EQUIVARIANT INTERSECTION THEORY
§6: MORE ON LOCALIZATION

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We return to the discussion of localization at fixed points begun in §4.6.

1. Fixed points on flag varieties

Let $X = Fl(n)$, and fix the standard flag

$$E_\bullet = p_{id} = (\langle e_1\rangle \subset \langle e_1, e_2\rangle \subset \cdots).$$

The torus $T = (\mathbb{C}^*)^n$ acts on X with fixed points p_v, the flags corresponding to ordered bases $\{e_{v(1)}, e_{v(2)}, \ldots, e_{v(n)}\}$. (The ith part of the flag is the span of the first i basis vectors.) Thus, $p_v \in X_v^o$; in fact, X_v^o is the B-orbit of p_v. Similarly, let q_v be the flag whose ith part is the span of the *last* i vectors in the basis $\{e_{v(1)}, \ldots, e_{v(n)}\}$. (That is, q_v corresponds to the ordered basis $\{e_{v(n)}, e_{v(n-1)}, \ldots, e_{v(1)}\}$.) Thus $q_v \in \Omega_v^o$.

We have $\mathfrak{S}_w(x;t) = [\Omega_w]^T = \sigma_w^{eq} = \sigma_w$.

Proposition 1.1. *The class $\sigma_w = [\Omega_w]^T$ restricts to*

$$\sigma_w|_v = \mathfrak{S}_w(t_{v(1)}, \ldots, t_{v(n)}; t_1, \ldots, t_n)$$

under the map $H_T^*X \to H_T^*(q_v) = \Lambda_T$.

Proof. We consider the case of a general flag bundle $X = \mathbf{Fl}(E) \to Z$, where E is a rank-n vector bundle with a fixed split flag of subbundles $E_1 \subset \cdots \subset E_n = E$ on Z; thus E_i is a sum of line bundles $L_1 \oplus \cdots \oplus L_i$. (The proposition will follow by taking Z to be $Fl(1, \ldots, n; \mathbb{C}^m)^s$, the variety of split flags, so $H_T^i Fl(n) = H^i X$ for $i \ll m$.) As usual, we use the same notation to denote these bundles pulled back to X. Let $S_1 \subset \cdots \subset S_n = E$ be the tautological subbundles on X.

The "point" q_v is the image of the section $s_v : Z \to X$ corresponding to the flag

$$L_{v(n)} \subset L_{v(n-1)} \oplus L_{v(n)} \subset \cdots \subset E$$

on Z; that is, $s_v^* S_i = L_{v(n+1-i)} \oplus \cdots \oplus L_{v(n)}$. Therefore $x_i = c_1(S_{n+1-i}/S_{n-i})$ maps (by s_v^*) to $c_1(L_{v(i)}) = t_{v(i)} \in H^2 Z$, and so $\sigma_w = \mathfrak{S}_w(x_1, \ldots, x_n; t_1, \ldots, t_n)$ maps to $\mathfrak{S}_w(t_{v(1)}, \ldots, t_{v(n)}; t_1, \ldots, t_n)$. □

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We will sometimes write "$\mathfrak{S}_w(t_v;t)$" for this polynomial with permuted t's.

Challenge: (Exercise) For $v = w$,

$$\begin{aligned} \sigma_w|_w &= \mathfrak{S}_w(t_w;t) \\ &= \prod (t_{w(i)} - t_{w(j)}), \end{aligned}$$

where the product is over inversions of w (i.e., $i < j$ such that $w(i) > w(j)$).

Corollary 1.2. *$\mathfrak{S}_w(t_v;t) = 0$ if $q_v \notin \Omega_w$.*

In §5, we mentioned the *Bruhat order*, characterized by the following equivalent properties, among others:

(1) $p_w \in X_v$.
(2) $X_w \in X_v$.
(3) $\Omega_v \subseteq \Omega_w$.

We add to this list the condition $q_v \in \Omega_w$.

In general, we can describe the image of the class of a T-invariant subvariety as follows:

Lemma 1.3. *Let T be a torus acting on X, with a codimension-d invariant subvariety $V \subset X$. Let $p \in V$ be a nonsingular point of V (and of X). Then the image of $[V]^T$ in $H^*_T(p) = \Lambda$ is the product of the weights of the action of T on $N_p = T_pX/T_pV$; in fact, it is an element of $\Lambda^d = \operatorname{Sym}^d M$.*

Proof. We first reduce to the case where V is nonsingular. In general, if G acts on X, $V \subset X$ is a G-invariant closed subvariety, and $p \in V$ is nonsingular in V and X, then there is a G-invariant neighborhood X^o of p such that $V^o = X^o \cap V$ is nonsingular. (Note that $V^o \subset X^o$ will be a closed inclusion of nonsingular varieties.) Indeed, X_{reg}, X_{sing}, V_{reg}, and V_{sing} are all preserved by the action of G – this is true in any setting where the statement makes sense. Set $X^o = X_{reg} \smallsetminus V_{sing}$; one easily checks this works.

Now consider the diagram

$$\begin{tikzcd}
& H^*_TX \arrow[r] \arrow[d] & H^*_TV \arrow[r] & H^*_T(p) \\
H^*_TV^o \arrow[r, "\iota_*"] & H^*_TX^o \arrow[r, "\iota^*"] & H^*_TV^o. \arrow[ur] &
\end{tikzcd}$$

where $\iota : V^o \hookrightarrow X^o$ is the (closed) embedding. We have $[V^o]^T = \iota_*(1)$, so $\iota^*[V^o]^T = \iota^*\iota_*(1) = c^T_d(N) \in H^{2d}_T(V^o)$, where N is the normal bundle. Restricting to $H^*_T(p)$, then, we have $c^T_d(N_p)$, as desired. □

For a nonsingular point p as in the lemma, $c^T_d(N_p)$ is not zero – that is, none of the weights is 0. However, the following example shows that this fails for singular p.

Example 1.4. (Cf. [Brion].) Let $T = (\mathbb{C}^*)^2$ act on $X = \mathbb{P}^4$ by

$$(z_1, z_2) \cdot [y_1 : \cdots : y_5] = [y_1 : z_1\, y_2 : z_1^{-1}\, y_3 : z_2\, y_4 : z_2^{-1}\, y_5].$$

Let $V \subset X$ be the hypersurface defined by the equation $Y_2\, Y_3 - Y_4\, Y_5 = 0$, and let $p = p_1 = [1 : 0 : \cdots : 0]$ be the singular point of V. Then $[V]^T \mapsto 0$ in $H_T^*(p)$.

In fact, $V = \text{Zeroes}(s)$, where s is a section of $\mathcal{O}(2)$. If $\zeta = c_1^T(\mathcal{O}(1))$, then

$$H_T^* X = \Lambda[\zeta]/(\zeta(\zeta + t_1)(\zeta - t_1)(\zeta + t_2)(\zeta - t_2)).$$

Thus $\zeta \mapsto (0, -t_1, t_1, -t_2, t_2)$ in $H_T^* X^T$, and $[V]^T = 2\zeta$.

A natural question arising here is the following: For which singularities $p \in V$ is it necessary that $[V]^T$ not restrict to 0 in $H_T^*(p)$? For example, is this true for all Schubert varieties, or for rational singularities, etc.?

Example 1.5. Let $X = Fl(n)$, $T = (\mathbb{C}^*)^n$, $p = q_w$, and $V = \Omega_w = \Omega_w(std)$. Then

$$[\Omega_w]^T \mapsto \prod_{i<j,\, w(i)>w(j)} (t_{w(i)} - t_{w(j)}) = \sigma_w|_w \in H_T^*(q_w),$$

and therefore

$$\mathfrak{S}_w(t_{w(1)}, \ldots, t_{w(n)}; t_1, \ldots, t_n) = \prod_{i<j,\, w(i)>w(j)} (t_{w(i)} - t_{w(j)}).$$

Claim: The weights of T on N_p are $t_{w(i)} - t_{w(j)}$, for those $i < j$ with $w(i) > w(j)$.

Recall that $\Omega_w = X_v$, where $v = w\, w_0$, and $q_w = p_v$, the flag corresponding to $(e_{v(1)}, \ldots, e_{v(n)})$. The claim is equivalent to the following: $c_{top}^T(N_{p_v}(X_v))$ is the product of all $t_{v(i)} - t_{v(j)}$ with $i < j$, $v(i) < v(j)$.

For example, let $v = 3\ 1\ 6\ 4\ 2\ 5$. There is a neighborhood $U_v \cong \mathbb{A}^N$ (with $N = \binom{n}{2}$) of p_v, in which p_v corresponds to the origin. This can be described in terms of matrices:

$$U_v = \begin{pmatrix} * & 1 & 0 & 0 & 0 & 0 \\ * & * & * & * & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 1 & 0 & 0 \\ * & * & * & * & * & 1 \\ * & * & 1 & 0 & 0 & 0 \end{pmatrix}.$$

In this description, $X_v^o \subset U_v$ is the locus where entries below 1's are set equal to 0. Thus these entries (below and left of 1's) give a parametrization of the normal space $N_{q_v}(X_v^o)$. Now $\text{diag}(z_1, \ldots, z_n) \in (\mathbb{C}^*)^n$ acts on U_v by multiplying the kth row by z_k and dividing the jth column by $z_{v(j)}$. (The division is done to preserve the 1's in the specified positions, and does not change the flag represented by the matrix.) For example,

$$(z_1, z_2, z_3, z_4, z_5, z_6) \cdot \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & d & e & 1 \\ a & 0 & 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{z_2}{z_1} b & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{z_4}{z_1} c & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{z_5}{z_4} d & \frac{z_5}{z_2} e & 1 \\ \frac{z_6}{z_3} a & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Thus the weight on position (k, j) is $t_k - t_{v(j)}$. There is an entry in position (k, j) in $N(X_v^o)$ exactly when $k > v(j)$, and $k = v(i)$ for some $i > j$, so the claim is proved.

2. Localization and fixed points

We begin with an example. Let T be a torus acting on $\mathbb{P}^1$, with fixed points $0 = [1 : 0]$ and $\infty = [0 : 1]$. Specifically, let ψ_1 and ψ_2 be two characters of T. (That is, $\psi_i : T \to \mathbb{C}^*$.) Write M for the lattice of characters.

Letting T act by

$$t \cdot [z_1 : z_2] = [\psi_1(t)\, z_1 : \psi_2(t)\, z_2], \tag{1}$$

we see that if $\psi_1 \neq \psi_2$, the fixed-point set is precisely $\{0, \infty\}$. (If $\psi_1 = \psi_2$, then T acts trivially – and hence the fixed-point set is not finite; we will only consider actions with finitely many fixed points.)

The induced action on the tangent space $T_0\mathbb{P}^1 = \mathbb{C}$ is given by the composition

$$t : z \mapsto [1 : z] \mapsto [\psi_1(t) : \psi_2(t)\, z] = [1 : \frac{\psi_2(t)}{\psi_1(t)}\, z] \mapsto \frac{\psi_2(t)}{\psi_1(t)}\, z. \tag{2}$$

That is, the action is by the character $\psi_2 - \psi_1$. Similarly, the action on $T_\infty\mathbb{P}^1$ is by the character $\psi_1 - \psi_2$.

Say $\chi = \psi_2 - \psi_1$ is *the character of the action.* (So χ is determined by the geometry up to sign, as we have seen.)

Lemma 2.1. *Any algebraic action of T on $\mathbb{P}^1$ with fixed points $0, \infty$ is isomorphic to an action as described above.*

(Note that the action with characters $\psi_1 + \psi$ and $\psi_2 + \psi$ is the same for any $\psi \in M$.)

Proof. (Sketch.) First, we show that the only action of T on $\mathbb{C} = \mathbb{A}^1$ fixing 0 is given by

$$t \cdot Z = \chi(t)\, z,$$

for some $\chi \in M$. Indeed, any algebraic action corresponds to a map of $\mathbb{C}$-algebras

$$\mathbb{C}[x] \to \mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[x].$$

Requiring that this be compatible with the group action and that 0 be fixed means the map is given by $x \mapsto \chi \otimes x$.

To get an action on $\mathbb{P}^1$, we just glue actions on the two copies of $\mathbb{A}^1$. $\square$

Lemma 2.2. *The map*

$$H_T^*\mathbb{P}^1 \to H_T^*(0) \oplus H_T^*(\infty) = \Lambda \oplus \Lambda$$

is injective, and its image is the set of pairs (u_0, u_∞) *such that* $u_\infty - u_0$ *is divisible by* χ*, the character of the action.*

Proof. Injectivity is clear, given that psi_1 and ψ_2 are distinct (and hence not both 0).

To describe the image, recall that $H_T^*\mathbb{P}^1 = \Lambda[\zeta]/(\zeta + \psi_1)(\zeta + \psi_2)$ maps to $\Lambda \oplus \Lambda$ by $\zeta \mapsto (-\psi_1, -\psi_2)$. Thus a general element $a + b\zeta$ maps to $(a - b\psi_1, a - b\psi_2) = (u_0, u_\infty)$; we see that $u_\infty - u_0 = b(\psi_1 - \psi_2)$ is divisible by $\chi = \psi_2 - \psi_1$.

Conversely, suppose $u_\infty - u_0 = g(\psi_2 - \psi_1)$ for some $g \in \Lambda$. Setting $b = -g$ and $a = u_0 - g\psi_1 = u_\infty - g\psi_2$, we get an element $a + b\zeta$ mapping to (u_0, u_∞). $\square$

Lemma 2.3. *For any ring extension* $\Lambda \to R$*, an element of* $H_T^*\mathbb{P}^1 \otimes_\Lambda R$ *is invertible if and only if its image in* $H_T^*(\mathbb{P}^1)^T \otimes_\Lambda R \cong R \oplus R$ *is invertible.*

Proof. Exercise. $\square$

The main lemma we will need is the following:

Lemma 2.4. *Let* T *act on an* m*-dimensional nonsingular variety* X*, and let* $E \subset X$ *be a* T*-invariant subvariety isomorphic to* $\mathbb{P}^1$*, with isoloated fixed points* $p, q \in E$*; let* $c = m - 1$ *be the codimension of* E*. Let* $(\pm)\chi$ *be the character of the action of* T *on* E*. Let the characters of the action of* T *on* T_pX *be* $\chi, \varphi_2, \ldots, \varphi_m$*, and let the characters of the action on* T_qX *be* $-\chi, \psi_2, \ldots, \psi_m$*.*

Then for any $u \in H_T^*E$*, the element* $\varphi_2 \cdots \varphi_m \cdot \psi_2 \cdots \psi_m \cdot u$ *is in the image of the restriction map* $H_T^*X \to H_T^*E$*. (Here we are identifying* Λ *with* $\operatorname{Sym}^\bullet M$*.)*

Proof. Let $\iota : E \hookrightarrow X$ be the inclusion, with Gysin map $\iota_* : H_T^*E \to H_T^*X$. For all i, we have commutative diagrams

$$\begin{tikzcd}
H_T^iE \arrow[r, "\iota_*"] \arrow[d, hook] & H_T^{i+2c}X \arrow[r, "\iota^*"] & H_T^{i+2c}E \arrow[d, hook] \\
\Lambda \oplus \Lambda \arrow[rr] & & \Lambda \oplus \Lambda.
\end{tikzcd}$$

The map $\iota^* \circ \iota_* : \Lambda \oplus \Lambda \to \Lambda \oplus \Lambda$ is multiplication by $c_{top}^T(N)$, where N is the normal bundle to E in X. (So "*top*" is the codimension $c = m - 1$.) The image of $c_{top}^T(N)$ in $H_T^*(p) = \Lambda$ is $\varphi_2 \cdots \varphi_m$, and in $H_T^*(q)$, it is $\psi_2 \cdots, \psi_m$.

The lemma follows from a general algebra fact: Let F be a free, finitely generated module over a commutative ring R, and let $A : F \to F$ be an R-linear map. Then $\det(A)$ annihilates $\operatorname{coker}(A)$, and for all $u \in F$, $\det(A)\, u$ is in the image $\operatorname{im}(A)$.

Applying this to $\iota^* \circ \iota_*$, whose determinant is $\varphi_2 \cdots \varphi_m \cdot \psi_2 \cdots \psi_m$, we deduce the lemma. □

If a nonsingular point $p \in X$ is an isolated fixed point for the T-action, then the weights $\psi_1, \dots, \psi_m$ of T on the tangent space T_pX must be nonzero. In the case of T acting on $\mathbb{P}^{n-1}$ by characters $\psi_1, \dots, \psi_n$, the standard coordinate points are the fixed points if and only if all the ψ_i's are distinct. (Indeed, the action on $T_{p_i}\mathbb{P}^{n-1}$ is by $\psi_1 - \psi_i, \dots, \psi_n - \psi_i$.)

Theorem 2.5. *Let $T = (\mathbb{C}^*)^n$ be a torus acting on a nonsingular variety X with H_T^*X free over Λ (taking coefficients in $\mathbb{Q}$ if necessary). Suppose $H_T^*X \hookrightarrow H_T^*(X^T)$, with X^T finite, and assume the following hold:*

1. *For each $p \in X^T$, the characters of the action of T on T_pX are pairwise relatively prime** *in $\Lambda = R[t_1, \dots, t_n] = \mathrm{Sym}_R^\bullet M$.*
2. *For $p \neq q$ in X^T, the characters of T_pX and T_qX are either relatively prime, or they differ by a sign.*
3. *For any χ occurring as a character in any T_pX, let $T(\chi) = \{t \in T \mid \chi(t) = 1\}$. Assume $X^{T(\chi)}$ is a disjoint union of curves $E \cong \mathbb{P}^1$ joining two distinct points of X^T, together with a finite set of isolated points of X^T.*

With these hypotheses, for $u = \oplus u_p \in H_T^*(X^T) = \bigoplus H_T^*(p) = \bigoplus \Lambda$ to be in H_T^*X it is necessary and sufficient that for any such E joining p and q in $X^{T(\chi)}$, $u_p - u_q$ is divisible by the character χ.

Remark 2.6. In this form, the theorem is usually attributed to [Goresky-Kottwitz-MacPherson], although something similar can be found in work of Borel and his contemporaries in the 1960s.

In principle, this reduces any problem in the equivariant cohomology ring of a sufficiently nice variety to a calculation in a direct sum of polynomial rings.

Proof. If $u \in H_T^*X$, its restriction to $H_T^*E = H_T^*\mathbb{P}^1 \hookrightarrow H_T^*(p) \oplus H_T^*(q)$ shows that $u_p - u_q$ is divisible by χ; indeed, χ is the character of the action on E.

Conversely, suppose for any such E, $u_p - u_q$ is divisible by χ. If P is the product of all characters occurring at all fixed points (with multiplicity), we know that $P \cdot u$ is in H_T^*X. In fact, $P \cdot u$ comes from $H_T^*(X^T)$ via a Gysin map:

$$\begin{tikzcd}
u \in H_T^*(X^T) \arrow[r, "\text{Gysin}"] \arrow[d, equal] & H_T^*X \arrow[r] & H_T^*(X^T) \arrow[d, equal] \\
\displaystyle\bigoplus_{p \in X^T} H_T^*(p) \arrow[rr] & & \displaystyle\bigoplus_{p \in X^T} H_T^*(p).
\end{tikzcd}$$

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*Here R is to be $\mathbb{Z}$ or $\mathbb{Q}$; note that "relatively prime" means something different depending on the choice of R!

As in Lemma 2.4, the composed map $\bigoplus H_T^*(p) \to \bigoplus H_T^*(p)$ is diagonal; each $H_T^*(p) \to H_T^* X \to H_T^*(p)$ is given by multiplication by $c_{top}^T(N)$, which is the product of characters in T_pX; and the determinant of the map annihilates the cokernel. Since P is this determinant, $P \cdot u$ is in the image.

Now consider a character χ occurring in some T_pX. Let $S = T(\chi)$, so

$$X^S = E_1 \sqcup E_2 \sqcup \cdots \sqcup E_s \sqcup \{p_1\} \sqcup \cdots \sqcup \{p_r\}.$$

For E joining T-fixed points p and q, say T_pX has characters $\chi, \varphi_2, \ldots, \varphi_m$, and T_qX has characters $-\chi, \psi_2, \ldots, \psi_m$ (for a suitable ordering of p and q). Also, χ is the character of the action of T or T/S on $E \cong \mathbb{P}^1$. Therefore $T_pE \subset T_pX$ has weight χ, and $T_qE \subset T_qX$ has weight $-\chi$.

The theorem follows from the following claim:

Claim: For any character χ occurring at any fixed point, there is a polynomial P which is a product of characters relatively prime to χ, such that $P \cdot u$ is in $H_T^* X$.

To deduce the theorem, take a basis $\{e_1, \ldots, e_N\}$ for H_T^*X over Λ (since H_T^*X is free), and write $u = \sum f_i e_i$ with $f_i \in \mathrm{Frac}(\Lambda)$. We want to show that $f_i \in \Lambda$, and by the above discussion, we know $\tilde{P} \cdot f_i \in \Lambda$, where $\tilde{P}$ is the product of all characters (including χ). The claim says that χ need not appear in $\tilde{P}$, but since this holds for any given χ, we can replace $\tilde{P}$ with 1.

The proof of the claim is as follows. Let $\iota : X^S \hookrightarrow X$ be the inclusion. We know $u \in H_T^*(X^S)$, and we have maps

$$\begin{tikzcd}
H_T^*(X^S) \arrow[r, "\iota_*"] & H_T^*X \arrow[r, hook] & H_T^*(X^S) \arrow[r, hook] \arrow[d, equal] & H_T^*(X^T) \arrow[d, equal] \\
 & & \bigoplus H_T^*(E_i) \oplus H_T^*(p_j) \arrow[r, hook] & \bigoplus H_T^*(p_i).
\end{tikzcd}$$

We need to understand the last map, the restriction from X^S to X^T. Since $u_p - u_q$ is divisible by χ, (u_p, u_q) comes from $H_T^*(E_i)$.

Let

$$P = \left(\prod_{i=1}^{s} \varphi_2^{(E_i)} \cdots \varphi_m^{(E_i)} \cdot \psi_2^{(E_i)} \cdots \psi_m^{(E_i)} \right) \cdot \left(\prod_{j=1}^{r} (\text{characters at } p_j) \right).$$

Notice that this is relatively prime to χ. Now $P \cdot u$ comes from $\iota_* : H_T^*(X^S) \to H_T^* X$, since P is the determinant of $\iota^* \circ \iota_* : H_T^*(X^S) \to H_T^*(X^S)$. The claim is proved. □

Example 2.7. Let $X = Gr(r, n)$ and $T = (\mathbb{C}^*)^n$. The fixed points are $p_I = \langle e_{i_1}, \ldots, e_{i_r} \rangle$, and

$$T_{p_I} X \cong \mathrm{Hom}(S_{p_I}, Q_{p_I}) = \mathrm{Hom}(\langle e_i \rangle_{i \in I}, \langle e_j \rangle_{j \notin I}).$$

The weights are therefore $t_j - t_i$, for $i \in I$, $j \notin I$, and the characters χ are all $t_j - t_i$ for $j \neq i$.

Exercise 2.8. Find the corresponding E's for $\chi = t_j - t_i$.

Exercise 2.9. Do the same for the case $X = Fl(n)$.

One might ask for relaxations of the hypotheses of Theorem 2.5, but we do not know the right general theorem in this context. For example, suppose the torus T acts on X, with finitely many fixed points, such that $H_T^*X \hookrightarrow H_T^*(X^T)$, and both of these cohomology rings are free modules over $\Lambda = R[t_1, \dots, t_n]$ for some UFD R.

(1) When is H_T^*X given by the conditions of the theorem; that is, when does it consist of those elements $u = \oplus u_p$ such that $u_p - u_q$ is divisible by χ when p and q are connected by $E \cong \mathbb{P}^1$ with character χ?

(2) Can one describe H_T^*X when the theorem fails?

One can already test generalizations in the case $X = \mathbb{P}^{n-1}$, with T acting by distinct characters $\psi_1, \dots, \psi_n$. (That is, $t \cdot [z_1, \dots, z_n] = [\psi_1(t)\, z_1, \dots, \psi_n(t)\, z_n]$.) The characters of the action on the E's are $\chi = \psi_i - \psi_j$, for $i \neq j$. We have an algebraic description of the cohomology ring: $H_T^*\mathbb{P}^{n-1} = \Lambda[\zeta] / \prod(\zeta + \psi_i)$, and the localization map is given by $\zeta \mapsto (-\psi_1, \dots, -\psi_n)$ in $\bigoplus_p H_T^*(p)$.

Exercise 2.10. Let $X = \mathbb{P}^2$. If some non-unit $r \in R$ divides $\psi_k - \psi_i$ and $\psi_j - \psi_i$ (where $\{i, j, k\} = \{1, 2, 3\}$), then the image of the localization map is not given by the "GKM" recipe.

References

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