EQUIVARIANT INTERSECTION THEORY
§5: EQUIVARIANT SCHUBERT CALCULUS

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1. Classical Schubert calculus on $Gr(r, \mathbb{C}^n)$

Set $X = Gr(r, \mathbb{C}^n)$. For a (complete) flag $F_\bullet = (0 \subset F_1 \subset \cdots \subset F_n = V = \mathbb{C}^n)$ (with $\dim F_i = i$), and $L \in X$, consider the *attitude* of L with respect to $F_\bullet$ – that is, the sequence of dimensions

$$0 = \dim(L \cap F_0) \leq \dim(L \cap F_1) \leq \dim(L \cap F_2) \leq \cdots \leq \dim(L \cap F_n) = r.$$

Note that these dimensions increase by at most 1 at each step. For example, with $r = 5$ and $n = 10$, a possible sequence is $0, 1, 1, 2, 2, 2, 3, 4, 4, 5$. (We omit $0 = \dim(L \cap F_0)$.)

Let $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ be the *jumping sequence*, defined by $i_a = \min\{i \mid \dim(L \cap F_i) = a\}$. (These are the positions where the dimension "jumps.") Let $I = \{i_1 < \cdots < i_r\} \subset [n] := \{1, \ldots, n\}$. For the above example, the corresponding I is $\{2, 4, 7, 8, 10\}$.

Define

$$X_I^o = X_I^o(F_\bullet) = \{L \in X \mid I \text{ is the jumping sequence of } L \text{with respect to } F_\bullet\}.$$

This subset $X_I^o(F_\bullet)$ is called a *Schubert cell*, terminology justified by the following:

Claim: X_I^o is a locally closed subvariety of X, isomorphic to $\mathbb{A}^N = \mathbb{C}^N$, with $N = (i_1 - 1) + (i_2 - 2) + \cdots + (i_r - r)$.

This can be seen from "reduced echelon form" of a matrix. The subspace L is spanned by the column vectors of a unique $n \times r$ matrix with a 1 in the i_ath row of the ath column (for $1 \leq a \leq r$, and with 0's below and right of these 1's. (Here we are taking F_i to be the span of the first i standard basis vectors of $\mathbb{C}^n$.)

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Here is an example:

$$\begin{pmatrix} * & * & * & * & * \\ 1 & 0 & 0 & 0 & 0 \\ 0 & * & * & * & * \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The number of $*$'s is $(i_1 - 1) + (i_2 - 2) + \cdots + (i_r - r)$, and the $*$'s can be arbitrary elements of $\mathbb{C}$ (or any other field). Therefore this is isomorphic to $\mathbb{A}^N$.

Remark 1.1. For this choice of $F_\bullet$, these Schubert cells are exactly the B-orbits in X, where B is the group of upper-triangular matrices in GL_n. Indeed, for $g \in B$, we have $g \cdot F_i = F_i$ for all i. Thus $g(L \cap F_i) = g(L) \cap g(F_i) = g(L) \cap F_i$, and the attitude is unchanged.

We define the *Schubert variety* $X_I = X_I(F_\bullet)$ to be the closure of the corresponding Schubert cell: $X_I(F_\bullet) = \overline{X_I^o(F_\bullet)}$.

Remark 1.2. Consider the point corresponding to the matrix where all $*$'s are zero (and with 1's in the (i_a, a) positions). An affine open neighborhood U_I of this point, isomorphic to $\mathbb{A}^{r(n-r)}$, is given by only requiring zeroes to the right and left of the 1's. That is, it is the set of all matrices such that the $r \times r$ submatrix on rows indexed by I is the identity matrix. In this neighborhood, the Schubert variety X_I is given by setting some coordinates equal to zero. That is, $X_I \cap U_I = X_I^o$ is a linear subspace of U_I. This fact will make it easy to calculate some intersection numbers.

Claim: $X_I(F_\bullet) = \{L \in X \mid \dim(L \cap F_{i_a}) \geq a \text{ for } 1 \leq a \leq r\}$.

In fact, $X_I(F_\bullet)$ is the union of all $X_J^o(F_\bullet)$ with $J = \{j_1 < \cdots < j_r \mid j_a \leq i_a \text{ for all } a\}$.

Exercise 1.3. Prove this.

Claim: The homology classes $[X_I(F_\bullet)]$, as I varies over all r-subsets of $[n]$, form a basis for H_*X. Those with $\sum(i_a - a) = k$ form a basis for $H_{2k}X$.

Proof. Let

$$X_k = \bigcup_{\sum(i_a - a) \leq k} X_I(F_\bullet) = \coprod_{\sum(i_a - a) \leq k} X_I^o(F_\bullet),$$

so there is a filtration by closed subsets $\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_{r(n-r)} = X$, with $X_k \smallsetminus X_{k-1} = \coprod_{\sum(i_a-a)=k} X_I^o = \coprod \mathbb{A}^k$. It follows that the closures X_I give a basis for $\overline{H}_{2k}X = H_{2k}X = H^{2r(n-r)-2k}X$. $\square$

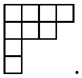
There are $\binom{n}{r}$ Schubert varieties, indexed by the r-subsets I of $[n]$. Another useful way of labelling them is as follows. Given $I = \{i_1 < \cdots < i_r\} \subset [n]$, define a corresponding partition $\lambda = \{\lambda_1 \geq \cdots \geq \lambda_r\}$, with $n - r \geq \lambda_1$, by $\lambda_a = n - r + a - i_a$. This sets up a bijection between r-subsets of $[n]$ and partitions with at most r parts, and largest part at most $n - r$. Set $\Omega_\lambda(F_\bullet) = X_I(F_\lambda)$, and let $\sigma_\lambda = [\Omega_\lambda] \in H^{2k}X = H_{2r(n-r)-2k}X$, where $k = |\lambda| = \sum \lambda_i$. These classes are the *Schubert classes.*

One way to remember the definition of λ is to draw the matrix corresponding to I, and see how far the 1's are above the lowest possible position. Another is to note that this says

$$\Omega_\lambda(F_\bullet) = \{L \in X \mid \dim(L \cap F_{n-r+i-\lambda_i}) \geq i \text{ for } 1 \leq i \leq r\},$$

and this is an empty condition when $\lambda_i = 0$. (Thus $\Omega_\emptyset = X$, where $\emptyset = (0, 0, \ldots)$ is the zero partition.)

It is often convenient to use a pictoral representation for partitions. The *Young diagram* of a partition λ is a collection of boxes, left-justified, with λ_i boxes in the ith row. For example, with $\lambda = (4, 3, 1, 1)$ the corresponding Young diagram is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We will often identify partitions with their Young diagrams. Note that the conditions on the parts of our partitions amount to requiring that the Young diagram fit inside a $r \times n - r$ rectangle.

Exercise 1.4. The Schubert variety $\Omega_\lambda(F_\bullet)$ is defined by those conditions coming from the "outer corners" of λ, i.e., those i with $\lambda_i > \lambda_{i+1}$. (This is true even scheme-theoretically: the minors corresponding to these conditions suffice to locally generate the ideal of Ω_λ.)

Claim: The classes $[X_I(F_\bullet)]$ and $[\Omega_\lambda(F_\bullet)]$ are independent of the choice of $F_\bullet$. (This is particular to the classical case – and false in equivariant cohomology! Recall the situation with $\mathbb{P}^{n-1}$.)

This follows from the fact that $X_I(F'_\bullet) = g \cdot X_I(F_\bullet)$ for some $g \in GL_n$, since GL_n acts transversally on X. (An action of a connected group on a space induces the trivial action on cohomology.)

The standard problems in Schubert calculus – which are solved for the Grassmannian, but remain open in many other cases – are the following:

(1) "Giambelli problem."* Express σ_λ in terms of generators of H^*X – say, the Chern classes $c_1(Q), \ldots, c_{n-r}(Q)$, or $\pm c_1(S), \ldots, \pm c_r(S)$.

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*So called, of course, because Schubert originally solved it.

(Here, as usual, S and Q are the tautological subbundle and quotient bundle, respectively.)

(2) "Structure constant problem." Writing $\sigma_\lambda \cdot \sigma_\mu = \sum_\nu c_{\lambda\mu}^\nu \sigma_\nu$, give a formula for the coefficients $c_{\lambda\mu}^\nu$.

Example 1.5. Consider $\sigma_k = \sigma_{(k,0,\ldots,0)}$, and

$$\Omega_k(F_\bullet) = \{L \mid \dim(L \cap F_{n-r+1-k}) \geq 1\}.$$

Then $\sigma_k = c_k(Q)$, where Q is the rank-$(n-r)$ universal quotient bundle on X. Indeed, we have

$$\begin{tikzcd} & F_{n-r+1-k} = \mathbb{C}\cdot e_1 \oplus \cdots \oplus \mathbb{C}\cdot e_{n-r+1-k} \arrow[d, hook] \arrow[dr, two heads] & \\ S \arrow[r, hook] & \mathbb{C}^n_X \arrow[r, two heads] & Q, \end{tikzcd}$$

and corresponding sections $s_1, \ldots, s_{n-r+1-k}$ of Q. Notice that $\Omega_k(F_\bullet)$ is the locus where the composed map $F_{n-r+1-k} \to \mathbb{C}^n_X \to Q$ is not injective – i.e., where $s_1, \ldots, s_{n-r+1-k}$ are not linearly independent.

For $k = 1$, this is the zero locus of the section $s_1 \wedge \cdots \wedge s_{n-r}$ of $\Lambda^{n-r}Q$, which represents $c_1(\Lambda^{top}Q) = c_1(Q)$. (To make this precise, one must check that this locus is reduced, i.e., $s_1 \wedge \cdots \wedge s_{n-r}$ cuts out Ω_1 without multilpicity.)

For $k = n - r$, this is just the zero locus of s_1, representing $c_{top}(Q)$.

Example 1.6. Similarly, consider $\sigma_{(1^k)}$, and $\Omega_{(1^k)}(F_\bullet) = \{L \mid \dim(L \cap F_{n-r+k-1}) \geq k\}$. (This follows from Exercise 1.4.) This is the locus where $\mathrm{rk}(F_{n-r+k-1} \to Q) \leq n - r - 1$. Equivalently, considering the diagram

$$\begin{tikzcd} & F_{n-r+k-1} \arrow[d, hook] \arrow[dr, two heads] & \\ S \arrow[r, hook] \arrow[dr] & \mathbb{C}^n_X \arrow[r, two heads] \arrow[d] & Q \\ & \mathbb{C}^n_X / F_{n-r+k-1} =: E, & \end{tikzcd}$$

we see that $\Omega_{(1^k)}$ is the locus where $\mathrm{rk}(S \to E) \leq r - k$, or the locus where $\mathrm{rk}(E^\vee \to S^\vee) \leq r - k$. Thus, as in the previous example, we have $\sigma_{(1^k)} = c_k(S^\vee)$.

Example 1.7. The "Pieri rule" gives the structure constants for multiplication with a special Schubert class:

$$\sigma_k \cdot \sigma_\lambda = \sum \sigma_\mu, \tag{1}$$

where the sum is over all partitions μ obtained from λ by adding a *horizontal strip* of size k. A horizontal strip is a collection of boxes with no two in the

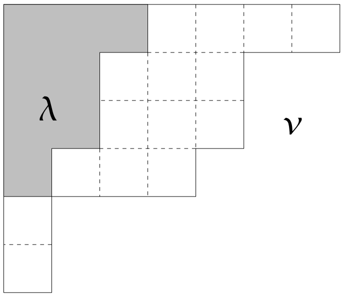
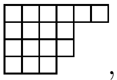


FIGURE 1. The skew shape ν/λ

same column. For example, with $k = 5$ and λ given by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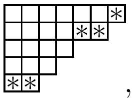
μ could be



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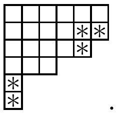
but not



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The "dual Pieri rule" says $\sigma_{(1^k)} \cdot \sigma_\lambda = \sum \sigma_\mu$, with the sum over all μ obtained by adding a vertical strip of size k, which is defined similarly: no two boxes should be added to the same row.

The numbers $c_{\lambda\mu}^{\nu}$, called *Littlewood-Richardson coefficients*, are among the most interesting combinatorial numbers. From their description as structure constants, it is clear that $c_{\lambda\mu}^{\nu} = 0$ unless $|\lambda| + |\mu| = |\nu|$, and that $c_{\lambda\mu}^{\nu} = c_{\mu\lambda}^{\nu}$.[†] In fact, $c_{\lambda\mu}^{\nu} = 0$ unless $\lambda \subset \nu$ and $\mu \subset \nu$; this will be clear from the following combinatorial description. (Containment of partitions is defined as containment of the corresponding Young diagrams.)

If λ and ν are partitions, with $\lambda \subset \nu$, then the *skew shape* ν/λ is defined to be the complement of the diagram of λ inside the diagram of ν. (In Figure 1, ν/λ is the unshaded region.) The number $c_{\lambda\mu}^{\nu}$ is equal to the number of ways to fill the boxes of ν/λ with μ_1 1's, μ_2 2's, μ_3 3's, etc., such that

1. The entries are weakly increasing across rows.
2. The entries are strictly increasing down columns.

[†] Actually, there is a full S_3 symmetry: If we write $c_{\lambda\mu\nu} = \int \sigma_\lambda \cdot \sigma_\mu \cdot \sigma_\nu$, then Poincaré duality shows that $c_{\lambda\mu}^{\nu} = c_{\lambda\mu\nu^\vee}$, and these latter numbers are clearly symmetric in the three partitions.

(3) The result of reading the rows from right to left, starting with the top row, is a "Yamanouchi word" (also called a "reverse lattice word"): If this word is $w_1\, w_2\, \ldots w_p$, then for each $q \leq p$,

$$\#(1\text{'s in } \{w_1, \ldots, w_q\}) \geq \#(2\text{'s in } \{w_1, \ldots, w_q\}) \geq \cdots .$$

A filling of the shape ν/λ which satisfies conditions (1) and (2) is called a *semistandard Young tableau.* (Note that the symmetry $c^{\nu}_{\lambda\mu} = c^{\nu}_{\mu\lambda}$, obvious from the geometric interpretation, is a nontrivial combinatorial identity!)

There are many "equivalent" combinatorial formulas for the Littlewood-Richardson coefficients, including many which have appeared in recent years. The first formulas came from representation theory, where the Littlewood-Richardson numbers are multiplicities arising in tensor products of irreducible representations of GL_n. The formula given above is the original one given by Littlewood and Richardson; the original proofs were quite flawed, though, and it was not until work of Lascoux and Schützenberger that this formula was rigorously established.

The solution to the Giambelli problem is the following "Giambelli formula": For $\lambda = (\lambda_1, \ldots, \lambda_r)$, we have

$$\sigma_\lambda = |\sigma_{\lambda_i+j-i}| = \begin{vmatrix} \sigma_{\lambda_1} & \sigma_{\lambda_1+1} & \cdots & \sigma_{\lambda_1+r} \\ \sigma_{\lambda_2-1} & \sigma_{\lambda_2} & & \\ \vdots & & \ddots & \\ & & & \sigma_{\lambda_r} \end{vmatrix} . \tag{2}$$

Here we use the convention that $\sigma_0 = 1$ and $\sigma_i = 0$ if $i < 0$ or $i > n - r$. (Recall that $\sigma_i = c_i(Q)$.)

Also, suppose $\mu = \lambda'$ is the *transpose* or *conjugate* partition of λ, obtained by reflecting the diagram of λ along the main diagonal. (So $\mu_j = \#\{i \,|\, \lambda_i \leq j\}$.) Then we have

$$\sigma_\lambda = |\tau_{\mu_i+j-i}|, \tag{3}$$

where $\tau_k = \sigma_{(1^k)} = c_k(S^\vee)$.

It is not easy to see a combinatorial formula for the structure constants $c^{\nu}_{\lambda\mu}$ directly from the geometry; however, Vakil has recently given such a formula. See [Vakil].

The usual approach to establishing the above combinatorial formula is the following. One proves the formula (called the "Littlewood-Richardson rule") for multiplication of certain symmetric functions, and then shows there is a homomorphism from the ring of symmetric functions to H^*X which takes these to the Schubert classes. We refer to [Macdonald 1995], [Stanley], or [Fulton 1997] for details, but state the basic results here.

More specifically, let $\Lambda_m = \mathbb{Z}[x_1, \ldots, x_m]^{S_m}$ be the ring of symmetric polynomials in m variables. (Letting $m \to \infty$, one gets the ring Λ of symmetric functions; however, as with the sequence of approximation spaces $E_m \to E$, the resulting ring is not just a limit of the approximation rings. In any case, we shall only need polynomials in finitely many variables.) This ring has a $\mathbb{Z}$-basis of *Schur polynomials*, which are indexed by partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$, with $m \geq \lambda_1$.

For example, $s_{(1)} = x_1 + x_2 + \cdots + x_m$, $s_{(1^k)} = e_k$, and $s_{(k)} = h_k$, where e_k and h_k are the elementary and complete homogeneous symmetric polynomials, respectively.

The are generating functions for the elementary and complete homogeneous symmetric polynomials are

$$E(t) = \sum_{k=0}^{m} e_k(x)t^k = \prod_{i=1}^{m}(1 + x_i t); \tag{4}$$

$$H(t) = \sum_{k\geq 0} h_k(x)t^k = \prod_{i=1}^{m}(1 - x_i t)^{-1}. \tag{5}$$

The Schur polynomials s_λ can be defined in several equivalent ways, among them the following:

1. $s_\lambda = |h_{\lambda_1+j-i}|$.
2. $s_\lambda = |e_{\lambda'_i+j-i}|$.
3. $s_\lambda = \sum_T x^T$, where the sum is over all semistandard Young tableaux on the shape λ, and $x^T := \prod x_i^{\#\{i\in T\}}$.

The Schur polynomials satisfy a Pieri rule:

$$s_k \cdot s_\lambda = \sum s_\mu, \tag{6}$$

with the sum over all μ obtained by adding a horizontal k-strip to λ, as before. More generally, their multiplication is given by the Littlewood-Richardson coefficients:

$$s_\lambda \cdot s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu. \tag{7}$$

Remark 1.8. In fact, $s_\lambda(x_1, \ldots, x_m)$ is the trace of the diagonal matrix with entries $x_1, \ldots, x_m$ in GL_m acting on an irreducible representaton V_λ. (One sees this by using definition (3) and the construction of V_λ via Young symmetrizers.) The numbers $c^\nu_{\lambda\mu}$ are therefore also determined by

$$V_\lambda \otimes V_\mu = \bigoplus_\nu V_\nu^{\oplus c^\nu_{\lambda\mu}}.$$

(This was the original context of the "Littlewood-Richarson rule.")

Claim: There is a homomorphism from Λ_m to $H^*Gr(r,n)$ (for $m \geq n-r$) mapping

$$s_\lambda \quad \mapsto \quad \begin{cases} \sigma_\lambda & \text{if } \lambda \subset ((n-r)^r)\,; \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, there exists such a map as $\mathbb{Z}$-modules: both sides are bases. The problem is to show this map respects multiplication. Using the "Giambelli" formulas for s_λ and σ_λ, it is enough to check that $s_k \cdot s_\lambda \mapsto \sigma_k \cdot \sigma_\lambda$ is well-defined. But the "Pieri" rules guarantee this, so the problem is reduced to proving the Pieri rule.

We refer to [Macdonald 1995] or [Stanley] for proofs of the Pieri rule for Schur polynomials, but outline the proof for Schubert classes here. (See also [Fulton 1997, §9.4].) First, we need the following strong form of Poincaré duality:

Lemma 1.9. *For $\sigma_\lambda, \sigma_\mu \in H^*Gr(r,n)$, we have*

$$\langle \sigma_\lambda, \sigma_\mu \rangle = \begin{cases} 1 & \text{if } \mu = \lambda^\vee; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $F_\bullet$ be the standard flag in $\mathbb{C}^n$, with $F_i = \langle e_1, \dots, e_i \rangle$, and let $F_\bullet^{opp}$ be the opposite flag, with $F_i^{opp} = \langle e_n, \dots, e_{n+1-i} \rangle$. Use $\sigma_\lambda = [\Omega_\lambda(F_\bullet)]$, and $\sigma_\mu = [\Omega_\mu(F_\bullet^{opp})]$. Now we establish the following claim:

Claim: Let $(\mu)^{180^o}$ denote the 180^o-rotation of μ inside the $r \times (n-r)$ box. If λ and $(\mu)^{180^o}$ overlap, then $\Omega_\lambda(F_\bullet) \cap \Omega_\mu(F_\bullet^{opp}) = \emptyset$.

The condition that λ and $(\mu)^{180^o}$ overlap is exactly that $\lambda_i + \mu_{r+1-i} > n - r$ for some i. Suppose $L \in \Omega_\lambda(F_\bullet) \cap \Omega\mu(F_\bullet^{opp})$. Then $\dim(L \cap F_{n-r+i-\lambda_i}) \geq i$ and $\dim(L \cap F^{opp}_{n-r+(r+1-i)-\mu_{r+1-i}}) \geq r+1-i$. Since L has dimension r, these two subspaces ($L \cap F_{n-r+i-\lambda_i}$ and $L \cap F^{opp}_{n-r+(r+1-i)-\mu_{r+1-i}}$) must meet in a nontrivial subspace. Therefore

$$\begin{aligned} n &> (n-r+i-\lambda_i) + (n-r+r+1-i-\mu_{r+1-i}) \\ &= 2n-r+1-\lambda_i-\mu_{r+1-i}, \end{aligned}$$

and hence $\lambda_i + \mu_{r+1-i} < n-r+1$. The claim follows.

When $\mu = \lambda^\vee$, the intersection is one point, namely the point corresponding to $L = \langle e_{i_1}, \dots, e_{i_r} \rangle$, with $i_j = n-r+j-\lambda_j$. Indeed, $\Omega_\lambda(F_\bullet) = \bigcup_{\nu \supseteq \lambda} \Omega^o_\nu$, so by the above claim, we see that

$$\Omega_\lambda(F_\bullet) \cap \Omega_{\lambda^\vee}(F_\bullet^{opp}) = \Omega^o_\lambda(F_\bullet) \cap \Omega^o_{\lambda^\vee}(F_\bullet^{opp}).$$

In the affine neighborhood containing these cells, it is easy to see that this is the (transversal) intersection of two complementary-dimensional coordinate subspaces. (More explicitly, the affine neighborhood is the set of matrices whose submatrix on rows $\{i_1, \dots, i_r\}$ is the identity matrix; the cell $\Omega^o_\lambda(F_\bullet)$ is obtained by setting all $*$'s below 1's equal to 0, and the cell $\Omega^o_\lambda(F_\bullet^{opp})$ is obtained similarly, by setting $*$'s above 1's equal to 0.) Thus $\langle \sigma_\lambda, \sigma_{\lambda^\vee} \rangle = 1$.

Finally, when $|\lambda| + |\mu| < r(n-r)$, we have $\langle \sigma_\lambda, \sigma_\mu \rangle = 0$ for dimension reasons. This, together with the above observations, proves the lemma. □

Remark 1.10. Note that in this case (the classical situation), dimension considerations also imply $\langle \sigma_\lambda, \sigma_\mu \rangle = 0$ for $|\lambda| + |\mu| > r(n-r)$ – because $H^*(pt) = \mathbb{Z}$ is zero except in degree 0. Since this is no longer true in the equivariant setting, we will need the stronger statement appearing in the above proof: the corresponding varieties are actually disjoint for $|\lambda| + |\mu| \geq r(n-r)$, unless $\mu = \lambda^\vee$.

To prove the Pieri formula,

$$\sigma_k \cdot \sigma_\lambda = \sum \sigma_\mu,$$

summing over μ obtained by adding a horizontal k-strip to λ, we compute $\langle \sigma_k \cdot \sigma_\lambda, \sigma_{\mu^\vee} \rangle = p_*(\sigma_k \cdot \sigma_\lambda \cdot \sigma_{\mu^\vee})$, a triple intersection number. (In general, these are much harder to compute than double intersections. The issue is in moving three subvarieties to general position with respect to one another.) Do this by intersecting $\Omega_\lambda(F_\bullet)$ with $\Omega_{\mu^\vee}(F_\bullet^{opp})$, for any flag $F_\bullet$, and $\Omega_k(G_\bullet)$, where $G_\bullet$ is "general" with respect to $F_\bullet$ and $F_\bullet^{opp}$. In this case, one can describe the intersection $\Omega_\lambda(F_\bullet) \cap \Omega_{\mu^\vee}(F_\bullet^{opp})$ explicitly. For more details, see [Fulton 1997, pp. 150-152].

2. Equivariant Schubert calculus on Grassmannians

As usual, the equivariant story is essentially a globalization of the classical version. We will see this in several examples.

Our first goal is to study H_T^*X, where $X = Gr(r, \mathbb{C}^n)$, and $T = (\mathbb{C}^*)^n$, acting as usual. Let S and Q be the universal bundles on X (of ranks r and $n-r$, respectively; these are T-equivariant vector bundles. Thus we have Chern classes $c_i^T(S)$ and $c_i^T(Q)$ in H_T^*X. Recall that $H_T^*X = H_B^*X$, where B is the group of upper-triangular matrices; we will often look at B-invariant subvarieties.

Let $F_\bullet$ be the standard flag, with $F_i = \langle e_1, \ldots, e_i \rangle$ spanned by the first i standard basis vectors. Then the Schubert varieties $\Omega_\lambda(F_\bullet)$ are B-invariant subvarieties of codimension $|\lambda|$, and so determine classes in $H_B^{2|\lambda|}X = H_T^{2|\lambda|}X$. Write $[\Omega_\lambda(F_\bullet)]^B = [\Omega_\lambda(F_\bullet)]^T = \sigma_\lambda^{eq} = \sigma_\lambda$. (We will usually suppress the superscript, since the meaning will be clear.) These classes restrict to $\sigma_\lambda^{cl} \in H^*X$, so by the usual Leray-Hirsch argument, they give a basis for H_T^*X over $\Lambda = \mathbb{Z}[t_1, \ldots, t_n]$.

As we saw in §4 for the case of projective space, the flags which may be used to define (invariant) Schubert varieties depend on whether one considers B or T. The "standard" Schubert varieties $\Omega_\lambda(F_\bullet)$ are exactly the B-orbit closures, so these are the only B-invariant subvarieties. For T, let $F_\bullet^w$ be the flag with $F_i^w = \langle e_{w(1)}, e_{w(2)}, \ldots, e_{w(i)} \rangle$, for any $w \in S_n$. (These are exactly the T-invariant flags.) Then $\Omega_\lambda(F_\bullet^w)$ is T-invariant, so there is a corresponding class $\sigma_\lambda^w = [\Omega_\lambda(F_\bullet^w)]^T$ in $H_T^{2|\lambda|}X$. Note that these depend on the choice of w! For any fixed w, they form a basis.

In particular, the opposite flag $F_\bullet^{opp}$ is just $F_\bullet^{w_0}$, where $w_0(i) = n+1-i$. We get the classes $\sigma_\lambda^{opp} = [\Omega_\lambda(F_\bullet^{opp})]^T$. (These are not B-invariant, but they are B_--invariant, where B_- is the group of lower-triangular matrices.)

As in the classical case, the opposite Schubert varieties are Poincaré dual to the standard ones:

Lemma 2.1.

$$\langle \sigma_\lambda, \sigma_\mu^{opp} \rangle \quad = \quad \begin{cases} 1 & \textit{if } \mu = \lambda^\vee; \\ 0 & \textit{otherwise.} \end{cases}$$

(Note that this statement has more content than classical Poincaré duality, since degree considerations alone do not guarantee $\langle \sigma_\lambda, \sigma_\mu^{opp} \rangle = 0$ when $|\lambda| + |\mu| > r(n-r)$.)

Proof. For $|\lambda| + |\mu| < r(n-r)$, $p_*(\sigma_\lambda \cdot \sigma_\mu) \in \Lambda$ has negative degree, and is therefore 0.

For $|\lambda| + |\mu| \geq r(n-r)$, we have $\Omega_\lambda(F_\bullet) \cap \Omega_\mu(F_\bullet^{opp}) = \emptyset$ unless $\lambda = \mu^\vee$, in which case the intersection is transversal in the T-invariant point $L = \langle e_{i_1}, \ldots, e_{i_r} \rangle$. (We saw this in the proof of Lemma 1.9.) Therefore $\sigma_\lambda \cdot \sigma_\mu^{opp} = 0$ if $|\lambda| + |\mu| \geq r(n-r)$ and $\mu \neq \lambda^\vee$, and $\langle \sigma_\lambda, \sigma_{\lambda^\vee}^{opp} \rangle = p_*([pt]^T) = 1$. □

Remark 2.2. We can generalize this Poincaré duality statement as follows. Consider any vector bundle E on a nonsingular variety Z, with $E = L_1 \oplus \cdots \oplus L_n$ a sum of line bundles. On $X = \mathbf{Gr}(r, E) \xrightarrow{p} Z$, we have the tautological sequence

$$0 \to S \to E_X \to Q \to 0,$$

with S of rank r, and Q of rank $n-r$. Writing $E_i = L_1 \oplus \cdots \oplus L_i$, and $E_i^{opp} = L_n \oplus \cdots \oplus L_{n+1-i}$, there are degeneracy loci on X:

$$\Omega_\lambda(E_\bullet) = \{\mathrm{rk}(E_{i_a} \to Q) \leq n - r - \lambda_a\}$$

and

$$\Omega_\lambda(E_\bullet^{opp}) = \{\mathrm{rk}(E_{i_a}^{opp} \to Q) \leq n - r - \lambda_a,$$

for where $1 \leq a \leq r$, and $i_a = n - r + a - \lambda_a$. These define classes σ_λ and σ_λ^{opp} in H^*X, and we have

$$p_*(\sigma_\lambda \cdot \sigma_\mu^{opp}) \quad = \quad \begin{cases} 1 & \text{if } \mu = \lambda^\vee; \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

The proof of Lemma 1.9 works to establish the case $\mu \neq \lambda^\vee$; indeed, the intersection is empty.

When $\mu = \lambda^\vee$, the intersection of these loci in X is the image of the section $s : Z \to X$ corresponding to the vector bundle $L_{i_1} \oplus \cdots \oplus L_{i_r} \to Z$. Therefore $p_*([\Omega_\lambda(E_\bullet) \cap \Omega_\mu(E_\bullet^{opp})]) = p_*(s_*(1)) = 1$.

The equivariant case is a consequence of this more general relative statement.

We now address the Giambelli problem: we seek formulas for σ_λ^{eq}.

The geometry is given by the following setup. Taking approximation spaces, we have diagrams

$$\begin{tikzcd}
M^o_{m,n} \times X \arrow[r] \arrow[d] & M^o_{m,n} \arrow[d] \\
M^o_{m,n} \times_B X \arrow[r] \arrow[d, phantom, "\|"] & M^o_{m,n}/B \arrow[d, phantom, "\|"] \\
\mathbf{Gr}(r,U) \arrow[r] & Fl(1,\ldots,n;\mathbb{C}^m).
\end{tikzcd}$$

Here U is the rank-n bundle in the tautological sequence $U_1 \subset U_2 \subset \cdots \subset U_n = U \subset \mathbb{C}^m_{Fl}$ on Fl. The horizontal maps are locally trivial fiber bundles, with fiber X; the map $M^o_{m,n} \times X \to \mathbf{Gr}(r,U)$ is given by $(A,L) \mapsto ([A], A(L))$, where $[A]$ is the flag determined by A. On $\mathbf{Gr}(r,U)$, there is the universal sequence $0 \to \tilde{S} \to U \to \tilde{Q} \to 0$; the Chern classes $c_i(\tilde{S})$ and $c_i(\tilde{Q})$ are the equivariant Chern classes $c_i^T(S)$ and $c_i^T(Q)$.

We want to understand $M^o_{m,n} \times_B \Omega_\lambda(F_\bullet)$, since the classes of these varieties in $H^*(\mathbf{Gr}(r,U))$ are the classes $\sigma_\lambda \in H_T^{2|\lambda|}X$.

Exercise 2.3. The Schubert variety $M^o_{m,n} \times_B \Omega_\lambda(F_\bullet)$ is the locus where $\dim(S \cap E_{n-r+i-\lambda_i}) \geq i$ for $1 \leq i \leq r$. Equivalently, it is the locus where the map $E_{n-r+i-\lambda_i} \to Q$ has rank at most $n - r - \lambda_i$, for $1 \leq i \leq r$.

The class of such a degeneracy locus is given by the *Kempf-Laksov formula* [Kempf-Laksov]. For vector bundles E and F, let $c_i(F - E)$ denote the coefficient of t^i in the formal quotient

$$\frac{1 + c_1(F)t + c_2(F)t^2 + \cdots}{1 + c_1(E)t + c_2(E)t^2 + \cdots},$$

i.e., in

$$1 + (c_1(F) - c_1(E))t + (c_2(F) + c_1(E)^2 - c_2(E) - c_1(E)c_1(F))t^2 + \cdots .$$

The formula is the following:

$$\sigma_\lambda = \det(c_{\lambda_a+b-a}(Q - E_{n-r+a-\lambda_a}))_{1\leq a,b\leq r}. \tag{9}$$

If the bundles E_i are trivial (as is the case for the classical Grassmannian), then this is the usual Giambelli formula: $\sigma_\lambda = \det(c_{\lambda_a+b-a})$. This is just a Schur polynomial, when the c_i's are elementary or complete homogeneous symmetric functions. The Kempf-Laksov formula therefore gives polynomials which generalize Schur polynomials – the variables in the defining determinant change from row to row. If we write

$$c(E_i) = \prod_{j=1}^{i}(1 + t_j),$$

the corresponding polynomials are called "multi-Schur polynomials" or "factorial Schur polynomials." (Perhaps a more natural name would have been "Kempf-Laksov polynomials," though.)

For proofs, see [Kempf-Laksov] or [Fulton 1984, §14.3].

3. Schubert polynomials and classical Schubert calculus on $Fl(n)$

Now let $X = Fl(n)$. If $S_1 \subset \cdots \subset S_n = \mathbb{C}^n_X$ is the tautological sequence of subbundles on X, and $\mathbb{C}^n_X = Q_n \to \cdots \to Q_1$ is the tautological sequence of quotient bundles (so $Q_i = \mathbb{C}^n/S_{n-i}$), let $x_i = c_1(\ker(Q_i \to Q_{i-1}) = c_1(S_{n-i+1}/S_{n-i})$; recall that $H^*X = \mathbb{Z}[x_1, \ldots, x_n]/(e_1, \ldots, e_n)$. (We have changed notation from §3.) We now develop a more geometric description of this cohomology ring, along the lines of our treatment of $H^*Gr(r,n)$.

For each $w \in S_n$, define $r_w(q,p) = \#\{j \leq q \,|\, w(j) \leq p\}$. Define a 01-matrix A_w so that A_w has 1's in positions $(w(i), i)$ and 0's elsewhere. Equivalently, $w \mapsto A_w$ is a homomorphism $S_n \hookrightarrow GL_n$ – that is, $A_w(e_i) = e_{w(i)}$. Thus $r_w(q,p)$ is the rank of the upper-left $p \times q$ submatrix of A_w.

Caution: Beware that these conventions may be changed later in the notes. In particular, the main issues concern transposing these matrices, and the resulting confusion between $r_w(q,p)$ and $r_w(p,q)$, w and w^{-1}, $w_0 w$ and $w w_0$, etc. The current version should be consistent, but double-checking is in progress. Look for `caveats` in the text.

Now fix a flag $E_\bullet$ – we will take the standard flag $E_i = \langle e_1, \ldots, e_i \rangle$. With respect to this flag, the *Schubert cell* corresponding to w is

$$X^o_w = X^o_w(E_\bullet) = \{W_\bullet \in X \mid \dim(W_q \cap E_p) = r_w(q,p)\},$$

and one sees that X^o_w is the set of flags $W_\bullet$ such that W_i is spanned by the first i columns of a matrix with 1's in positions $(w(i), i)$ and 0's below and to the right of these 1's. In fact, this sets up a one-to-one correspondence between X^o_w and the set of such matrices: each flag in X^o_w has a unique such matrix representative. Moreover, the entries not required to be 1 or 0 are free to vary, and there are $\ell(w)$ of them, where $\ell(w) = \#\{i < j \,|\, w(i) > w(j)\}$ is the *length* of the permutation w. Thus $X^o_w \cong \mathbb{A}^{\ell(w)}$.

For example, writing the permutation $w = w(1)\; w(2) \;\cdots\; w(n)$ (in "one-line notation"), take $w = 2\;4\;3\;1$. The corresponding Schubert cell is the set of matrices of the form

$$\begin{pmatrix} * & * & * & 1 \\ 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

so $X^o_{2431} \cong \mathbb{A}^{\ell(w)} = \mathbb{A}^4$.

One way to verify these statements is to notice that the Schubert cells are exactly the B-orbits, where the group B of upper-triangular matrices acts by multiplication on the left on $X \cong GL_n(\mathbb{C})/B$. Indeed, requiring that $\dim(W_q \cap E_p) = r_w(q,p)$ is the same as asking that a matrix representative for $W_\bullet$ has upper-left $p \times q$ submatrix of rank $r_w(q,p)$. Now GL_n acts transitively on X (on the left), and B is the largest subgroup which preserves the rank of every upper-left $p \times q$ submatrix. If $p_w \in X$ is the flag $(\langle e_{w(1)}\rangle \subset \langle e_{w(1)}, e_{w(2)}\rangle \subset \cdots)$, then certainly $p_w \in X_w^o$; thus $X_w^o = B \cdot p_w$. Since p_w is represented by the matrix A_w, we can find matrix representatives for points in X_w^o by filling in arbitrary entries above the 1's in A_w, and rescaling columns by the right-action of B to get the form described above. For the dimension count, simply note that a free entry (‘$*$’) appears exactly when $w(i) > w(j)$, with $i < j$. `Here some trouble with the conventions shows up, as mentioned in the above ‘‘caution.’’ Should be fixed now, though!`

The *Schubert varieties* are $X_w = \overline{X_w^o}$. It is not hard to check that

$$X_w = \{W_\bullet \mid \dim(W_q \cap E_p) \geq r_w(q,p)\}; \tag{10}$$

in fact, X_w is the disjoint union of all Schubert cells X_v^o contained in the closure $\overline{X_w^o}$. Therefore (10) can be checked by finding those points p_v which lie in X_w. (We will see several alternate descriptions of this condition shortly.)

As for the Grassmannian – and for the same reasons – the classes $[X_w]$ form a basis for $H_{2\ell(w)}X$. We write $\Omega_w = \Omega_w(E_\bullet) = X_{w\cdot w_0}(E_\bullet)$, where $w_0 = n\ (n-1)\ \cdots\ 2\ 1$ is the “longest permutation,” and set

$$\sigma_w = \sigma_w^{cl} = [\Omega_w].$$

The classes σ_w form a basis for $H^{2\ell(w)}X$.

Exercise 3.1. For permutations $w, v \in S_n$, the following are equivalent:

(1) $p_w \in X_v$.
(2) $X_w \subseteq X_v$.
(3) $\Omega_v \subseteq \Omega_w$.
(4) $r_w(q,p) \leq r_v(q,p)$ for all q, p.
(5) For $1 \leq p \leq n$, $\{w(1),\dots,w(p)\} \leq \{v(1),\dots,v(p)\}$; that is, when these sets are sorted in increasing order, the first is termwise less than or equal to the second.
(6) For some (or equivalently, any) description of v as a reduced product of simple transpositions $v = s_{i_1} \cdot s_{i_2} \cdots s_{i_l}$ (where $l = \ell(v)$), w can be written as a subword – that is, remove some of the s_{i_j}'s.

Any of these conditions defines a partial order on S_n, called the *Bruhat order*; one writes $w \leq v$. This poset is graded by the length of permutations. (See [Fulton 1997, pp. 173-177].)

A major open problem here is the “structure constant problem”: writing

$$\sigma_u \cdot \sigma_v = \sum_{w \in S_n} c_{uv}^w \sigma_w, \tag{11}$$

the problem is to find a (positive) combinatorial formula for the coefficients c_{uv}^w.

On the other hand, the "Giambelli problem" – to write σ_w as a polynomial in the Chern roots $x_1, \dots, x_n$ – has been solved. Of course, there is no unique answer, but a "best" solution is given by the *Schubert polynomials* of Lascoux-Schützenberger.

The recipe for computing these is the following. For $w_0 \in S_n$, we have $\Omega_{w_0} = X_{id} = p_{id}$, the standard flag.

Exercise 3.2. Show that $x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1$ represents the class of a point in H^*X. [Hint/solution: Show $p_*(x_1^{n-1}x_2^{n-2}\cdots x_{n-1}^1) = 1$ in $H^0(pt) = \mathbb{Z}$, using the construction of X as a sequence of projective bundles.]

We will define the Schubert polynomial

$$\mathfrak{S}_{w_0}(X_1, \dots, X_n) = X_1^{n-1} X_2^{n-2} \cdots X_{n-1} \in \mathbb{Z}[X_1, \dots, X_n],$$

and work up from the class of a point to define the other Schubert polynomials $\mathfrak{S}_w$.[‡]

For an arbitrary w, write $w = w_0 \cdot s_{i_1} \cdot s_{i_2} \cdots s_{i_l}$, where s_i is the simple transposition interchanging i and $i+1$, and where $l = \ell(w_0) - \ell(w) = \binom{n}{2} - \ell(w)$. (The requirement on the length means that $s_{i_1} \cdot s_{i_2} \cdots s_{i_l}$ is a *reduced word* for $w_0 \cdot w$.) Define the *divided difference operators* ∂_i on the polynomial ring $\mathbb{Z}[X_1, \dots, X_n]$ by

$$\partial_i(P) = \frac{P - s_i \cdot P}{X_i - X_{i+1}} = \frac{P(\dots, X_i, X_{i+1}, \dots) - P(\dots, X_{i+1}, X_i, \dots)}{X_i - X_{i+1}},$$

and define the Schubert polynomial $\mathfrak{S}_w(X)$ to be

$$\mathfrak{S}_w(X) = \partial_{i_l} \circ \partial_{i_{l-1}} \circ \cdots \circ \partial_{i_1}(X_1^{n-1} \cdots X_{n-1}). \tag{12}$$

Theorem 3.3. *The Schubert polynomial $\mathfrak{S}_w$ represents the class σ_w, i.e., the natural map $\mathbb{Z}[X_1, \dots, X_n] \to H^*X$ sends $\mathfrak{S}_w$ to σ_w.*

In fact, the Schubert polynomials multiply with the same structure constants *as polynomials.* That is,

$$\mathfrak{S}_u \cdot \mathfrak{S}_v = \sum_{w \in S^\infty} c_{uv}^w \mathfrak{S}_w, \tag{13}$$

as an identity in a polynomial ring in sufficiently many variables. Note that ff $u, v \in S_n$, w need not be in S_n for c_{uv}^w to be nonzero. In particular, when $\ell(u) + \ell(v) > \dim(Fl(n))$, $\sigma_u \cdot \sigma_v = 0$ – but $\mathfrak{S}_u \cdot \mathfrak{S}_v$ is certainly not zero. (In fact, this product of Schubert polynomials can be expanded as a sum of $\mathfrak{S}_w$ with $w \in S_{2n-1}$.) However, when $w \in S_n$, the coefficient c_{uv}^w in (13) is the same as that in (11).

Before discussing the proof of Theorem 3.3, we give an example.

[‡] This is very strange from the geometric point of view, where one usually works down from the classes of divisors! We will exploit the fact that X is a sequence of $\mathbb{P}^k$-bundles and we know how to push forward cohomology classes across these bundles.

Example 3.4. We compute the Schubert polynomials for $n = 3$. We have $\mathfrak{S}_{321} = x_1^2 x_2$, and applying the divided difference operators, we get the following:

$$\begin{tikzcd}
 & \mathfrak{S}_{321} = x_1^2 x_2 \arrow[dl, "\partial_1"'] \arrow[dr, "\partial_2"] & \\
\mathfrak{S}_{231} = x_1 x_2 \arrow[d, "\partial_2"'] & & \mathfrak{S}_{312} = x_1^2 \arrow[d, "\partial_1"'] \\
\mathfrak{S}_{213} = x_1 \arrow[dr, "\partial_1"'] & & \mathfrak{S}_{132} = x_1 + x_2 \arrow[dl, "\partial_2"] \\
 & \mathfrak{S}_{123} = 1. &
\end{tikzcd}$$

These are best understood by working out examples. In computations, the main points to note are that $w \cdot s_i$ swaps the entries in positions i and $i+1$ (so $2\ 3\ 1\ 4 \cdot s_2 = 2\ 1\ 3\ 4$), and if Q is symmetric in x_i and x_{i+1}, then $\partial_i(Q \cdot P) = Q \cdot \partial_i(P)$.

Exercise 3.5. Work out some (or all) of the Schubert polynomials $\mathfrak{S}_w$ for $w \in S_4$.

Remark 3.6. There are several formulas for writing $\mathfrak{S}_w$ as a linear combination of monomials $x_1^{i_1} \cdots x_n^{i_n}$, with $i_j \leq n - j$, including combinatorial formulas showing that the coefficients are nonnegative. One does not have to work from the definition to compute a given Schubert polynomial in S_9, say, which would be somewhat inconvenient!

The definition apparently depends on the choice of a word for w, but it is easily checked that $\mathfrak{S}_w$ is independent of $s_{i_1}, \ldots, s_{i_l}$. In fact,

$$\begin{aligned}
\partial_i \circ \partial_j &= \partial_j \circ \partial_i \text{ for } |i - j| > 1; \\
\partial_i \circ \partial_{i+1} \circ \partial_i &= \partial_{i+1} \circ \partial_i \circ \partial_{i+1}.
\end{aligned}$$

(We leave the verification as an exercise.)

This means that given $u = s_{i_1} \cdots s_{i_l}$, with $l = \ell(u)$, we can define $\partial_u := \partial_{i_1} \circ \cdots \circ \partial_{i_l}$ unambiguously. (Indeed, all reduced words for u are related by sequences of commutations and braid moves.) In this notation, $\mathfrak{S}_w = \partial_{w^{-1} w_0}(x_1^{n-1} \cdots x_{n-1})$.

To begin the proof of Theorem 3.3, we need a Poincaré duality lemma:

Lemma 3.7. *For $\sigma_u, \sigma_v \in H^* X$, we have*

$$\langle \sigma_u, \sigma_v \rangle = \begin{cases} 1 & \text{if } v = w_0 \cdot u; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. (See [Fulton 1997, p. 159], but beware of differing notation.) First one shows that $\Omega_{w_0\,v\,w_0}(E_\bullet^{opp})$ meets $X_u(E_\bullet)$ only if $\ell(v) \leq \ell(u)$. If $\ell(v) = \ell(u)$, then they meet only for $v = u$, and in that case they meet transversally at the point $p_u = u \cdot E_\bullet$. (This is the flag corresponding to the basis $(e_{u(1)}, e_{u(2)}, \ldots, e_{u(n)})$.) In fact, one can see this directly from the matrix representatives in the relevant Schubert cells, as in the Grassmannian case. $\square$

`Here again there is some convention trouble. Things should be consistent as stated, though.`

Example 3.8. In $H^*(Fl(3)) = \mathbb{Z}[x_1, x_2, x_3]/(e_1, e_2, e_3)$, we have

$$\begin{aligned} \langle \sigma_{231}, \sigma_{213} \rangle &= \langle x_1 x_2, x_1 \rangle \\ &= p_*(x_1^2 x_2) \\ &= 1, \end{aligned}$$

and

$$\begin{aligned} \langle \sigma_{312}, \sigma_{213} \rangle &= \langle x_1^2, x_1 \rangle \\ &= p_*(x_1^3) \\ &= 0. \end{aligned}$$

(In fact, $x_1^3 = 0$ in $H^*(Fl(3))$.)

4. Double Schubert polynomials and equivariant Schubert calculus on $Fl(n)$

We now turn to the problem of describing $H_T^* X = H_B^* X$, for $X = Fl(n)$. We have the usual setup:

$$\begin{tikzcd}
M^o_{m,n} \times X \arrow[r] \arrow[d] & M^o_{m,n} \arrow[d] \\
M^o_{m,n} \times_B X \arrow[r] \arrow[d, equal] & M^o_{m,n}/B \arrow[d, equal] \\
\mathbf{Fl}(E) \arrow[r] & Fl(1, \ldots, n; \mathbb{C}^m).
\end{tikzcd}$$

The vertical arrows are B-bundles, and the horizontal arrows are $Fl(n)$-bundles; here E is the bundle in the tautological sequence on $Fl(1, \ldots, n; \mathbb{C}^m)$, $E_1 \subset \cdots \subset E_n = E \subset \mathbb{C}^m_{Fl}$.

(In fact, for most of our discussion, the base $Fl(1, \ldots, n; \mathbb{C}^m)$ could be replaced with any variety – the only requirement is that it have a flag of vector bundles.)

On $\mathbf{Fl}(E)$, we have the tautological bundles $S_1 \subset \cdots \subset S_n = E$. Let $Q_i = E/S_{n-i}$, so we have a sequence (on $\mathbf{Fl}(E)$)

$$E_1 \hookrightarrow E_2 \hookrightarrow \cdots \hookrightarrow E_{n-1} \hookrightarrow E \to Q_{n-1} \to \cdots \to Q_1.$$

Set $t_i = c_1(E_i/E_{i-1})$, and $x_i = c_1(\ker(Q_i \to Q_{i-1})) = c_1(S_{n-i+1}/S_{n-i})$. (When using subbundles instead of quotients, number the x's top-down.)

We have classes $\sigma_w = \sigma_w^{eq} = [\Omega_w]^B = [\Omega_w]^T$ in H_T^*X. Specifically, $[\Omega_w]^B$ is the class of $M_{m,n}^o \times_B \Omega_w$ in $H^{2\ell(w)}(\mathbf{Fl}(E))$. This is the locus

$$\{x \in \mathbf{Fl}(E) \mid \dim(S_q(x) \cap E_p(x)) \geq r_{w\cdot w_0}(q,p) \text{ for all } p,q\},$$

or equivalently,

$$\{x \in \mathbf{Fl}(E) \mid \mathrm{rk}(E_p(x) \to Q_q(x)) \leq r_w(q,p) \text{ for all } p,q\}.$$

Fiberwise, these are just the usual Schubert varieties, with $E_\bullet$ acting as the reference flag, and $Q_\bullet$ as the "moving flag." That is, the fiber of $M_{m,n}^o \times_B \Omega_w$ over a point $x \in Fl$ is just $\Omega_w(E_\bullet(x))$.

These equivariant classes are represented by the *double Schubert polyomials*, which we now define.

Definition. Given $w \in S_n$, write $w = w_0 \cdot s_{i_1} \cdots s_{i_l}$, where $l = \ell(w_0) - \ell(w)$. The double Schubert polynomial $\mathfrak{S}(x;t)$ is defined by

$$\mathfrak{S}(x;t) = \partial_{i_l} \circ \cdots \circ \partial_{i_1} \left(\prod_{i+j<n} (x_i - t_j) \right), \tag{14}$$

where $\partial_i = \partial_i^x$ is the usual divided difference operator, acting only on the x variables. (That is,

$$\partial_i(P(x;t)) = \frac{P(x;t) - P(\ldots, x_{i+1}, x_i, \ldots; t)}{x_i - x_{i+1}}.)$$

Notice that these specialize to the ordinary Schubert polynomials defined above: $\mathfrak{S}_w(x;0) = \mathfrak{S}_w(x)$.

Exercise 4.1. Write out the double Schubert polynomials for $w \in S_3$. (Notice that the specializations $\mathfrak{S}_w(0;t)$ are $\pm\mathfrak{S}_u(t)$, for appropriate u!)

Eventually, we will prove the following fact, generalizing Theorem 3.3

Theorem 4.2. *The double Schubert polynomial $\mathfrak{S}_w(x;t)$ represents the equivariant class $\sigma_w = [\Omega_w]^T$. That is, the natural map $\mathbb{Z}[t_1, \ldots, t_n][x_1, \ldots, x_n] \to H_T^*X$ sends $\mathfrak{S}_w(x;t)$ to σ_w.*

For this, we will need "equivariant Poincaré duality," as in the case of $H_T^*(Gr(r,n))$. We refine our notation, and write $\sigma_w = \sigma_w(std)$ for the equivariant class taken with respect to the standard flag, and $\sigma_w(opp) = [\Omega_w(opp)]^T$ for the class with respect to the opposite flag. (Note that $\Omega_w(opp)$ is T-invariant, but not B-invariant.)

Lemma 4.3. *We have*

$$\langle \sigma_u(std), \sigma_v(opp) \rangle = \begin{cases} 1 & \text{if } v = w_0 \cdot u; \\ 0 & \text{otherwise.} \end{cases}$$

The proof is analogous to that of Lemma 2.1 (the Grassmannian case). `Check to make sure this is okay with our conventions!`

Exercise 4.4. Expand all products $\mathfrak{S}_u(x,t) \cdot \mathfrak{S}_v(x,t)$ as polynomials in $\mathfrak{S}_w(x,t)$, for $u, v \in S_3$. For example,

$$\begin{aligned}
\mathfrak{S}_{213} \cdot \mathfrak{S}_{213} &= (x_1 - t_1) \cdot (x_1 - t_1) \\
&= \mathfrak{S}_{312} + (t_2 - t_1)\mathfrak{S}_{213}; \\
\mathfrak{S}_{213} \cdot \mathfrak{S}_{132} &= \mathfrak{S}_{231} + \mathfrak{S}_{312}; \\
\mathfrak{S}_{213} \cdot \mathfrak{S}_{312} &= (t_3 - t_1)\mathfrak{S}_{312} + \mathfrak{S}_{4123}.
\end{aligned}$$

Remark 4.5. The ultimate goal in this setting would be to understand the coefficients $c_{uv}^w(t)$ for multiplication of double Schubert polynomials; setting $t = 0$ would recover the coefficients c_{uv}^w for classical Schubert calculus.

5. PROPERTIES OF SCHUBERT POLYNOMIALS

Some basic references for Schubert polynomials are [Lascoux-Schützenberger] and [Macdonald 1991]. We review some of them here.

The double Schubert polynomials can be expressed in terms of the ordinary polynomials as follows:

$$\mathfrak{S}_w(x,y) = \sum (-1)^{\ell(v)}\, \mathfrak{S}_u(x) \cdot \mathfrak{S}_v(y), \tag{15}$$

where the sum is over u, v such that $\ell(u) + \ell(v) = w$ and $v^{-1}u = w$.

By reindexing, it follows from (15) that

$$\mathfrak{S}_{w^{-1}}(x,y) = (-1)^{\ell(w)}\, \mathfrak{S}_w(y,x). \tag{16}$$

Thus one can obtain $\mathfrak{S}_w(x,y)$ via divided difference operators "∂_i^y" with respect to the y-variables instead of the usual ∂_i^x's – a symmetry which is not at all obvious from the definitions.

The Schubert polynomial $\mathfrak{S}_w(x,y)$ is symmetric in x_i and x_{i+1} if and only if $w(i) < w(i+1)$, or equivalently, $\partial_i \mathfrak{S}_w(x,y) = 0$. This is easily checked from the definition. Using (16), we also see that $\mathfrak{S}_w(x,y)$ is symmetric in y_i and y_{i+1} if and only if $w^{-1}(i) < w^{-1}(i+1)$.

Exercise 5.1. If $w = s_{i_l} \cdots s_{i_1} \cdot w_0$, with $\ell(w) = \binom{n}{2} - l$, then

$$\mathfrak{S}_w(x,y) = (-1)^l\, \partial_{i_l}^y \circ \cdots \circ \partial_{i_1}^y \left(\prod_{i+j \le n} (x_i - y_j) \right).$$

We have $\mathfrak{S}_{id} = 1$, and for the simple transposition $s_k = (k\ k+1)$, we have $\mathfrak{S}_{s_k} = x_1 + \cdots x_k - y_1 - \cdots - y_k$.

There is *Monk's formula* for multiplication by a divisor class $\mathfrak{S}_{s_k}$:

$$\mathfrak{S}_{s_k}(x)\, \mathfrak{S}_w(x) = \sum \mathfrak{S}_v(x), \tag{17}$$

where the sum is over those v obtained from w as $v = w \cdot t$ for some transposition $t = (i\ j)$, with $i \le k < j$ and $\ell(v) = \ell(w) + 1$. That is, v is obtained

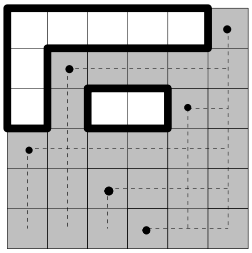

FIGURE 2. $D(w)$ for $w = 4\ 2\ 5\ 6\ 3\ 1$.

from w by interchanging entries in positions i and j, with $i \leq k < j$, such that $w(i) < w(j)$ and for all s between i and j, $w(s)$ is not between $w(i)$ and $w(j)$. For example,

$$\mathfrak{S}_{s_3} \cdot \mathfrak{S}_{416523} = \mathfrak{S}_{516423} + \mathfrak{S}_{456123} + \mathfrak{S}_{426513}\,(+\mathfrak{S}_{4175236}).$$

(Note that although the original polynomials are for $w \in S_6$, one needs a polynomial for S_7 to carry out the multiplication in $\mathbb{Z}[x]$. Of course, in $H^*(Fl(6))$, the last term would not appear.)

The Schubert polynomials $\mathfrak{S}_w(x)$ form a basis for $\mathbb{Z}[x_1, x_2, \ldots]$, with $w \in S_\infty := \bigcup_n S_n$. More precisely, those Schubert polynomials $\mathfrak{S}_w$ with $w(n+1) < w(n+2) < w(n+3) < \cdots$ form a basis for $\mathbb{Z}[x_1, \ldots, x_n]$, and when $w \in S_n$, $\mathfrak{S}_w$ is a positive linear combination of monomials $x_1^{i_1} \cdots x_n^{i_n}$, with $i_j \leq n - j$.

Remark 5.2. Schubert polynomials are very much a type A phenomenon. It is not possible to find such wonderful polynomial representatives for Schubert classes in flag varieties of other Lie types: see [Fomin-Kirillov 1996].

We now mention some combinatorial formulas for expressing Schubert polynomials as sums of monomials. One of the most appealing is "Kohnert's formula," described as follows. Given $w \in S_n$, define the *diagram* of w to be the configuration $D(w)$ of $\ell(w)$ boxes obtained by placing dots in positions $(w(i), i)$ of an $n \times n$ grid, and crossing out the boxes appearing below or to the right of these dots. For example, with $w = 4\ 2\ 5\ 6\ 3\ 1$, the diagram $D(w)$ is the unshaded region in Figure 2. There is a one-to-one correspondence between such diagrams and permutations.

Kohnert's recipe says to associate to each diagram D a monomial

$$x^D := \prod x_i^{\#\{\text{boxes in row } i \text{ of } D\}}.$$

To compute $\mathfrak{S}_w(x)$, form $D(w)$, and perform moves on the diagram of the following type: Take a rightmost box in any row, and move it up to the next empty space. The Schubert polynomial $\mathfrak{S}_w(x)$ is the sum over x^D for the diagrams obtained from $D(w)$ in this way:

$$\mathfrak{S}_w(x) = \sum_D x^D. \tag{18}$$

Unfortunately, a proof that this formula works remains at large!

Another explicit formula is given in [Billey-Jockusch-Stanley]. Let $l = \ell(w)$. Then

$$\mathfrak{S}_w(x) = \sum_{a_1,\dots,a_l} \sum_{i_1,\dots,i_l} x_{i_1} \cdots x_{i_l}, \tag{19}$$

where the sum is over $a_1, \dots, a_l$ such that $w = s_{a_1} \cdots s_{a_l}$, and $1 \le i_1 \le \cdots \le i_l \le n$ such that $i_j \le a_j$, and $i_j < i_{j+1}$ if $a_j < a_{j+1}$.

6. Schubert polynomials and Schubert classes

We now turn to the task of proving that Schubert polynomials represent the classes of Schubert varieties in $Fl(n)$ (Theorems 3.3 and 4.2).

One way to characterize the polynomials $\mathfrak{S}_w(x)$ is to say that they "stabilize," in two (compatible) senses. Algebraically, we can regard a permutation $w \in S_n$ as an element of S_{n+1}, S_{n+2}, etc., by considering $S_n \subset S_{n'}$ as the subgroup that fixes the letters $n+1, n+2, \dots, n'$ (for $n' \ge n$). Then $\mathfrak{S}_w(x)$ and $\mathfrak{S}_w(x,y)$ are independent of n. Indeed, start with $x_1^n x_2^{n-1} \cdots x_n$, which is $\mathfrak{S}_{w_0}$ for $w_0 = (n+1)\ n \cdots 1 \in S_{n+1}$. Then one computes

$$\partial_n \circ \cdots \circ \partial_2 \circ \partial_1 (x_1^n x_2^{n-1} \cdots x_n) = x_1^{n-1} x_2^{n-2} \cdots x_{n-1},$$

which is $\mathfrak{S}_{w_0}$ for $w_0 \in S_n$. (This is the same as $n\ (n-1)\ \cdots 1\ (n+1) \in S_{n+1}$.) The situation with double Schubert polynomials is similar.

Geometrically, the Schubert classes are also stable. There are natural inclusions

$$Fl(n) \subset Fl(n+1) \subset \cdots \subset Fl(n'),$$

via

$$S_1 \subset \cdots \subset S_n = \mathbb{C}^n \subset \mathbb{C}^{n+1} \subset \cdots \subset \mathbb{C}^{n'}.$$

The inclusion $\iota : Fl(n) \hookrightarrow Fl(n')$ is equivariant with respect to the inclusion $GL_n \hookrightarrow GL_{n'}$,

$$A \mapsto \begin{array}{|c|c|} \hline A & 0 \\ \hline 0 & I \\ \hline \end{array}\,.$$

The inclusion of tori $T = (\mathbb{C}^*)^n \hookrightarrow T' = (\mathbb{C}^*)^{n'}$ induces a map $\Lambda' \to \Lambda$ mapping $t_i \mapsto t_i$ if $i \le n$, and $t_i \mapsto 0$ if $i > n$. We obtain a map

$$\iota^* : H^*_{T'}(Fl(n')) \to H^*_T(Fl(n))$$

by

$$x_i \mapsto \begin{cases} x_i & \text{if } i \le n; \\ 0 & \text{if } i > n, \end{cases}$$

which is a homomorphism over $\Lambda' \to \Lambda$.

Claim: For $w \in S_{n'}$¡ the class $\sigma'_w \in H^*_{T'}(Fl(n'))$ maps to $\sigma_w \in H^*_T(Fl(n))$ if $w \in S_n \subset S_{n'}$, and maps to 0 otherwise.

It follows that the coefficients c_{uv}^{w} are preserved – that is, they correspond via the natural map $\Lambda' \to \Lambda$.

In fact, $\iota^{-1}(\Omega'_w) = \Omega_w$ when $w \in S_n$, and $\iota^{-1}(\Omega'_w) = \emptyset$ otherwise. The reason for this is the following. Restrict the tautological sequence of $Fl(n')$ to $Fl(n)$; it becomes

$$E_1 \subset \cdots \subset E_n \subset \cdots \subset E_n \oplus \mathbb{C}^d \to E_n \oplus \mathbb{C}^{d-1} \to \cdots \to Q_n \to \cdots \to Q_1.$$

For $w \in S_n$, the conditions imposed on the ranks of the maps $E_n \oplus \mathbb{C}^i \to E_n \oplus \mathbb{C}^j$ are trivial, so the locus Ω_w is cut out by the same equations on $Fl(n)$ and $Fl(n')$.

On the other hand, if $w \notin S_n$, then the condition on $E_n \to Q_n$ is that this map have rank at most $\#\{i \,|\, i \leq n, w(i) \leq n\}$, and this number is strictly less than n. This is a contradiction, though, since the map $E_n \to Q_n$ is the identity.

Lemma 6.1. *$\mathfrak{S}_w$ is the only polynomial which represents $\sigma_w \in H^*Fl(n')$ for all $n' \geq n$.*

The reason for this is that for $n' \gg n$, the monomials $x_1^{i_1} \cdots x_n^{i_n}$, with $i_j \leq N$, are linearly independent in $\mathbb{Z}[x_1, \ldots, x_{n'}]/(e_1, \ldots, e_{n'})$. In fact, we have seen that this ring has a basis of monomials $x_1^{k_1} \cdots x_{n'}^{k_{n'}}$, with $k_j \leq n' - j$, and the monomials in question are among these for large enough n'. (Taking $n' > n + N$ will do.)

The basic geometry involved in proving Theorem 4.2 is the following setup. Let Z be a variety (or scheme), with a rank-n vector bundle E and a complete flag of subbundles $E_1 \subset \cdots \subset E_n = E$, and let $y_i = c_1(E_i/E_{i-1})$. (For us, Z will be the mixing space $Fl(1, 2, \ldots, n; \mathbb{C}^m) = M_{m,n}^o \times_B Fl(n)$, with its tautological sequence of bundles, and $y_i = t_i$.) Consider $X = \mathbf{Fl}(E) \to Z$, with the tautological bundles $S_1 \subset \cdots \subset S_n = E_X$, and let $Q_i = E/S_{n-1}$, so there is a sequence $E \to Q_{n-1} \to \cdots \to Q_1 \to 0$. On X, then, there is a sequence

$$E_1 \subset \cdots \subset E_n = E \to Q_{n-1} \to \cdots \to Q_1.$$

Let $\tilde{x}_i = c_1(\ker(Q_i \to Q_{i-1})) = c_1(S_{n-i+1}/S_{n-1})$. (We use a tilde to distinguish the Chern classes belonging on X from those which are pulled back from Z.)

The general statement we will prove is the following:

Theorem 6.2. *For $w \in S_n$, let*

$$\mathbf{\Omega}_w = \{x \in X \mid \operatorname{rk}(E_p(x) \to Q_q(x)) \leq r_w(q,p) \text{ for } 1 \leq p, q \leq n\}.$$

This is an irreducible subvariety of X, of codimension $\ell(w)$, and

$$[\mathbf{\Omega}_w] = \mathfrak{S}_w(\tilde{x}, y).$$

(Recall that $r_w(q,p) = \#\{i \leq q \,|\, w(i) \leq p\}$.) Applying this to the case $Z = Fl(1, \ldots, n; \mathbb{C}^m)$, we obtain $\sigma_w^{eq} = \mathfrak{S}_w(x, t)$, as in Theorem 4.2.

In fact, $\mathbf{\Omega}_w$ has a very explicit local description: where the flag $E_1 \subset \cdots \subset E_n$ is trivial on Z, the bundle $X \to Z$ looks like $Z \times Fl(n)$, and here $\mathbf{\Omega}_w$ looks like $Z \times \Omega_w$, where $\Omega_w \subset Fl(n)$ is the classical Schubert variety.

We need a lemma describing the tangent bundle.

Lemma 6.3. *With notation as above, the relative tangent bundle to $X \to Z$ is*

$$K = \ker \left(\bigoplus_{p=1}^{n-1} \mathrm{Hom}(S_p, Q_{n-p}) \to \bigoplus_{p=1}^{n-1} \mathrm{Hom}(S_p, Q_{n-p-1}) \right),$$

where the map takes $(S_p \to Q_{n-p})$ to $(S_p \to Q_{n-p} \to Q_{n-p-1}) - (S_{p-1} \hookrightarrow S_p \to Q_{n-p})$.

Exercise 6.4. The map described in the lemma is surjective, so K has rank $\binom{n}{2}$.

Proof. The relative tangent bundle $T_{X/Z}$ is the normal bundle to the diagonal $\delta : X \hookrightarrow X \times_Z X$. On $X \times_Z X$, one has

$$\widetilde{K} = \ker \left(\bigoplus_p \mathrm{Hom}(p_1^* S_p, p_2^* Q_{n-p}) \to \bigoplus_p (p_1^* S_p, p_2^* Q_{n-p-1}) \right).$$

This comes with a section s, coming from the defining sequence

$$p_1^* S_1 \subset \cdots \subset p_1^* S_{n-1} \subset p_1^* E = p_2^* E \to p_2^* Q_{n-1} \to \cdots \to p_2^* Q_1.$$

(That is, the maps in $\bigoplus_p \mathrm{Hom}(p_1^* S_p, p_2^* Q_{n-p})$ are to be those appearing in this sequence.)

Notice that $\mathrm{Zeroes}(s) = \delta(X)$, and therefore

$$T_{X/Z} = N_{\delta(X)/X \times_Z X} = \delta^* \widetilde{K} = K.$$

$\square$

We now prove Theorem 6.2.

Proof. Case 1: $w = w_0$ (“a point”). We show that

$$[\mathbf{\Omega}_{w_0}] = \mathfrak{S}_{w_0}(\tilde{x}, y) = \prod_{i+j \le n} (\tilde{x}_i - y_j).$$

In fact, $\mathbf{\Omega}_{w_0}$ is $s(Z)$, where $s : Z \to X$ is the section of the flag bundle given by the flag $E_1 \subset \cdots \subset E_n = E$ on Z. It is also the zero locus of a section of the bundle K on X, where

$$K = \ker \left(\bigoplus_p \mathrm{Hom}(E_p, Q_{n-p}) \to \bigoplus_p \mathrm{Hom}(E_p, Q_{n-p-1}) \right).$$

Therefore $[\mathbf{\Omega}_{w_0}] = c_{top}(K) = \prod_{i+j \le n} (\tilde{x}_i - y_j)$.

For the general case, we work up from the “class of a point.” Write $v = w \cdot s_k$. Our strategy will be to show that $[\mathbf{\Omega}_v] = \partial_k([\mathbf{\Omega}_w])$ in the case $\ell(v) < \ell(w)$.

The geometry is as follows. Let $Z_k \subset X \times_Z X$ be the set of pairs $(W_\bullet, W'_\bullet)$ of flags in the same fiber over Z, such that $W_i = W'_i$ for all $i \neq n-k$. Let Y_k be the partial flag bundle

$$Y_k = \mathbf{Fl}(1, \ldots, n-k-1, n-k+1, \ldots, n; E).$$

Then we have a diagram:

$$\begin{tikzcd}
& Z_k \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
X \arrow[dr, "\pi"'] & & X \arrow[dl, "\pi"] \\
& Y_k \arrow[d] & \\
& Z, &
\end{tikzcd}$$

and we claim that the diamond at top is a fiber square, where all the maps are $\mathbb{P}^1$-bundles. Indeed, if

$$U_1 \subset \cdots \subset U_{n-k-1} \subset U_{n-k+1} \subset \cdots \subset U_n = E$$

is the tautological sequence of bundles on Y_k, then $\pi : X \to Y_k$ is just the bundle $\mathbb{P}(U_{n-k+1}/U_{n-k-1}) \to Y_k$. One checks that Z_k is precisely the fiber product over Y_k.

The theorem follows from Lemmas 6.5(1) and 6.6 below:

Lemma 6.5.
(1) *If $w(k) > w(k+1)$, then p_1 maps $p_2^{-1}(\mathbf{\Omega}_w)$ birationally onto $\mathbf{\Omega}_v = \mathbf{\Omega}_{w \cdot s_k}$. Thus*

$$(p_1)_*(p_2)^*[\mathbf{\Omega}_w] = [\mathbf{\Omega}_v].$$

(2) *If $w(k) < w(k+1)$, then p_1 maps $p_2^{-1}(\mathbf{\Omega}_w)$ into $\mathbf{\Omega}_w$, so by dimension considerations,*

$$(p_1)_*(p_2)^*[\mathbf{\Omega}_w] = 0.$$

Recall that $H^*X = H^*Z[\tilde{x}_1, \ldots, \tilde{x}_n]/(e_i(\tilde{x}) - c_i(E) \,|\, 1 \leq i \leq n)$. Thus the operator ∂_k, acting with respect to the variables $\tilde{x}_i$ (and regarding elements of H^*Z as scalars), induces an operator $\partial_k : H^*X \to H^*X$ that reduces degrees by 2. Our second lemma relates this to geometry:

Lemma 6.6. $(p_1)_* \circ (p_2)^* = \partial_k$.

To deduce the theorem, use descending induction on the length of permutations. Assume we know that $[\mathbf{\Omega}_w] = \mathfrak{S}_w(\tilde{x}, y)$. If $v = w \cdot s_k$, with $w(k) > w(k+1)$, then $\ell(v) = \ell(w) - 1$, and this is a reduced expression. We have $\partial_k(\mathfrak{S}_w) = \mathfrak{S}_v$; on the other hand, Lemmas 6.5 and 6.6 combine to show that $\partial_k([\mathbf{\Omega}_w]) = [\mathbf{\Omega}_v]$. (Additionally, Lemma 6.5(1) says that $\partial_k[\mathbf{\Omega}_w] = 0$ when $w(k) < w(k+1)$, and we see a geometric reason for $\partial_k \mathfrak{S}_w = 0$.) $\square$

It remains to prove the lemmas. For these, we give sketches, and refer to [Fulton 1997, §10].

Lemma 6.5 is local on Z, so it suffices to treat the case where Z is a point. Here one looks at the classical Schubert varieties Ω_w, which are B-orbit-closures. In fact, the projection maps p_1 and p_2 are B-equivariant (for the diagonal action of B on $Fl \times Fl$), so one can restrict attention to the point $p_w \in \Omega_w^o$. Now $p_1(p_2^{-1}(p_w))$ is the T-invariant $\mathbb{P}^1$ connecting p_w and p_v, so $p_1(p_2^{-1}(\Omega_w^o)) = \Omega_w^o \cup \Omega_v^o$, and the lemma follows.

Lemma 6.6 comes from a general fact about $\mathbb{P}^1$-bundles. Let F be a rank-2 vector bundle on a variety Y, and consider the diagram

$$\begin{tikzcd}
 & \mathbb{P}(F) \times_Y \mathbb{P}(F) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
\mathbb{P}(F) \arrow[dr, "\pi"'] & & \mathbb{P}(F) \arrow[dl, "\pi"] \\
 & Y. &
\end{tikzcd}$$

On $\mathbb{P}(F)$, there is the universal quotient bundle $F \to Q \to 0$; set $x = c_1(Q)$. Then $H^*(\mathbb{P}(F)) = H^*Y \cdot 1 \oplus H^*Y \cdot x$. In this setup, $(p_1)_*(p_2)^*(a + b \cdot x) = b$. That is, $(p_1)_*(p_2)^*(1) = (p_1)_*(1) = 0$ (by degree considerations), and $(p_1)_*(p_2)^*(x) = 1 \in H^0(Y)$. To see this, restrict to a fiber of p_1. Here $(p_2)^*(x)$ is the class of a point in $\mathbb{P}^1$, and $(p_1)_*([pt]) = 1$.

We apply this to the situation in the Lemma, with

$$\pi : X = \mathbb{P}(U_{n-k+1}/U_{n-k-1}) \to Y_k.$$

The universal quotient is $\ker(Q_k \to Q_{k-1})$, so $x = \tilde{x}_k$. H^*X has a H^*Z-basis of monomials $\tilde{x}_1^{i_1} \cdots \tilde{x}_n^{i_n}$ $(i_j \leq n-j)$, but one can use the relations to reindex and obtain a basis $\tilde{x}_{\sigma(1)}^{i_1} \cdots \tilde{x}_{\sigma(n)}^{i_n}$ for any $\sigma \in S_n$. In particular, one can express any class in terms of monomials with at most a linear factor of $\tilde{x}_k$, and no $\tilde{x}_{k+1}$. One checks that $(p_1)_*(p_2)^*$ and ∂_k agree on such monomials.

Remark 6.7. The geometry described here also resolves singularities for any Schubert variety. The idea is as follows: Start with the point Ω_{w_0}, and obtain a $\mathbb{P}^1$-bundle mapping birationally to all curves $\Omega_{w_0 \cdot s_k}$. (In fact, all the Schubert curves are already isomorphic to $\mathbb{P}^1$.) Then, starting with a (nonsingular) curve, obtain a $\mathbb{P}^1$-bundle mapping birationally to all surfaces $\Omega_{w_0 \cdot s_k \cdot s_l}$. Continuing in this fashion, one constructs nonsingular models for all Schubert varieties. The intermediate varieties arising in this construction are called *Bott-Samelson varieties.*

Remark 6.8. Recall that the class $[\mathbf{\Omega}_w]$ actually comes from

$$\overline{H}_{2\dim \mathbf{\Omega}}(\mathbf{\Omega}_w) = H^{2\ell(w)}(X, X \smallsetminus \mathbf{\Omega}_w) \to H^{2\ell(w)}X.$$

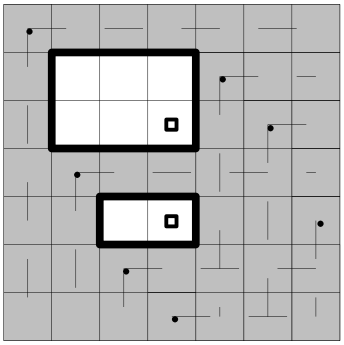

FIGURE 3. $D(w)$ for $w = 1\ 4\ 6\ 7\ 2\ 3\ 5$.

If $f : Y \to X$ is any map of varieties over Z, with $f^{-1}(\mathbf{\Omega}_w) = \emptyset$, then $f^*\mathfrak{S}_w(\tilde{x}, y) = 0$.

7. APPLICATIONS TO GRASSMANNIANS

There is a natural map $Fl(n) \to Gr(r,n)$ taking a flag $W_\bullet = (W_1 \subset \cdots \subset W_n = E)$ to the rth vector space $W_r \subset E$; more generally, there is a globalized map $\mathbf{Fl}(E) \to \mathbf{Gr}(r,E)$ for a vector bundle $E \to Z$. Given any flag of bundles $E_1 \subset \cdots \subset E_n = E$ on Z, there are loci $\Omega_\lambda = \Omega_\lambda(E_\bullet)$ in $\mathbf{Gr}(r,E)$. Here λ is a partition inside the $r \times (n-r)$ rectangle. Setting $i_a = n - r + a - \lambda_a$, the locus is defined by

$$\Omega_\lambda = \{x \mid \mathrm{rk}(E_{i_a}(x) \to Q) \leq n - r - \lambda_a \text{ for } 1 \leq a \leq r\},$$

where Q is the rank-$(n-r)$ universal quotient bundle on $\mathbf{Gr}(r,E)$, appearing in the sequence $0 \to S \to E \to Q \to 0$.

We would like to compare this locus with the loci on flag varieties (or bundles), using the projection map $f : \mathbf{Fl}(E) \to \mathbf{Gr}(r,E)$. To do this, associate to $\lambda \subset r \times (n-r)$ a permutation $w(\lambda) = j_1\, j_2 \cdots j_{n-r}\ i_1\, i_2 \cdots i_r$, where $(j_1 < \cdots < j_{n-r})$ is the complement to $(i_1 < \cdots < i_r)$ in $\{1, \ldots, n\}$.

Proposition 7.1. $f^{-1}(\Omega_\lambda) = \Omega_{w(\lambda)}$.

Example 7.2. Let $r = 3$ and $n = 7$. Consider $\Omega_\lambda \subset Gr(3,7)$, for $\lambda = (3,3,2)$. Here $I = I(\lambda) = \{2,3,5\}$, so $w(\lambda) = 1\ 4\ 6\ 7\ 2\ 3\ 5$. Look at the diagram $D(w)$, shown in Figure 3. The rank conditions coming from the lower-right corners of the two pieces of $D(w)$ suffice to define the Schubert variety Ω_w; that is,

$$\Omega_w = \{x \mid \mathrm{rk}(E_p(x) \to Q_q(x)) \leq r_w(q,p) \text{ for } (p,q) = (3,4) \text{ and } (5,4)\}.$$

Notice that exactly the same conditions define Ω_λ, by Exercise 1.4.

Exercise 7.3. For any $w \in S_n$, the rank conditions $\mathrm{rk}(E_p \to Q_q) \leq r_w(q,p)$, with (p,q) a lower-right corner of a piece of $D(w)$, suffice to define $\Omega_w \subset Fl(n)$ scheme-theoretically.

[Recall that we may change the definition of $D(w)$ to have dots in $(i, w(i))$; the statement of the exercise should be changed accordingly.]
The proof of Proposition 7.1 is a direct generalization of the above example, using Exercise 7.3.

We obtain a formula for $[\Omega_\lambda]$, for $\Omega_\lambda \subset \mathbf{Gr}(r, E) \to Z$. Let $y_i = c_1(E_i/E_{i-1})$, for $1 \le i \le n$, be Chern classes on Z. Let $x_1, \ldots, x_{n-r}$ be the Chern roots of the universal quotient bundle Q on $\mathbf{Gr}(r, E)$, and let $x_{n-r+1}, \ldots, x_n$ be the Chern roots of the tautological subbundle S. Then we have

$$[\Omega_\lambda] = \mathfrak{S}_{w(\lambda)}(x, y). \tag{20}$$

(More explicitly, one forms the flag bundle $\mathbf{Fl}(E) \to Z$, with the sequence of bundles

$$E_1 \subset \cdots \subset E_n = E \to Q_{n-1} \to \cdots Q_{n-r} = Q \to Q_{n-r-1} \to \cdots Q_1,$$

where $Q = Q_{n-r}$ is pulled back from $\mathbf{Gr}(r, E)$. Then $x_i = c_1(\ker(Q_i \to Q_{i-1})$ on $\mathbf{Fl}(E)$, and we consider $H^* \mathbf{Gr}(r, E) \subset H^* \mathbf{Fl}(E)$ via the canonical injection.)

Remark 7.4. The polynomial $\mathfrak{S}_{w(\lambda)}(x, y)$ is symmetric in x_i and x_{i+1} for $1 \le i \le n-r$ and $r \le i \le n$, so $\mathfrak{S}_{w(\lambda)}(x, y)$ can be written as a polynomial in such elementary symmetric functions. In fact, we have given formulas in Eq. (9). The above discussion shows the double Schubert polynomials indexed by $w(\lambda)$ can be expressed as multi-Schur polynomials:

$$\mathfrak{S}_{w(\lambda)}(x, y) = \det(c_{n-r+b-i_a}(Q - E_{i_a}))_{1 \le a,b \le r}. \tag{21}$$

For more formulas of this type, see [Macdonald 1991].

There is a notion of stability for Schubert classes in the Grassmannian, but it is slightly different from the flag case. We want a map $\iota : Gr(r, n) \hookrightarrow Gr(r, n')$ such that

$$\iota^{-1}(\Omega_\lambda) \quad = \quad \begin{cases} \Omega_\lambda & \text{if } \lambda \subset r \times (n-r); \\ \emptyset & \text{otherwise.} \end{cases}$$

To do this, define ι by including $\mathbb{C}^n \subset \mathbb{C}^{n'}$ as $\mathbb{C}^{n'} = \mathbb{C}^d \oplus \mathbb{C}^n$. (The other choice would be $\mathbb{C}^{n'} = \mathbb{C}^n \oplus \mathbb{C}^d$.) This is equivariant with respect to the inclusion $GL_n \hookrightarrow GL_{n'}$ given by

$$A \mapsto \begin{array}{|c|c|} \hline I & 0 \\ \hline 0 & A \\ \hline \end{array}.$$

The corresponding map $\Lambda' \to \Lambda$ is given by

$$t_i \mapsto \begin{cases} t_{i-d} & \text{if } i > d; \\ 0 & \text{if } i \le d. \end{cases}$$

If we write $i_a = n - r + a - \lambda_a$, $i'_a = n' - r + a - \lambda_a = i_a + d$, then $c(E_{i'_a}) \mapsto c(E_{i_a})$ and $c(Q') \mapsto c(Q)$.

Exercise 7.5. Fill in the details of the above stability statements, and check that they are false for the other choice of inclusion: $\mathbb{C}^n \hookrightarrow \mathbb{C}^n \oplus \mathbb{C}^d$.

8. Double Schubert polynomials and matrix Schubert varieties

The simplest – and most recent – realization of double Schubert polynomials is in terms of *matrix Schubert varieties.* (Cf. [Knutson-Miller]. See also [Fulton-Pragacz] and [Fulton 1991].) Let E and F be a n- and m-dimensional vector spaces over $\mathbb{C}$, and let $H = \mathrm{Hom}_{\mathbb{C}}(E,F) \cong M_{m,n}$. Then $G = \mathrm{GL}(E) \times \mathrm{GL}(F)$ acts on H by

$$(g', g'') \cdot \varphi = g'' \circ \varphi \circ (g')^{-1}.$$

Choose (complete) flags $E_1 \subset \cdots \subset E_n = E$ and $S_1 \subset \cdots \subset S_m = F$, and put $Q_i = F/S_{m-i}$, so there is a quotient flag $F = Q_m \to Q_{m-1} \to \cdots Q_1$. Let $B' \subset \mathrm{GL}(E)$ be the Borel subgroup preserving $E_\bullet$, let $B'' \subset \mathrm{GL}(F)$ be the Borel subgroup preserving $S_\bullet$ (or equivalently, preserving $Q_\bullet$), and let $T' \subset B'$ and $T'' \subset B''$ be the maximal tori. Write $B = B' \times B'' \subset G$.

The orbits of B on H are exactly the loci

$$\Omega^o_w = \{\varphi \in H \mid \mathrm{rk}(E_p \hookrightarrow E \xrightarrow{\varphi} F \to Q_q) = r_w(q,p)\}.$$

Here w is a *partial permutation*, given by an $m \times n$ matrix M_w which has at most one 1 in each row and column, and 0's elsewhere. For example, M_w could be

$$M_w = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The rank $r_w(q,p)$ is defined to be the number of 1's in the upper-left $q \times p$ submatrix of M_w.

The closure $\Omega_w = \overline{\Omega^o_w}$ is a B-equivariant subvariety of the affine space $H \cong \mathbb{A}^{mn}$. Notice that

$$H^*_B(H) = H^*_B(pt) = H^*_{T' \times T''}(pt) = \mathbb{Z}[x_1, \ldots, x_m; y_1, \ldots, y_n],$$

where the x's and y's are characters of T' and T'', respectively.

Theorem 8.1. *The class $[\Omega_w]^B \in H^{2\ell(w)}_B(H)$ is the double Schubert polynomial $\mathfrak{S}_w(x_1, \ldots, x_m; y_1, \ldots, y_n)$.*

Proof. On H, there is a sequence of (trivial) vector bundles

$$E_1 \subset \cdots \subset E_n = E \xrightarrow{\Phi} F = Q_m \to \cdots \to Q_1,$$

where Φ is the universal map, whose fiber over $\varphi \in H$ is just $\varphi : E \to F$. We apply the general degeneracy locus formula, with $x_i = c_1^T(\ker(Q_i \to Q_{i-1}))$ and $y_i = c_1^T(E_i/E_{i-1})$. Take $E = \mathbb{C}^n$, with the standard flag $E_1 \subset \cdots \subset E_n$, $E_i = \langle e_1, \ldots, e_i \rangle$; and take $F = \mathbb{C}^m$ with the opposite flag $S_1 \subset \cdots \subset S_m$,

$S_i = \langle e_{m+1-i}, \ldots, e_m \rangle$. Thus $\ker(Q_i \to Q_{i-1}) = \mathbb{C} \cdot e_i$, and $E_i/E_{i-1} = \mathbb{C} \cdot e_i$. Notice that $B' = B_+$ (upper-triangular matrices), while $B'' = B_-$ (lower-triangular matrices).

`This will probably be moved to a different section and completed there.` □

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