and the bundles have the same fibers (namely, G), so naturality of the exact homotopy sequence gives isomorphisms $\pi_i(E \times_G X) \to \pi_i(EG \times_G X)$ for $i \leq N$.

This implies that the same is true for the homology groups, by naturality of the Hurewicz homomorphism $\pi_i(E \times_G X) \to H_i(E \times_G X)$: and the universal coefficient theorem then gives an isomorphism of cohomology groups for $i \leq N$. □

Now suppose $E \to E/G = B$ and $E' \to E'/G = B'$ are two G-bundles. Then G acts diagonally on $E \times E'$, and we have a G-bundle $E \times E' \to (E \times E')/G$. Indeed, to see this, apply Lemma 1.6 to get the fiber square

$$\begin{tikzcd}
E \times E' \arrow[r] \arrow[d] & E \times B' \arrow[d] \\
E \times_G E' \arrow[r] & B \times B'.
\end{tikzcd}$$

(To get the exact hypotheses of the proposition, one needs to change the right action of G on E' to a left action.)

Given two such bundles E, E', with $H^iE = 0$ for $0 < i \leq N$ and $H^iE' = 0$ for $0 < i \leq N'$, Lemma 1.6 gives us

$$\begin{tikzcd}
E \times X \arrow[d] & E \times E' \times X \arrow[l] \arrow[r] \arrow[d] & E' \times X \arrow[d] \\
E \times_G X & (E \times E') \times_G X \arrow[l] \arrow[r] & E' \times_G X,
\end{tikzcd}$$

where the vertical maps are locally trivial fiber bundles with fiber G, the horizontal maps on the left are locally trivial with fiber E', and the horizontal maps on the right are locally trivial with fiber E.

Since $(E \times E') \times_G X \to E' \times_G X$ is a fiber bundle with fiber E, we have an immediate corollary of the Leray-Hirsch theorem: $H^i(E) = 0$ for $0 < i \leq N$ implies $H^i(E' \times_G X) \to H^i((E \times E') \times_G X)$ is an isomorphism for $i \leq N$. Indeed, the Leray-Hirsch theorem says

$$H^i((E \times E') \times_G X) = \bigoplus_{j=0}^{i} H^j(E' \times_G X) \otimes H^{i-j}(E),$$

and the only nonzero summand is the $j = i$ term.

This gives us isomorphisms $H^i(E\times_G X) \xrightarrow{\sim} H^i(E'\times_G X)$ for $i \leq \min(N, N')$. These are natural, in the sense that if $E'' \to B''$ is a third such bundle, with $H^i(E'') = 0$ for $0 < i \leq N''$, then for $i \leq \min(N, N', N'')$, the diagram

$$\begin{tikzcd}
H^i(E \times_G X) \arrow[rr, "\sim"] \arrow[dr, "\sim"'] & & H^i(E' \times_G X) \arrow[dl, "\sim"] \\
& H^i(E'' \times_G X) &
\end{tikzcd}$$

commutes.

Thus we see that for the purposes of calculation, we can forget the classifying bundle $EG \to BG$, and just use the finite-dimensional spaces $EG_m \to BG_m$ for computations in finite degrees.

We now treat two important special cases of G-actions.

Lemma 1.9. *If G acts freely on X (on the left), and $X \to G\backslash X$ is a locally trivial fiber bundle, then*

$$H^i_G X = H^(G\backslash X).$$

In particular, $H^i_G X = 0$ for $i > \dim X$ (or even $i > \dim(G\backslash X)$).

Proof. Consider the diagram

$$\begin{tikzcd}
E \times X \arrow[r] \arrow[d] & X \arrow[d] \\
E \times_G X \arrow[r] & G\backslash X,
\end{tikzcd}$$

in which the vertical arrows are G-bundles, and the horizontal arrows are E-bundles. As in the above discussion, the Leray-Hirsch theorem implies that if $H^i E = 0$ for $0 < i \leq N$, then the map $H^i(G\backslash X) \to H^i(E\times_G X) = H^i_G(X)$ is an isomorphism. □

Lemma 1.10. *If G acts trivially on X, and $H^*_G(pt)$ (or $H^*(X)$) is free over the coefficient ring, then*

$$H^*_G X \cong H^*_G(pt) \otimes H^* X$$

as skew-commutative graded algebras.

Proof. For $E \to B$, we have

$$\begin{tikzcd}
E \times X \arrow[d] \arrow[dr] & \\
E \times_G X \cong B \times X, &
\end{tikzcd}$$

where the diagonal map is $(e,x) \mapsto (\overline{e},x)$. Thus for $i \leq N$ (where $H^iE = 0$ for $0 < i \leq N$),

$$\begin{aligned} H^i_G(X) &= H^i(B \times X) \\ &= \bigoplus_{j+k=i} H^jB \otimes H^kX, \end{aligned}$$

since either of the hypotheses of the lemma are sufficient for this Künneth formula. ☐

Our next task is to develop an equivariant analogue of the situation of §1 – in particular, we will have equivariant Chern classes of vector bundles, and equivariant fundamental classes of invariant subvarieties.

2. Equivariant Chern classes and fundamental classes

Another special case of Lemma 1.6 is the situation of an equivariant vector bundle. Let $F = X' \to X$ be an equivariant vector bundle on X, with fiber $\mathbb{C}^r$. Then $E \times_G F \to E \times_G X$ is an equivariant vector bundle on $E \times_G X$, by Lemma 1.6, and we have Chern classes $c_i(E \times_G F) \in H^{2i}(E \times_G X)$.

Definition. We define the *equivariant Chern classes* of an equivariant vector bundle $F \to X$ to be $c_i^G(F) = c_i(EG \times_G F) \in H^{2i}(EG \times_G X) = H^{2i}_G X$.

As usual, any "suitably contractible" G-bundle E can be used in place of EG.

These Chern classes satisfy the usual properties:

- They are natural for equivariant maps. If $X \to X'$ is equivariant with respect to $G \to G'$, and $F \to X$ is the pullback of $F' \to X'$, i.e., there is an equivariant fiber square

$$\begin{tikzcd} F \arrow[r] \arrow[d] & F' \arrow[d] \\ X \arrow[r] & X', \end{tikzcd}$$

  then the map $H^{2i}_{G'}X' \to H^{2i}_G X$ sends $c_i^{G'}(F')$ to $c_i^G(F)$.
- There is a Whitney formula for an exact sequence $0 \to F' \to F \to F'' \to 0$ of equivariant vector bundles on X:

$$c(F) = c(F')c(F'').$$

Now let V be a codimension-d subvariety of a nonsingular variety X, and let G be a linear algebraic group acting on X and preserving V. Then there is an *equivariant fundamental class* $[V]^G \in H^{2d}_G X$, with the standard properties discussed in §1. It is constructed as follows.

Take an approximating bundle $EG_m \to BG_m$, where these are smooth algebraic varieties. (Since we assume $G \subseteq GL_n$, we can and usually will take $EG_m = M^o_{m,n}$.) Then we have a diagram

$$\begin{tikzcd}
EG_m \times V \arrow[r, hook] \arrow[d] & EG_m \times X \arrow[d] \\
EG_m \times_G V \arrow[r, hook] & EG_m \times_G X,
\end{tikzcd}$$

where the vertical arrows are smooth maps (since they are G-bundles). Since $EG_m \times V \hookrightarrow EG_m \times X$ looks like $V \hookrightarrow X$ and the projection maps are smooth, it follows that locally the same is true for $EG_m \times_G V \hookrightarrow EG_m \times_G X$. Therefore $EG_m \times_G V \subset EG_m \times_G X$ is a codimension-d subvariety, and we get a fundamental class $[EG_m \times_G V] \in H^{2d}(EG_m \times_G X)$.

Definition. We define the *equivariant fundamental class* $[V]^G \in H^{2d}_G X$ of an invariant subvariety $V \subset X$ to be $[EG_m \times_G V] \in H^{2d}(EG_m \times_G X)$, for m large enough so that $H^{2d}(EG_m \times_G X) = H^{2d}_G X$.

Of course, for this definition to make sense, we need to check that this class is compatible with the maps induced by $EG_m \to EG_{m+1}$. More precisely, we need to check this inclusion induces a commutative diagram

$$\begin{tikzcd}
EG_m \times_G V \arrow[r, hook] \arrow[d, hook] & EG_m \times_G X \arrow[d, hook] \\
EG_{m+1} \times_G V \arrow[r, hook] & EG_{m+1} \times_G X,
\end{tikzcd}$$

and the cohomology pullback $H^{2d}(EG_{m+1} \times_G X) \to H^{2d}(EG_m \times_G X$ takes $[EG_{m+1} \times_G V]$ to $[EG_m \times_G V]$.

Many properties of classical Chern classes and fundamental classes extend to the equivariant setting. We discuss a few of these.

If V and W are G-invariant subvarieties of X, and $V \cap W = \emptyset$, then $[V]^G \cdot [W]^G = 0$. Indeed, $(EG_m \times_G V) \cap (EG_m \times_G W) = \emptyset$, so this reduces to the classical case.

If V and W intersect properly, then there is an intersection cycle

$$V \cdot W = \sum_j i(U_j, V \cdot W; X) \cdot U_j,$$

where the U_j are the irreducible components of $V \cap W$, each of which is G-invariant. (Here we use the fact that G is connected, hence irreducible.) Then $[V]^G \cdot [W]^G = \sum_j i(U, V \cdot W; X) \cdot [U]^G$. To see this, reduce to the classical case as before, by looking at $EG_m \times_G V \cap EG_m \times_G W$.

Exercise 2.1. If L is an equivariant line bundle on X, and $s : X \to L$ is an equivariant section, then $Z(s)$ is an equivariant divisor on X, assuming it has the correct dimension. (If it is not irreducible, write $Z(s) = \sum m_i D_i$, with D_i an irreducible divisor.) Show that $[Z(s)]^G = c_1^G(L) \in H^2_G(X)$.

[Hint: Look at the corresponding line bundle $EG_m \times_G L \to EG_m \times_G X$, with section s'. Then $Z(s') = EG_m \times_G Z(s)$.]

Remark 2.2. This section is an actual *equivariant* section, with respect to given actions on L and X. If G acts on L by a character, then one gets a section on a different line bundle. (Namely, twist L by the line bundle associated to the character.)

Associated to the standard representation of $\mathbb{C}^*$ on $\mathbb{C}$, there is an equivariant line bundle L on a point. We have $c_1^{\mathbb{C}^*}(L) = t \in \mathbb{Z}[t] = H^*_{\mathbb{C}^*}(pt)$. For the action of $\mathbb{C}^*$ given by $z \cdot v = z^a v$, with $a \in \mathbb{Z}$, there is the equivariant line bundle L_a, and $c_1^{\mathbb{C}^*}(L_a) = at$. In fact, $L_a \cong L^{\otimes a}$.

Generalizing this, we have the following:

Corollary 2.3. *Suppose $\mathbb{C}^*$ acts on $\mathbb{C}^r$ by $z \cdot (v_1, \dots, v_r) = (z^{a_1} v_1, \dots, z^{a_r} v_r)$. (In fact, all algebraic actions look like this.) Then this makes $\mathbb{C}^r$ into an equivariant vector bundle $E_{a_1,\dots,a_r}$ on a point, with*

$$c_i^{\mathbb{C}^*}(E_{a_1,\dots,a_r}) = e_i(a_1 t, \dots, a_r t) = e_i(a_1, \dots, a_r) t^i.$$

(Equivalently, $c^{\mathbb{C}^*} = \prod_i (1 + a_i t)$.)

More generally still:

Corollary 2.4. *Let $T = (\mathbb{C}^*)^n$ act on $\mathbb{C}^r$ by*

$$(z_1, \dots, z_n) \cdot (v_1, \dots, v_r) = (\prod_{i=1}^n z_i^{a_{i1}} v_1, \dots, \prod_{i=1}^n z_i^{a_{ir}} v_r),$$

where $A = (a_{ij})$ is an $n \times r$ matrix of weights. Then we get an equivariant vector bundle E_A on a point, with

$$c^T(E_A) = \prod_{j=1}^r (1 + \sum_{i=1}^n a_{ij} t_i)$$

in $H^*_T(pt) = \mathbb{Z}[t_1, \dots, t_n]$.

Proof. (Sketch.) Reduce to the case $r = 1$, using the splitting principle (and the fact that any representation of T decomposes as a sum of one-dimensional representations). Now $c_1^T(E_A) = \sum_{i=1}^n b_i t$, with $b_i \in \mathbb{Z}$. Restricting to the ith copy of $\mathbb{C}^*$ in $(C^*)^n$ via $z \mapsto (1, \dots, 1, z, 1, \dots, 1)$, notice that $b_i = a_{i1}$. Indeed, the induced map $H^*_T(pt) = \mathbb{Z}[t_1, \dots, t_n] \to \mathbb{Z}[t] = H^*_{C^*}(pt)$ is given by $t_k \mapsto \delta_{ik} t$. $\square$

Exercise 2.5. Let a group homomorphism $(C^*)^n \to (C^*)^r$ be given by

$$(z_1, \dots, z_n) \mapsto (\prod_{i=1}^n z_i^{a_{i1}}, \dots, \prod_{i=1}^n z_i^{a_{ir}}),$$

where $A = (a_{ij}) \in M_{n,r}(\mathbb{Z})$. Show that the induced map

$$H^*_{(C^*)^r}(pt) = \mathbb{Z}[s_1, \dots, s_r] \to \mathbb{Z}[t_1, \dots, t_n] = H^*_{(C^*)^n}(pt)$$

is given by $s_j \mapsto \sum_{i=1}^n a_{ij} t_i$.

[Hint: Reduce to the case of $\mathbb{C}^* \to \mathbb{C}^*$ using $\mathbb{C}^* \hookrightarrow (\mathbb{C}^*)^n \to (\mathbb{C}^*)^r \to \mathbb{C}^*$.]

We make a few more general comments on torus-equivariant line bundles. Let T be a torus – that is, an algebraic group isomorphic to $(\mathbb{C}^*)^n$ – and let M be the group of characters of T, so $M = \mathrm{Hom}_{\text{alg. gp.}}(T, \mathbb{C}^*)$. ($M$ is isomorphic to $\mathbb{Z}^n$ after choosing an isomorphism $T \cong (\mathbb{C}^*)^n$.)

For each $\chi \in M$, there is an equivariant line bundle L_χ on a point. This is $L_\chi = \mathbb{C}$, with the action $t \cdot v = \chi(t)v$ for $t \in T$. Thus we get a class $c_1^T(L_\chi) \in H_T^2(pt)$.

Claim: This gives canonical isomorphisms

$$M \xrightarrow{\sim} H_T^2(pt)$$

and

$$\mathrm{Sym}^\bullet M = \bigoplus_{k\geq 0} \mathrm{Sym}^k M \xrightarrow{\sim} H_T^*(pt) = \Lambda_T.$$

One way to prove this is by choosing a basis, i.e., fixing $T \cong (\mathbb{C}^*)^n$. Then, as we have already seen, $\mathbb{Z}[t_1, \ldots, t_n] \cong H^*_{(\mathbb{C}^*)^n}(pt)$ via $t_i = c_1^T(L_i)$, where $L_i = L_{\chi_i}$ and $\{\chi_i\}$ is the basis of M corresponding to the choice of basis for T.

Here is a more intrinsic way to see it. A homeomorphism of tori $\varphi : T \to T'$ corresponds to a map $M' \to M$, by pulling back characters. ($\chi' \mapsto \chi' \circ \varphi$.) We have a commutative diagram

$$\begin{tikzcd}
\mathrm{Sym}^\bullet M' \arrow[r] \arrow[d, "\sim"'] & \mathrm{Sym}^\bullet M \arrow[d, "\sim"'] \\
H^*_{T'}(pt) \arrow[r] & H^*_T(pt) \\
\| \arrow[r] & \| \\
H^*(BT') \arrow[r] & H^*(BT)
\end{tikzcd}$$

Indeed, the map $BT \to BT'$ makes L_χ the pullback of $L_{\chi'}$. Use the fact $\chi = \chi' \circ \varphi : T \to \mathbb{C}^*$, and the corresponding map of BT's.

Exercise 2.6. Let $T = (\mathbb{C}^*)^n$, considered as the subgroup of diagonal matrices in B, the group of invertible upper-triangular $n \times n$ matrices. There is an obvious retraction $B \to T$, which in fact is a group homomorphism. If B acts on a space X, show that the induced map $H_T^* X \to H_B^* X$ is an isomorphism.

[Hint/solution: Use approximations $E_m \times_B X \to E_m \times_T X$, with $E_m = M^o_{m,n}$ as usual. In fact, this is easily seen to be a vector bundle of rank $\begin{pmatrix} n \\ 2 \end{pmatrix}$, so the corresponding map on cohomology is an isomorphism.]

Remark 2.7. We will often have a sequence of groups acting on a given space X, and we can relate the equivariant cohomology of X with respect to the various groups. The general philosophy is that for theory, it is better to work with groups at the "right end" of the sequence, while for computations, it is easier to use groups at the "left end."

Any finite-dimensional representation $\rho : G \to GL(V)$ of a group G can be considered as an equivariant vector bundle on a point: $V \to pt$. Therefore there are equivariant Chern classes $c_i(\rho) := c_i^G(V) \in H_G^{2i}(pt)$. When $G = GL_n(\mathbb{C})$, the irreducible representations are exactly $\rho_\lambda : G \to GL(V_\lambda)$, where $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$ is a "highest weight."

For example, for $\lambda = (k, 0, \ldots, 0)$, $V_\lambda = \mathrm{Sym}^k(\mathbb{C}^n)$; for $\lambda = (1, \ldots, 1, 0, \ldots, 0)$ (k 1's), this is $V_\lambda = \Lambda^k(\mathbb{C}^n)$. So if $\lambda = (1, \ldots, 1)$, $V_\lambda = \Lambda^n(\mathbb{C}^n)$ is the "determinant" representation, denoted det. In fact, $V_\lambda \otimes \det \cong V_{(\lambda_1+1,\ldots,\lambda_n+1)}$, so it is enough to study those V_λ with $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$, i.e., polynomial representations.

Challenge: Compute $c_i(\rho_\lambda) \in \Lambda_{GL_n} = \mathbb{Z}[c_1, \ldots, c_n]$ – that is, write $c_i(\rho_\lambda) = p_i(c_1, \ldots, c_n)$ as a polynomial of degree $2i$.

Here is an equivalent formulation of this problem. Note that GL_n-bundles correspond to vector bundles, as discussed in §3. For any vector bundle E of rank n on a space X, and $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$, there is a bundle $S^\lambda E$. (These generalize the cases $Sym^k E$ and $\Lambda^k E$, for $\lambda = (k, 0, \ldots, 0)$ and $\lambda = (1, \ldots, 1, 0, \ldots, 0)$, respectively. See [Fulton, §8.1].) What is $c_i(S^\lambda E)$ as a polynomial $p_i(c_1(E), \ldots, c_n(E))$?

Lascoux has given formulas for the cases of $\mathrm{Sym}^2 E$ and $\Lambda^2 E$, but little is known beyond some trivial cases. (E.g., $c_1(\Lambda^n \mathbb{C}^n) = c_1$.)

In fact, V_λ has a basis $\{e_T\}$ indexed by *semi-standard tableaux* T on λ, with entries in $\{1, \ldots, n\}$. It is not obvious how GL_n acts on this, but the maximal torus $(\mathbb{C}^*)^n$ acts by

$$(z_1, \ldots, z_n) \cdot e_T = \prod_{i=1}^n z_i^{\#(i\text{'s in T})} e_T.$$

So in $\mathbb{Z}[t_1, \ldots, t_n] \supset \mathbb{Z}[c_1, \ldots, c_n]$, we have

$$c^{(\mathbb{C}^*)^n}(\rho_\lambda) = \prod_T (1 + \sum_{i \in T} t_i).$$

Thus one can compute any particular example.

Note, by the way, that this is a symmetric function (though not obviously so). There are *Schur polynomials*

$$s_\lambda(t_1, \ldots, t_n) = \sum_{T \in SSYT(\lambda)} \prod_{i \in T} t_i,$$

and as λ varies, these form a basis of the ring of symmetric polynomials, $\mathbb{Z}[t_1, \ldots, t_n]^{S_n}$. In fact, $s_\lambda = \det(c_{\lambda'_i+j-i})$, where λ' is the *conjugate* partition to λ. Then the following is true, but the only proof we know uses the Hard Lefschetz Theorem:

Theorem 2.8. *The symmetric function $c_i^G(\rho_\lambda)$ is* Schur-positive*. That is, when one expands it in the basis of Schur polynomials as $c_i^G(\rho_\lambda) = \sum_\mu b_{\lambda\mu} s_\mu$, the coefficients $b_{\lambda\mu}$ are nonnegative integers.*

Challenge: Is there a more elementary proof of Schur-positivity? Even better, give a combinatorial formula for the integers $b_{\lambda\mu}$. Which are nonzero?

Even more generally, one could consider an expansion $s_\nu(c(S^\lambda E)) = \sum_\mu b_{\lambda\mu\nu} s_\mu(c(E))$.

We will discuss the combinatorics of Young tableaux in slightly more detail in the next part of these lectures, on Schubert calculus. For definitions and discussions of the relation with representation theory, see [Fulton, §7-8] or [Fulton-Harris].

3. EQUIVARIANT COHOMOLOGY OF $\mathbb{P}^{n-1}$

We now introduce an extended example, and begin to study the equivariant cohomology of projective space in detail. Many of the phenomena particular to equivariant cohomology can be seen here, and we will return to this example several times in the sequel.

There are several groups acting naturally on $\mathbb{P}^{n-1}$. We have actions of $G = GL_n(\mathbb{C})$, the upper-triangular subgroup B, and the torus of diagonal matrices $T \cong (\mathbb{C}^*)^n$. In addition, there are actions of $T' = (\mathbb{C}^*)^n/\mathbb{C}^*$ and $PGL_n(\mathbb{C}) = GL_n(\mathbb{C})/\mathbb{C}^*$, but we will focus on the actions of the first three groups, $T \subset B \subset G$.

The tautological subbundle $\mathcal{O}(-1) \subset \mathbb{C}^n_{\mathbb{P}^{n-1}}$ is a G-equivariant line bundle; therefore the dual bundle $\mathcal{O}(1)$ is, too. (The action is by $(g \cdot s)(p) = s(g^{-1} \cdot p)$.) Let $\zeta = c_1^G(\mathcal{O}(1)) \in H^2_G \mathbb{P}^{n-1}$.

Proposition 3.1. $H^*_G \mathbb{P}^{n-1} = \mathbb{Z}[c_1, \dots, c_n][\zeta]/(\zeta^n + c_1 \zeta^{n-1} + \cdots + c_n)$.

Remark 3.2. Notice the analogy with projective bundles. This illustrates the theme that equivariant theory is "spread-out geometry," that is, geometry done relatively over non-trivial bases.

Proof. Look at the approximation space $E_m = M^o_{m,n}$. We have $M^o_{m,n} \times_G \mathbb{P}^{n-1} \cong \mathbb{P}(S)$, where S is the tautological subbundle on $Gr(n, \mathbb{C}^m)$. Indeed, consider the diagram

$$\begin{tikzcd}
M^o_{m,n} \times \mathbb{P}^{n-1} \arrow[r] \arrow[d] & M^o_{m,n} \arrow[d] \\
M^o_{m,n} \times_G \mathbb{P}^{n-1} \arrow[r] \arrow[d, "\cong"'] & M^o_{m,n}/G \arrow[d, equal] \\
\mathbb{P}(S) \arrow[r] & Gr(n, \mathbb{C}^m).
\end{tikzcd} \tag{2}$$

Here $M^o_{m,n}/G$ is identified with $Gr(n,\mathbb{C}^m)$ by considering a matrix $A \in M^o_{m,n}$ as a linear map $\mathbb{C}^n \to \mathbb{C}^m$, and sending A to $\mathrm{im}(A) \in Gr(n,\mathbb{C}^m)$. The map $M^o_{m,n} \times_G \mathbb{P}^{n-1} \to \mathbb{P}(S)$ is defined by

$$(A,[v]) \mapsto (\mathrm{im}(A),[A(v)]),$$

where v is a vector in $\mathbb{C}^m$ spanning the line $[v]$. This is well-defined (i.e., it passes to the quotient), since for any $B \in GL_n(\mathbb{C})$,

$$(\mathrm{im}(A\cdot B),[(A\cdot B)(v)]) = (\mathrm{im}(A),[A(B(v))]),$$

so $(A\cdot B,[v])$ and $(A, B\cdot[v])$ have the same image under this map. In fact, it is straightforward to check that this is a homeomorphism, and therefore an isomorphism of bundles, since the lower square of (2) commutes.

Now the tautological bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$ on $\mathbb{P}^{n-1}$ pulls back to the bundle $\mathcal{O}_{\mathbb{P}(S)}(-1)$ on $\mathbb{P}(S)$; therefore $\zeta = c_1^G(\mathcal{O}_{\mathbb{P}^{n-1}}(1)) = c_1(\mathcal{O}_{\mathbb{P}(S)}(1))$. We see that $H^*\mathbb{P}(S) = H^*Gr(n,\mathbb{C}^m)[\zeta]/(\zeta^n + c_1\zeta^{n-1} + \cdots + c_n)$, where $c_i = c_i(S)$. To complete the proof, it remains only to check that all this is compatible with the inclusions of approximations $EG_m \times_G \mathbb{P}^{n-1}$, as $m \to \infty$. $\square$

Corollary 3.3. *For $G = T = (\mathbb{C}^*)^n$, or for $G = B$ (upper-triangular matrices), we have*

$$H^*_G(\mathbb{P}^{n-1}) = \mathbb{Z}[t_1,\ldots,t_n][\zeta]/\prod_{i=1}^n(\zeta + t_i).$$

For either of these groups G, the inclusion $G \to GL_n$ induces a map $H^*_{GL_n}(pt) = \mathbb{Z}[c_1,\ldots,c_n] \to \mathbb{Z}[t_1,\ldots,t_n] = H^*_G(pt)$, given by $c_i \mapsto e_i(t_1,\ldots,t_n)$, where as usual e_i is the ith elementary symmetric polynomial. In the corollary above, we see this for $H^*_{GL_n}\mathbb{P}^{n-1} \to H^*_G\mathbb{P}^{n-1}$.

As a special case, note that for the group B of upper-triangular matrices, $M^o_{m,n}/B = Fl(1,\ldots,n;\mathbb{C}^m)$, with the tautological flag of bundles $S_1 \subset \cdots S_n = S \subset \mathbb{C}^m_{Fl}$. Since $M^o_{m,n} \times_B \mathbb{P}^{n-1} = \mathbb{P}(S)$, we see $t_i = c_1(S_i/S_{i-1})$.

For all these groups (GL_n, B, and T), we have a surjective map $H^*_G(\mathbb{P}^{n-1}) \to H^*(\mathbb{P}^{n-1})$ via "restriction to a fiber." Indeed, $H^*\mathbb{P}^{n-1} = H^*_G\mathbb{P}^{n-1}/\Lambda^+_G$, where $\Lambda^+_G \subset \Lambda_G$ is the ideal of positive-degree elements.

We now consider fixed points and invariant subvarieties of $\mathbb{P}^{n-1}$ under the three group actions we have studied.

For GL_n, there are no fixed points: $(\mathbb{P}^{n-1})^{GL_n} = \emptyset$.

For B, there is one fixed point: $p = [1:0:\cdots:0]$. The invariant subvarieties are the spaces $\mathbb{P}^0 \subset \mathbb{P}^1 \subset \cdots \subset \mathbb{P}^{n-1}$, where $\mathbb{P}^{k-1}$ is the subspace with the last $n-k$ coordinates zero (and the first k coordinates free).

For $T = (\mathbb{C}^*)^n$, all coordinate subspaces are invariant. We will write $\mathbb{P}_I = \{[x_1:\cdots:x_n] \in \mathbb{P}^{n-1} \,|\, x_i = 0 \text{ for } i \in I\}$, where $I \subset [n] = \{1,\ldots,n\}$. One should expect $[\mathbb{P}_I]^T \in H_T^{2\cdot|I|}\mathbb{P}^{n-1}$, and indeed, we will determine these equivariant classes.

We have $(\mathbb{P}^{n-1})^T = \{p_1, \ldots, p_n\}$, where $p_i = \mathbb{P}_{[n]\smallsetminus\{i\}}$. In fact, we have

$$\begin{tikzcd}
H_T^*\mathbb{P}^{n-1} \arrow[r, hook] \arrow[d, equal] & H_T^*(p_1) \oplus \cdots \oplus H_T^*(p_n) \arrow[d, equal] \\
\mathbb{Z}[t_1, \ldots, t_n][\zeta]/(\prod(\zeta + t_i)) \arrow[r, hook] & \mathbb{Z}[t_1, \ldots, t_n] \oplus \cdots \oplus \mathbb{Z}[t_1, \ldots, t_n],
\end{tikzcd}$$

via $\zeta + t_i \mapsto 0$ (i.e., $\zeta \mapsto -t_i$) on the ith factor. Indeed, the tautological buundle $\mathcal{O}(-1) \subset \mathbb{C}^n_{\mathbb{P}^{n-1}}$ restricts to the line bundle L_{χ_i} on the ith fixed point p_i. Thus $\zeta = c_1^T(\mathcal{O}(1)) \mapsto c_1^T(L_{\chi_i}^\vee) = -t_i$.

This map is injective; our next goal is to describe its image.

Claim: $[\mathbb{P}_I]^T = \prod_{i\in I}(\zeta + t_i)$.

(Note that for $I = [n]$, this says $[\emptyset]^T = [P_{[n]}]^T = 0$.) In particular, if $\mathbb{P}^{k-1} \subset \mathbb{P}^{n-1}$ is a B-invariant subvariety, $[\mathbb{P}^{k-1}]^B = \prod_{i=k+1}^n(\zeta + t_i)$. Geometrically, this is

$$\begin{tikzcd}
M^o_{m,n} \times_B \mathbb{P}^{k-1} \arrow[r, hook] \arrow[d, equal] & M^o_{m,n} \times_B \mathbb{P}^{n-1} \arrow[d, equal] \\
\mathbb{P}(S_k) \arrow[r, hook] & \mathbb{P}(S).
\end{tikzcd}$$

To prove the claim, recall the following general fact from §2:

Lemma 3.4. *Let $F \subset E$ be a subbundle on a nonsingular variety X, and let $d = \mathrm{rk}(E/F)$. Then the class $[\mathbb{P}(F)] \in H^{2d}(\mathbb{P}(E))$ is equal to $c_d(E/F \otimes \mathcal{O}_{\mathbb{P}(E)}(1))$.*

The claim follows by applying this to the case $X = Fl(1, \ldots, n; \mathbb{C}^m)$, with $E = S$ and $F = S_k$, where $S_1 \subset \cdots S_n = S \subset \mathbb{C}^m_X$ is the tautological flag bundle on X. Since $c(S/S_k) = \prod_{i=k+1}^n(1+t_i)$, we have $c_d(S/S_k \otimes \mathcal{O}_{\mathbb{P}(S)}(1)) = \prod_{i=k+1}^n(\zeta + t_i)$, as desired.

Challenge: Let $z_i = [\mathbb{P}^{n-1-i}]^B = \prod_{j=n+1-i}^n(\zeta + t_i) \in H_B^*(\mathbb{P}^{n-1})$. Find a good description of the coefficients in products of the z_i's. Specifically, since $\{1, z_1, \ldots, z_{n-1}\}$ is a basis for $H_B^*\mathbb{P}^{n-1}$ over $\Lambda_B = \mathbb{Z}[t_1, \ldots, t_n]$, we must have

$$z_i \cdot z_j = z_{i+j} + p^1_{ij}(t)z_{i+j-1} + p^2_{ij}(t)z_{i+j-2} + \cdots + p^{\min\{i,j\}}_{ij}(t)z_{\max\{i,j\}}.$$

What are the polynomials $p^k_{ij}(t)$?

A recipe for determining a given coefficient is the following. (This is not the same as a formula, though!)

Consider the more general problem of multiplying the classes $[\mathbb{P}_I]^T = \prod_{i\in I}(\zeta + t_i) \in H_T^*\mathbb{P}^{n-1}$. (Note, by the way, that these classes are all distinct – in contrast with the classical situation, where only the cardinality of I matters.) Multiplication of these classes (and therefore also of the z_i's) is determined by the products $[\mathbb{P}_{\{j\}}]^T \cdot [\mathbb{P}_I]^T$. If $j \notin I$, then $[\mathbb{P}_{\{j\}}]^T \cdot [\mathbb{P}_I]^T = [\mathbb{P}_{I\cup\{j\}}]^T$, as is easily seen by geometry or algebra.

On the other hand, if $j \in I$, then for any $k \notin I$, we have

$$\begin{aligned} [\mathbb{P}_{\{j\}}] \cdot [\mathbb{P}_I] &= (\zeta + t_j) \cdot \prod_{i\in I} (\zeta + t_i) \\ &= (\zeta + t_k) \cdot \prod_{i\in I} (\zeta + t_i) + (t_j - t_k) \cdot \prod_{i\in I} (\zeta + t_i) \\ &= [\mathbb{P}_{I\cup\{k\}}] + (t_j - t_k)[\mathbb{P}_I]. \end{aligned}$$

As a special case, we have $z_1 \cdot z_i = z_{i+1} + (t_n - t_{n-i})z_i$.

From these considerations, we see that all the coefficients $p_{ij}^k(t)$ are nonnegative linear combinations of monomials in the variables $\{t_n-t_{n-1}, \ldots, t_2-t_1\}$. A satisfying solution to the "Challenge" above would exhibit this positivity manifestly.

Remark 3.5. The torus $T = (\mathbb{C}^*)^n/\mathbb{C}^*$ acts on $\mathbb{P}^{n-1}$, with the same invariant subvarieties $\mathbb{P}_I$. The quotient map $(\mathbb{C}^*)^n \to (\mathbb{C}^*)^n/\mathbb{C}^*$ corresponds to the inclusion of character groups $\mathbb{Z}^n \hookleftarrow M$, where $M = \{(a_1, \ldots, a_n) \mid \sum a_i = 0\}$. M has a basis $\{t_1 - t_2, \ldots, t_{n-1} - t_n\}$, and

$$H_T^*(pt) = \mathrm{Sym}^\bullet M = \mathbb{Z}[t_1 - t_2, \ldots, t_{n-1} - t_n] \subset \mathbb{Z}[t_1, \ldots, t_n].$$

There is a map $H_T^*\mathbb{P}^{n-1} \hookrightarrow H_{(\mathbb{C}^*)^n}^*\mathbb{P}^{n-1} = \mathbb{Z}[t_1, \ldots, t_n][\zeta]/\prod(\zeta + t_i)$.

Exercise 3.6. Let $\zeta' = [\mathbb{P}_{\{n\}}]^T$. Show that this map sends ζ' to $\zeta + t_n$, and identify

$$H_T^*\mathbb{P}^{n-1} = H_T^*(pt)[\zeta']/\prod_{i=1}^n (\zeta' + t_i - t_n).$$

4. Equivariant Gysin maps

Suppose $f : X \to Y$ is the inclusion of a nonsingular subvariety X of codimension d into a nonsingular variety Y. Recall that in this situation, as well as for other suitably "nice" maps, there is a *Gysin pushforward* on cohomology: $f_* : H^iX \to H^{i+2d}Y$. (See §1.) There are equivariant versions of these Gysin maps, with analogous properties. If a group G acts on Y, and X is a G-invariant subvariety, then there are maps $f_* : H_G^iX \to H_G^{i+2d}Y$. Indeed, these come from the corresponding maps for $EG \times_G X \hookrightarrow EG \times_G Y$. (There are various ways of making sense of this map of infinite-dimensional spaces. One could consider a compatible family of embeddings of varieties $EG_m \times_G X \to EG_m \times_G Y$, and the corresponding Gysin maps. Perhaps a better way to construct the Gysin map is to note that the normal bundle N to X is an equivariant vector bundle, and the isomorphism of N with a tubular neighborhood U gives an isomorphism $E \times_G N \cong E \times_G U \subset E \times_G Y$. Then the Thom isomorphism gives the desired map on cohomology.)

Similarly, if $f : X \to Y$ is a proper map and a locally trivial fiber bundle whose fiber is an oriented n-manifold, and if f is equivariant with respect to an action of G on X and Y, then we have Gysin maps $f_* : H^i_G X \to H^{i-n}_G Y$. To construct this map, use the ordinary Gysin map for $E\times_G \to E \times_G Y$. Alternatively, one can consider the maps of approximation spaces $E_m \times_G X \to E_m \times_G Y$, and use the property of ordinary Gysin maps corresponding to (3) below to see that the result is compatible: Look at the diagrams

$$\begin{tikzcd}
E_m \times_G X \arrow[r] \arrow[d] & E_m \times_G Y \arrow[d] \\
E_{m+1} \times_G X \arrow[r] & E_{m+1} \times_G Y.
\end{tikzcd}$$

Just as for ordinary Gysin maps, these equivariant Gysin maps have the following properties:

(1) (Projection formula) $f_*(f^*(c) \cdot d) = c \cdots f_*(d)$, for $c \in H^*_G X$, $d \in H^*_G Y$.

(2) (Functoriality) Suppose $X \xrightarrow{f} Y \xrightarrow{g} Z$, with f and g as above. (Either both are inclusions, or both are fiber bundles.) Then $g \circ f$ is of the same type, and $(g \circ f)_* = g_* \circ f_*$.

(3) Suppose f is as above, and $g : Y' \to Y$ is an equivariant map, so there is a fiber square

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "h"'] & Y' \arrow[d, "g"] \\
X \arrow[r, "f"] & Y.
\end{tikzcd}$$

Then f' is of the same type as f, and $g^* \circ f_* = f'_* \circ h^*$.

Example 4.1. We compare the equivariant Gysin maps with the ordinary maps for the case where Y is a point. Consider $p : X \to pt$, where X is an n-dimensional compact manifold. The ordinary Gysin map gives $p_* : H^i X \to H^{i-n}(pt) = 0$ for $i \neq n$, and

$$H^n X \to \overline{H}_0 X \to H_0(pt) = H^0(pt) = \mathbb{Z}.$$

This map is often denoted $\int_X : H^* X \to \mathbb{Z}$; it has the effect of "throwing away" all but the top-degree part of an element of $H^* X$.

Consider $X = \mathbb{P}^{n-1}$, with the basis $\{1, \zeta, \ldots, \zeta^{n-1}\}$ for $H^* X$, where $\zeta = c_1(\mathcal{O}(1))$. Then $p_*(\sum_i a_i \zeta^i) = a_{n-1}$.

In the equivariant case, we get $p_* : H^*_G X \to H^*_G(pt) = \Lambda_G$. The ring Λ_G is graded; write $\Lambda_G = \bigoplus_{i \geq 0} \Lambda^i_G$. Then the Gysin map p_* has graded degree $-n$ – that is, $H^i_G X \to \Lambda^{i-n}_G$. Note that $H^i_G X$ maps to 0 for $i < n$, but *not* for $i > n$.

For $X = \mathbb{P}^{n-1}$ and $G = GL_n$, B, or T, with $\zeta = c^G_1(\mathcal{O}(1))$, we again have $p_*(\sum_i a_i \zeta^i) = a_{n-1}$, where $a_i \in \Lambda_G$. Indeed, $\zeta^i \mapsto 0$ for $i < n-1$, and

$\zeta^{n-1} \mapsto \Lambda^0_G = \mathbb{Z}$. To see that $\zeta^{n-1} \mapsto 1$, look at the fiber square

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & E \times_G X \arrow[d] \\
pt \arrow[r] & B,
\end{tikzcd}$$

and use the classical case.

5. Equivariant Poincaré duality

Let X be a compact, oriented, n-dimensional manifold, and assume that the map

$$H^i X \times H^{n-i} X \to H^n X \to \mathbb{Z}$$

given by $(c, d) \mapsto \int_X c \cdot d$ is a *perfect pairing* over $\mathbb{Z}$ – that is, it defines an isomorphism of $H^i X$ with the dual of $H^{n-i}X$. (In general, one may need to replace $\mathbb{Z}$ with $\mathbb{Q}$, killing any torsion in homology, to make this hold. We will deal with spaces whose homology is torsion-free, though.)

This actually makes H^*X into its own dual, but the duality pairing is trivial except in degrees $i, n-i$.

Lemma 5.1. *Assume there are homogeneous elements $x_1, \ldots, x_m \in H^*_G X$, with $x_i \in H^{k(i)}_G X$, such that their images $\bar{x}_i \in H^*X$ give a basis for H^*X over the ground ring (usually $\mathbb{Z}$). Then $\{x_1, \ldots, x_m\}$ is a basis for $H^*_G X$ over $\Lambda_G = H^*_G(pt)$.*

Moreover, for any map $G' \to G$, the images of the x_i in $H^*_{G'}X$ give a basis for this module over $\Lambda_{G'}$.

Proof. Apply the Leray-Hirsch theorem to the fiber bundle

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d] & E \times_G X \arrow[d] \\
pt \arrow[r, hook] & E/G.
\end{tikzcd}$$

□

Lemma 5.2. *Let $x_1, \ldots, x_m$ be as in the Lemma 5.1, and use the assumption above on classical Poincaré duality on X (i.e., that it gives a perfect pairing on cohomology). Then there are elements $y_1, \ldots, y_m \in H^*_G X$ with $y_i \in H^{n-k(i)}_G X$, such that*

$$\langle x_i, y_j \rangle = \delta_{ij}, \tag{3}$$

where $\langle x, y \rangle = p_*(x \cdot y) \in H^*_G(pt) = \Lambda_G$.

Each y_j is uniquely determined by condition (3), and $\{y_1, \ldots, y_m\}$ is a basis for $H^*_G X$ over Λ_G, called the "Poincaré dual basis" to $\{x_i\}$.

Proof. Take $\bar{y}_1, \ldots, \bar{y}_m$ to be the basis for H^*X dual to $\{x_i\}$ under ordinary Poincaré duality. Order the x_i's so that $x_i \in H_G^{k(i)}X$, with $n = k(1) \geq k(2) \geq \cdots \geq k(m) = 0$. Our goal is to use induction on r to find lifts $y_1, \ldots, y_r$ such that $\langle x_i, y_j \rangle = \delta_{ij}$ for all i, and for $j \leq r$.

For $r = 1$, $y_1 \in H_G^0X$ is the unique lift of $\bar{y}_1 \in H^0X$ via $H_G^0X \xrightarrow{\sim} H^0X$.

Now assume $y_1, \ldots, y_{r-1}$ have been found. Take any lift $y_r' \in H_G^{n-k(r)}X$ of $\bar{y}_r$, and set

$$y_r = y_r' - \sum_{j=1}^{r-1} a_j y_j,$$

where

$$a_j = (-1)^{k(j)\cdot(n-k(j))} \langle x_j, y_r' \rangle \in \Lambda_G^{k(j)-k(r)}.$$

A straightforward check shows that this choice of y_r works.

To see that the y_j's form a basis, it is enough to show they generate. Let F be a free Λ_G-module with generators $y_1, \ldots, y_m$, and consider the maps

$$F \hookrightarrow H = H_G^*X \to \operatorname{Hom}(H, \Lambda_G),$$

where the first is the obvious inclusion, and the second is given by $y \mapsto \langle \cdot, y \rangle$. The composition is surjective, because $y_i \mapsto x_i^*$. Thus the second map is surjective, but as a map of free modules of the same rank, it must be an isomorphism. ☐

Remark 5.3. Unlike the classical case, $\langle x, y \rangle$ is not necessarily 0 for $x \in H_G^iX$, $y \in H_G^jX$, with $i + j > n$ (though this is true for $i + j < n$). Indeed, $\langle x, y \rangle \in \Lambda_G^{i+j-n} \neq 0$ (in general). For example, while $\{1, \zeta, \ldots, \zeta^{n-1}\}$ is a basis for $H_G^*\mathbb{P}^{n-1}$, the set $\{\zeta^{n-1}, \ldots, 1\}$ is *not* the dual basis.

Example 5.4. On $\mathbb{P}^{n-1}$, a pair of "geometric" Poincaré dual bases are the following. Let $x_r = [\mathbb{P}_{\{n+1-r,\ldots,n\}}]^T$, where $\mathbb{P}_{\{n+1-r,\ldots,n\}} = (* : \cdots : * : 0 \cdots : 0)$ is the locus where the last r coordinates are 0. We have seen that $x_r = \prod_{i=n+1-r}(\zeta + t_i)$. The Poincaré dual basis to $x_0, \ldots, x_{n-1}$ is $y_{n-1}, \ldots, y_0$, where $y_s = [\mathbb{P}_{\{1,\ldots,s\}}]^T = \prod_{i=1}^{s}(\zeta + t_i)$. Note that $x_r y_s = 0$ for $r + s \geq n$. (One can see this geometrically – the corresponding subvarieties are disjoint – or algebraically: $\prod_{i=1}^{n}(\zeta + t_i) = 0$.)

Exercise 5.5. Show that the Poincaré dual basis for $\{1, \zeta, \zeta^2, \ldots, \zeta^{n-1}\}$ is $\{z_{n-1}, \ldots, z_1, z_0 = 1\}$, where $z_r = \zeta^r + c_1\zeta^{r-1} + c_2\zeta^{r-2} + \cdots + c_r$.

6. Localization

In studying the equivariant cohomology of $\mathbb{P}^{n-1}$, we saw that it was useful to consider the restriction map $H_T^*\mathbb{P}^{n-1} \to H_T^*(\mathbb{P}^{n-1})^T$. We will see that this is a general principle. First, we look at another example, in the spirit of the calculation of $H_T^*\mathbb{P}^{n-1}$ above.

Let $X = Gr(r, \mathbb{C}^n)$, with the standard action of $G = GL_n$. We will compute $H^*_G X$. As usual, use the approximation $EG_m = M^o_{m,n}$ (and always take $m \geq n$). As with the computation for $\mathbb{P}^{n-1}$, we have

$$\begin{tikzcd}
M^o_{m,n} \times Gr(r, \mathbb{C}^n) \arrow[r] \arrow[d] & M^o_{m,n} \arrow[d] \\
M^o_{m,n} \times_G Gr(r, \mathbb{C}^n) \arrow[r] & M^o_{m,n}/G,
\end{tikzcd}$$

where the vertical arrows are G-bundles and the horizontal arrows are $Gr(r, \mathbb{C}^m)$-bundles. We claim that the bottom row is canonically isomorphic to

$$Gr(r, S) \to Gr(n, \mathbb{C}^m),$$

where S is the tautological rank-n subbundle on $Gr(n, \mathbb{C}^m)$. Indeed, define a map $M^o_{m,n} \times_G Gr(r, \mathbb{C}^n) \to Gr(r, S)$ by $(A, L) \mapsto (\mathrm{im}(A), A(L))$. This is well-defined, since $(A \cdot B, L)$ and $(A, B(L))$ have the same image, for $B \in GL_n$. (Note that $A(L) \subset \mathrm{im}(A) \subset \mathbb{C}^m$ is a sequence of vector spaces of respective dimensions r, n, and m.)

On $Gr(r, S)$ there is a tautological rank-r subbundle $S' \subset S_{Gr(r,S)}$. As we have seen, $H^* \, Gr(r, S)$ is generated (as a ring) over $H^* Gr(n, \mathbb{C}^m)$ by the Chern classes of S' and/or the Chern classes of $Q = S/S'$. Letting m grow large, we get

$$\begin{aligned}
H^*_G X &= H^*_G(pt)[x_1, \ldots, x_r, y_1, \ldots, r_{n-r}]/(x_i = \pm c_i(S'), y_j = c_j(Q)) \\
&= \mathbb{Z}[c_1, \ldots, c_n][x_1, \ldots, x_r]/(r \text{ equations}) \\
&= \mathbb{Z}[c_1, \ldots, c_n][y_1, \ldots, y_{n-r}]/(n - r \text{ equations}).
\end{aligned}$$

The same is true if we replace G with B or T, and use $H^*_{\{B,T\}}(pt) = \mathbb{Z}[t_1, \ldots, t_n] \subset \mathbb{Z}[c_1, \ldots, c_n]$.

The T-fixed points in X are exactly the points p_I, for $I = \{i_1, \ldots, i_r\} \subset [n] := \{1, \ldots, n\}$, corresponding to the linear subspace spanned by the basis vectors $e_{i_1}, \ldots, e_{i_r}$ – that is, $p_I = \langle e_{i_1}, \ldots, e_{i_r} \rangle$. Thus there are $\binom{n}{r}$ fixed points.

We have a map

$$H^*_T X \to H^*_T(X^T) = \bigoplus_I H^*_T(p_I) = \Lambda_T^{\oplus \binom{n}{r}}.$$

Also, we know that $H^*_T X$ is free over Λ_T of rank $\binom{n}{r}$, by general facts about Grassmann bundles.

Proposition 6.1. *For an r-dimensional subspace L of a vector space $V = \mathbb{C}^n$, with $p_L \in X$ the corresponding point, there is a natural isomorphism $T_{p_L}(X) = \mathrm{Hom}(L, V/L)$.*

(For a proof, see the section on Grassmannians in [Griffiths-Harris].)

In our case, this says $T_{p_I} X = \mathrm{Hom}(\bigoplus_{i \in I} \mathbb{C}e_i, \bigoplus_{j \notin I} \mathbb{C}e_j)$. In fact, we have

Proposition 6.2. *There is a natural isomorphism of bundles* $TX \cong \operatorname{Hom}(S, Q)$ *on* X.

Proof. Since X is a nonsingular variety, TX is equal to the normal bundle to the diagonal embedding $\delta : X \hookrightarrow X \times X$. (Indeed, this is essentially one common definition of the tangent bundle: The sheaf of differentials is the pullback of the conormal sheaf $\mathscr{I}/\mathscr{I}^2$, where $\mathscr{I}$ is the ideal of the diagonal.) On $X \times X$, with projections $p_1, p_2 : X \times X \to X$, there are sequences

$$\begin{tikzcd}
0 \arrow[r] & p_1^*S \arrow[r] & p_1^*V_X \arrow[r] \arrow[d, equal] & p_1^*Q \arrow[r] & 0 \\
 & & V_{X\times X} \arrow[d, equal] & & \\
0 \arrow[r] & p_2^*S \arrow[r] & p_2^*V_X \arrow[r] & p_2^*Q \arrow[r] & 0,
\end{tikzcd}$$

pulled back from the universal sequence $0 \to S \to V_X \to Q \to 0$ on X.

The diagonal is exactly the locus where the composition $p_1^*S \to V_{X\times X} \to p_2^*Q$ is the zero map. That is, $\delta(X) = Z(s)$, where s is a given section of $\operatorname{Hom}(p_1^*S, p_2^*Q)$. (In fact, this is true scheme-theoretically and functorially.) This section s is transversal to the zero section. It is a general fact that given a vector bundle E on a manifold (or variety) Y, and a regular section s (i.e., a section transversal to 0), then the normal bundle to $Z(s) \subset Y$ is canonically isomorphic to $E|_{Z(s)}$. Applying this, we see that

$$\begin{aligned} TX &= \operatorname{Hom}(p_1^*S, p_2^*Q)|_{\delta(X)} \\ &= \operatorname{Hom}(S, Q). \end{aligned}$$

□

Remark 6.3. Similarly, when X is a partial or complete flag variety, $\delta(X) \subset X \times X$ is of the form $Z(s)$ for a section of some vector bundle of rank equal to the dimension of X. Where else is this true? For example, what about flag varieties in types B, C, or D?

Now consider the equivariant case. The inclusion of the T-fixed point p_I in X induces pullbacks of S' to the trivial bundle $L = \bigoplus_{i\in I} \mathbb{C} \cdot e_i$, and of Q to the trivial bundle $\mathbb{C}^n/L = \bigoplus_{j\notin I} \mathbb{C} \cdot e_j$. Thus the restriction map $H_T^*X \to H_T^*(p_I)$ sends $c_k(S') \mapsto e_k(\{e_i \,|\, i \in I\})$ and $c_k(Q) \mapsto e_k(\{t_j \,|\, j \notin I\})$. Note that the tangent bundle at p_I is $T_{p_I}X = \bigoplus_{i\in I,\, j\notin I} (\mathbb{C} \cdot e_i)^\vee \otimes \mathbb{C} \cdot e_j$, so

$$c^T_{top}(T_{p_I}X) = \prod_{i\in I,\, j\notin I} (t_j - t_i).$$

Also, pre-composing with the Gysin pushforward by the inclusion of p_I, the map

$$H_T^*(p_I) \to H_T^*X \to H_T^*(p_I)$$

is multiplication by $c^T_{top}(T_{p_I}X)$.

On the other hand, for $J \neq I$, the composition

$$H_T^*(p_I) \to H_T^* X \to H_T^*(p_J)$$

is the zero map. Assuming $c_{top}^T(T_{p_I}X) \neq 0$ (which we will show later), we see that $H_T^*X \to H_T^*(X^T)$ is injective and becomes an isomorphism after inverting $\prod_{i \neq j}(t_j - t_i)$.

Now we determine the equivariant class $[p_I]^T \in H_T^{2r(n-r)}X$. On X, there are the bundles

$$\begin{tikzcd}
 & & L = \bigoplus_{i \in I} \mathbb{C} \cdot e_i \arrow[d, hook] & & \\
0 \arrow[r] & S \arrow[r] & \mathbb{C}^n_X \arrow[r] & Q \arrow[r] & 0,
\end{tikzcd}$$

and p_I is the locus where $L \to \mathbb{C}^n_X \to Q$ vanishes. Thus

$$\begin{aligned}
[p_I]^T = c_{top}^T(\mathrm{Hom}(L, Q)) &= \prod_{i \in I} c_{top}^T((\mathbb{C} \cdot e_i)^\vee \otimes Q) \\
&= \prod_{i \in I} (y_{n-r} - t_i y_{n-r-1} + t_i^2 y_{n-r-2} - \cdots \pm t_i^{n-r}),
\end{aligned}$$

where $y_i = c_i(Q)$.

Alternatively, p_I is the zero locus of $S \to \mathbb{C}^n_X \to \mathbb{C}^n_X/L = \bigoplus_{j \notin I} \mathbb{C} \cdot e_j$. Thus

$$\begin{aligned}
[p_I]^T = c_{top}^T(\mathrm{Hom}(S, \bigoplus \mathbb{C} \cdot e_j)) &= \prod_{j \notin I} c_{top}^T(S^\vee \otimes \mathbb{C} \cdot e_j) \\
&= \prod_{j \notin I} (x_r + t_j x_{r-1} + \cdots + t_j^r),
\end{aligned}$$

where $x_j = c_j(S^\vee)$.

Exercise 6.4. Show directly that these expressions for $[p_I]^T$ are equal. [Hint: It is a fact from algebra that if

$$\prod_{i \in I}(1 + s_i) \prod_{j \in J}(1 + t_j) = \prod_{i \in I}(1 + \alpha_i) \prod_{j \in J}(1 + \beta_j)$$

in any commutative ring, then $\prod_{i,j}(\beta_j - s_i) = \prod_{i,j}(t_j - \alpha_i)$.]

Exercise 6.5. Find Poincaré dual bases for H_T^*X, where $X = Gr(r, n)$. (We'll see this later.)

Exercise 6.6. Compute $H_T^*(\mathbb{P}(E))$ and $H_T^*(Gr(r, E))$, where E is a vector bundle of the form $E = L_1 \oplus \cdots \oplus L_n$, and T acts on L_i with weight w_i. (We'll do this later, too.)

One has two general expectations about equivariant cohomology of "nice" spaces. First, $H_T^*X \to H^*X$ should be surjective; second, $H_T^*X \to H_T^*(X^T)$ should be injective, and an isomorphism after inverting some elements.

There are various hypotheses on X which allow one to prove this sort of statement, though often one can see it directly in any given example. For example, if X^T is finite and $c^T_{top}(T_pX)$ is nonzero in Λ_T for all $p \in X^T$, we know the map $H^*_TX \to H^*_T(X^T)$ is surjective after localizing (by the Gysin argument done above). Furthermore, if $\#(X^T) = \mathrm{rk}(H^*X)$, then $H^*_TX \hookrightarrow H^*_T(X^T)$ is injective, too, and therefore an isomorphism after localization.

In these nice situations, one has the following *localization formula*:

Proposition 6.7. *Let T be a torus acting on a oriented manifold (or nonsingular variety) X. Suppose X^T is finite, and additionally*

1. *$H^*_TX \to H^*X$ is surjective, with H^*X free over the coefficient ring; and*
2. *$H^*_TX \to H^*_T(X^T)$ is injective, and an isomorphism after localizing.*

Write $\Lambda_T = \mathrm{Sym}^\bullet M \cong \mathbb{Z}[t_1, \dots, t_n]$, and let $\rho : X \to pt$ induce the Gysin map $\rho_* : H^*_TX \to \Lambda_T$.

Then for $F \in H^*_TX$, we have

$$\rho_*(F) = \sum_{p \in X^T} \frac{\iota_p^*(F)}{c^T_{top}(T_pX)},$$

where $\iota_p : \{p\} \hookrightarrow X^T$ is the inclusion.

Proof. We have a diagram

$$\begin{tikzcd} \bigoplus_{p \in X^T} H^*_T(p) \arrow[r, "\oplus(\iota_p)_*"] & H^*_TX \arrow[r, "\oplus\iota_p^*"] \arrow[d, "\rho_*"'] & \bigoplus_{p \in X^T} H^*_T(p) \\ & H^*_T(pt). & \end{tikzcd}$$

As we have seen, the composed map in the top row is "diagonal," i.e., the factor $H^*_T(p)$ goes into the corresponding factor $H^*_T(p)$, with $\iota_p^* \circ (\iota_p)_* : H^*_T(p) \to H^*_T(p)$ given by multiplication by $c^T_{top}(T_pX)$. (We also claim this Chern class is nonzero, but postpone the proof of this fact.)

Use functoriality of Gysin maps: commutativity of

$$\begin{tikzcd} \{p\} \arrow[r, hook, "\iota_p"] \arrow[dr, hook, "id"'] & X \arrow[d, "\rho"] \\ & \{p\} \end{tikzcd}$$

implies that $\rho_* \circ (\iota_p)_* = id_* = id$ on $H^*_T(p) = \Lambda_T$. To prove the formula, then, it is enough to prove it for $F = (\iota_p)_*(a)$, for $a \in H^*_T(p)$.

With this F, the LHS of the formula is $\rho_*(F) = \rho_* \circ (\iota_p)_*(a) = a$. On the RHS, $\iota_q^*(F) = 0$ for $q \neq p$, and $\iota_p^*(F) = a \cdot c^T_{top}(T_pX)$. Dividing by $c^T_{top}(T_pX)$, we see that the RHS is also equal to a. $\square$

The general philosophy illustrated in the proof is that it is often better to think of elements of $H^*_T X$ as coming from $H^*_T(X^T)$, via Gysin maps.

Example 6.8. Let $X = \mathbb{P}^{n-1}$, and let $T = (\mathbb{C}^*)^n$ act in the usual way. Write $X^T = \{p_1, \dots, p_n\}$, and $H^*_T\mathbb{P}^{n-1} = \Lambda_T[\zeta]/\prod_i(\zeta + t_i)$, with basis $1, \zeta, \dots, \zeta^{n-1}$.

We have $\rho_*(\sum a_i\zeta^i) = a_{n-1}$, $c^T_{top}(T_{p_i}X) = \prod_{k\neq i}(t_k - t_i)$, and $\iota^*_{p_i}(\zeta^j) = (-t_i)^j$. This says, for $F = \sum_{j=0}^{n-1} a_j\zeta^j$, that

$$a_{n-1} = \sum_{i=1}^{n} a_j \frac{(-t_i)^j}{\prod_{k\neq i}(t_k - t_i)},$$

or equivalently,

$$\sum_{i=1}^{n} \frac{(-t_i)^j}{\prod_{k\neq i}(t_k - t_i)} \quad = \quad \begin{cases} 0 & \text{if } \ j < n-1; \\ 1 & \text{if } \ j = n-1. \end{cases}$$

(This is an interesting identity in $\mathbb{Q}(t_1, \dots, t_n)$. Can you give a direct proof?)

Example 6.9. Let T be any torus acting on $\mathbb{C}^n$ via characters $\chi_1, \dots, \chi_n$. (I.e., $t \cdot (z_1, \dots, z_n) = (\chi_1(t)z_1, \dots, \chi_n(t)z_n)$.) Thus T acts on $X = \mathbb{P}^{n-1}$, and we have the following:

1. $H^*_T X = \Lambda_T[\zeta]/\prod_{i=1}^n(\zeta + \chi_i)$.
2. X^T is finite if and only if $\chi_1, \dots, \chi_n$ are distinct, and in this case, $X^T = \{p_1, \dots, p_n\}$ as before.
3. $c^T_{top}(T_{p_j}X) = \prod_{k\neq j}(\chi_k - \chi_j)$, and the localization formula says
$$\sum_{i=1}^{n} \frac{(-\chi_i)^j}{\prod_{k\neq i}(\chi_k - \chi_i)} \quad = \quad \begin{cases} 0 & \text{if } \ j < n-1; \\ 1 & \text{if } \ j = n-1. \end{cases}$$
4. We have Poincaré duality as before.

Our next goal is to describe the image of the localization map $H^*_T X \to \bigoplus_{p\in X^T} H^*_T(p)$. In general, for any subtorus $T' \subset T$, T acts on $X^{T'}$ – that is, this is a T-invariant subset. Indeed, T is abelian, so for $x \in X^{T'}$, $t \cdot x = t(t' \cdot x) = t'(t \cdot x)$ is also in $X^{T'}$. We have $X^T \subset X^{T'} \subset X$, so

$$H^*_T(X) \to H^*_T(X^{T'}) \to H^*_T(X^T).$$

In general, then, the image of $H^*_T X$ in $H^*_T(X^T)$ is contained in the image of $H^*_T(X^{T'}) \to H^*_T(X^T)$ for every subgroup $T' \subset T$.

Example 6.10. Let us see this for $X = \mathbb{P}^{n-1}$, with $T = (\mathbb{C}^*)^n$ acting by characters as in the previous example. For $i \neq j$, let $T(i,j) = \{t \in T \mid \chi_i(t) = \chi_j(t)\}$. (When χ_i is just the ith coordinate character, this is $\{(*, \dots, *, t, *, \dots, *, t, * \dots, *)\}$, with t's in the ith and jth positions.) Thus $T(i,j) \subset T$ is a subgroup of codimension 1.

We have $X^{T(i,j)} = \{(0 : \cdots : 0 : z_i : 0 : \cdots : 0 : z_j : 0 \cdots : 0)\} \cong \mathbb{P}^1$; this is the line joining p_i and p_j. Write $\mathbb{P}^1(i,j)$ for this line. The map

$$\begin{tikzcd}
H_T^*(\mathbb{P}^{n-1}) \arrow[r] \arrow[d, equal] & H_T^*(\mathbb{P}^1(i,j) \arrow[d, equal] \\
\Lambda_T[\zeta]/\prod_k(\zeta+\chi_k) \arrow[r] & \Lambda_T[\zeta]/(\zeta+\chi_i)(\zeta+\chi_j)
\end{tikzcd}$$

is given by $\zeta \mapsto \zeta$, since $\zeta = c_1^T(\mathcal{O}(1))$, and $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ restricts to $\mathcal{O}_{\mathbb{P}^1(i,j)}(1)$.

The map $H_T^*(\mathbb{P}^1(i,j)) \to H_T^*(p_i) \oplus H_T^*(p_j)$ is given by $\zeta \mapsto (-\chi_i, -\chi_j)$, and the image of this is $A = \{(a,b) \in \Lambda_T \oplus \Lambda_T \mid a-b \text{ is divisible by } \chi_i - \chi_j\}$. Indeed, $c+d\zeta$ maps to $(c-d\chi_i, c-d\chi_j)$, so the image is certainly contained in A. On the other hand, if $a-b = e(\chi_i - \chi_j)$, then setting $c = a - e\chi_i = b - e\chi_j$ and $d = -e$, we see $c + d\zeta \mapsto (a,b)$.

Claim: The image of $H_T^* X \to H_T^*(X^T)$ is

$$\{(\alpha_1, \ldots, \alpha_n) \mid \alpha_i - \alpha_j \text{ is divisible by } \chi_i - \chi_j \text{ for all } i \neq j\}.$$

To see this, first note that the image is certainly contained in this set, by direct calculation. To show that equality holds, we seek an element $\sum a_i \zeta^i$ mapping to $(\alpha_1, \ldots, \alpha_n)$. Since $a \in \Lambda_T$ maps to $(a, *, \ldots, *) \in H_T^*(X^T)$, we may assume $\alpha_1 = 0$. (Do this by taking $a_0 = \alpha_1$ and subtracting.) Now $\alpha_2 = \alpha_2 - \alpha_1 = b(\chi_1 - \chi_2)$ for some $b \in \Lambda_T$, and $b(\zeta + \chi_1) \mapsto (0, b(\chi_1 - \chi_2), *, \ldots, *)$, so we may assume $\alpha_1 = \alpha_2 = 0$. Continuing in this manner, we find the desired element of $H_T^*(X)$.

The calculation of the image of $H_T^* X \to H_T^*(X^T)$ in the above example is a special case of a general fact, which we will return to later.

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