EQUIVARIANT INTERSECTION THEORY §7: AN EQUIVARIANT LITTLEWOOD-RICHARDSON RULE

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In this section, we present the combinatorial rule given in [Knutson-Tao] for computing the polynomial structure constants $c_{\lambda\mu}^{\nu}$ for multiplication in $H_T^*(Gr(r,n))$.

1. Puzzles

Let $X = Gr(r,n)$ and $T = (\mathbb{C}^*)^n$. (The theorems will also apply to $T' = (\mathbb{C}^*)^n/\mathbb{C}^*$.) Writing $\sigma_\lambda = [\Omega_\lambda(std)]^T \in H_T^{2|\lambda|}X$, we have

$$\sigma_\lambda \cdot \sigma_\mu = \sum_\nu c_{\lambda\mu}^{\nu}\sigma_\nu,$$

where $c_{\lambda\mu}^{\nu}$ is a homogeneous element of degree $|\lambda|+|\mu|-|\nu|$ in $\Lambda = \mathbb{Z}[t_1,\dots,t_n]$. (Since the case $T' = (\mathbb{C}^*)^n/\mathbb{C}^*$ is the same, we will see that actually $c_{\lambda\nu}^{\nu} \in \mathbb{Z}[t_2-t_1, t_3-t_2,\dots,t_n-t_{n-1}]$.) When $|\lambda|+|\mu|-|\nu| = 0$, these are the classical Littlewood-Richardson numbers. Knutson and Tao give a formula for all $c_{\lambda\mu}^{\nu}$.

We introduce some notation. Partitions λ fitting inside the $r \times (n-r)$ box correspond bijectively to sequences of r 1's and $n-r$ 0's, as follows. Starting in the northeast corner of the box, trace the border of λ; record a 0 for each step left, and a 1 for each step down. For example, the partition in Figure 1 corresponds to the sequence 010010011010. Note that the encoding depends not only on λ, but also on n and r.

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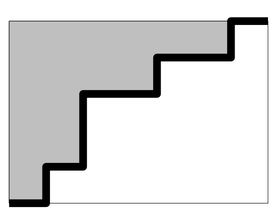

FIGURE 1. The border of $\lambda = (6,4,2,2,1)$, corresponding to the sequence 010010011010.

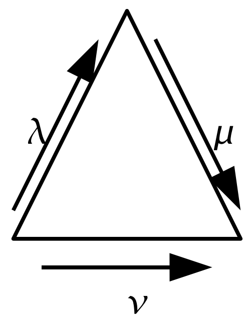

FIGURE 2.

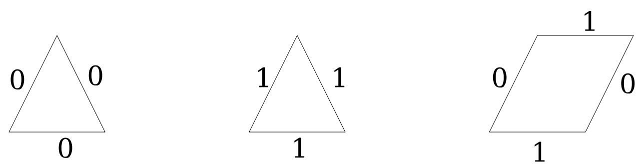

FIGURE 3. Classical puzzle pieces.

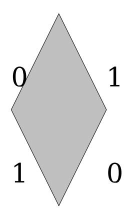

FIGURE 4. The equivariant puzzle piece.

A *puzzle of type* $\Delta^{\nu}_{\lambda\mu}$ is described as follows. Write the sequences corresponding to λ, μ, and ν around the border of an equilateral triangle of side length n as indicated in Figure 2: λ is written along the northwest edge from SW to NE, μ is written along the northeast edge from NW to SE, and ν is written along the bottom edge from left to right. To complete the puzzle, fill the triangle with the pieces shown in Figures 3 and 4, in a "jigsaw" fashion: shared edges must share the same label (0 or 1). The "classical pieces" of Figure 3 may be rotated; the "equivariant piece" of Figure 4 may only appear in the displayed orientation. An equivariant piece is said to be in position (i, j) if a line drawn SW from the piece meets the bottom edge i units from the left, and a line drawn SE from the piece meets the bottom edge j units from the left, as in Figure 5. The *weight* of a puzzle is $\prod(t_j - t_i)$, the product being taken over all (i, j) with an equivariant piece in position (i, j). An example is given in Figure 6.

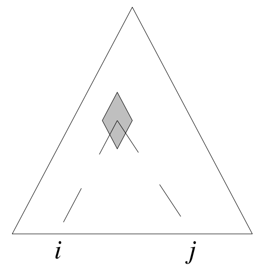

FIGURE 5. An equivariant piece in position (i, j).

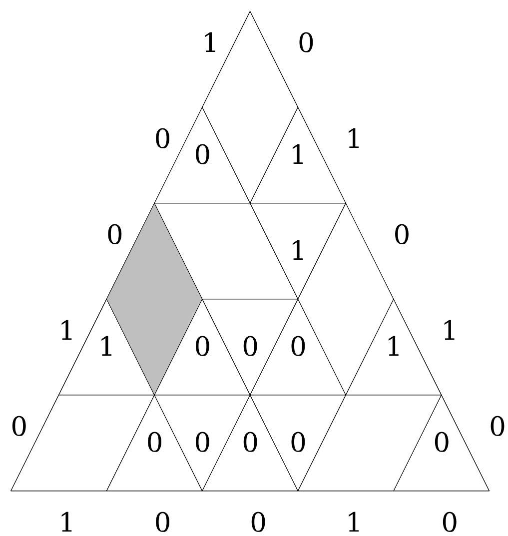

FIGURE 6. A puzzle of type $\Delta^{10010}_{01001,01010}$ and weight $t_3 - t_1$.

The main theorem of [Knutson-Tao] is the following:

Theorem 1.1. *The polynomial* $c^{\nu}_{\lambda\mu}$ *is the sum of the weights of all puzzles of type* $\Delta^{\nu}_{\lambda\mu}$*:*

$$c^{\nu}_{\lambda\mu} = \sum_{\textit{puzzles}} \prod_{\textit{eq. pieces}} (t_j - t_i).$$

To compute $\sigma_\lambda \cdot \sigma_\nu \in H^*_T X$, then, one writes λ along the northwest edge of a triangle and μ along the northeast edge, and forms all possible puzzles.

Exercise 1.2. Compute $\sigma_{(2,0)} \cdot \sigma_{(2,1)} \in H^*_T(Gr(2, 4))$ in this way.

As the following exercises show, it is not always clear how terms in the expansion of a product correspond to puzzles.

Exercise 1.3. With $r = 2, n = 5$,

$$\sigma_2 \cdot \sigma_{(2,1)} = \sigma_{(3,2)} + (t_5 - t_4 + t_3 - t_2)\sigma_{(2,2)} + (t_5 - t_4 + t_4 - t_3 + t_3 - t_1)\sigma_{(3,1)} + (t_3 - t_2)(t_5 - t_4 + t_3 -$$

Find the eight puzzles computing this product. Notice that the coefficient of $\sigma_{(3,1)}$ could also be written as $(t_5 - t_1)$, but there is no puzzle with an equivariant piece in position $(1,5)$.

Exercise 1.4. Compute the same product with $r = 2, n = 4$, and compare the puzzles to those appearing in the previous exercise.

2. On the proof of Theorem 1.1

In order to prove that puzzles compute equivariant Schubert calculus, one needs a list of properties of the structure constants $c^\nu_{\lambda\mu}$ that one can prove, and that characterize the $c^\nu_{\lambda\mu}$'s completely, without reference to cohomology. The basic properties we will use are the following:

1. Commutativity: $c^\nu_{\lambda\mu} = c^\nu_{\mu\lambda}$. (Note that this is not immediately clear in terms of puzzles.)
2. Associativity. (This is even harder to interpret in the combinatorial setting.)
3. Several formulas which we will discuss, and prove in the geometric setting.

We now list the eight formulas used to prove Theorem 1.1. Write $\sigma_\lambda|_\mu$ for the image of σ_λ in $H^*_T(p_\mu) \cong \Lambda$, and write $\sigma_\square = \sigma_1 = \sigma_{(1,0,\ldots,0)}$ for the divisor class in $H^2_T X$.

Lemma 2.1. *Let $I(\mu) = \{i_1 < \cdots < i_r\}$, where $i_a = n - r + a - \lambda_a$, and let $J(\mu) = \{j_1 < \cdots < j_{n-r}\}$ be the complement of $I(\mu)$ in $\{1, \ldots, n\}$. Then*

$$\sigma_\square|_\mu = \sum_{j \in J(\mu)} t_j - \sum_{i=1}^{n-r} t_i. \tag{1}$$

This was proved in §5. Note that it implies $\sigma_\square = x_1 + \cdots + x_{n-r} - (t_1 + \cdots + t_{n-r})$.

More important for our purposes, this says that the elements $\sigma_\square|_\mu$ are all known polynomials, and as such, can be used for an inductive characterization of the polynomials $c^\nu_{\lambda\mu}$.

Lemma 2.2. *We have the "Pieri-Monk" formula*

$$\sigma_\square \cdot \sigma_\lambda = \sum_{\tilde\lambda \to \lambda} \sigma_{\tilde\lambda} + (\sigma_\square|_\lambda)\, \sigma_\lambda. \tag{2}$$

, where the notation $\tilde\lambda \to \lambda$ means $\tilde\lambda \supset \lambda$ and $|\tilde\lambda| = |\lambda| + 1$.

Lemma 2.3. *We have $c^\nu_{\lambda\mu} = 0$ unless $\nu \supset \lambda$ and $\nu \supset \mu$.*

Proof. Fix λ. The images of the classes σ_α such that $\Omega_\alpha \not\subset \Omega_\lambda$ form a basis for $H^*_T(X \smallsetminus \Omega_\lambda)$; this can be seen by the cell decomposition of X.

Consider $\sigma_\lambda \cdot \sigma_\mu = \sum c_{\lambda\mu}^\nu \sigma_\nu$. Restricting to $H_T^*(X \smallsetminus \Omega_\lambda)$, this equation becomes

$$0 = \sum_{\Omega_\nu \not\subset \Omega_\lambda} c_{\lambda\mu}^\nu \sigma_\nu.$$

Therefore $c_{\lambda\mu}^\nu = 0$ if $\Omega_\nu \not\subset \Omega_\lambda$, i.e., if $\nu \not\supset \lambda$. Note that the same proof works for any G/P. □

We can deduce Lemma 2.2 from Lemma 2.3. We need only consider those ν such that $\nu \supset \lambda$, and $|\nu| \leq |\lambda| + 1$. This means either $\nu \to \lambda$ (which is the classical Pieri rule), or $\nu = \lambda$. The latter is then a special case of Lemma 2.4:

Lemma 2.4. *We have* $c_{\lambda\mu}^\mu = \sigma_\lambda|_\mu$.

Proof. Restrict the relation $\sigma_\lambda \cdot \sigma_\mu = \sum c_{\lambda\mu}^\nu \sigma_\nu$ to p_μ, to obtain

$$\sigma_\lambda|_\mu \cdot \sigma_\mu|_\mu = \sum c_{\lambda\mu}^\nu \, \sigma_\nu|_\mu. \tag{3}$$

But $\sigma_\nu|_\mu = 0$ unless $\nu \subset \mu$ (since this is equivalent to $p_\mu \in \Omega_\nu$). Also, Lemma 2.3 says $c_{\lambda\mu}^\nu = 0$ unless $\nu \supset \mu$, so (3) reduces to

$$\sigma_\lambda|_\mu \cdot \sigma_\mu|_\mu = c_{\lambda\mu}^\mu \, \sigma_\mu|_\mu. \tag{4}$$

Since $\sigma_\mu|_\mu \neq 0$ (see Lemma 2.5), these cancel to leave $\sigma_\lambda|_\mu = c_{\lambda\mu}^\mu$. □

Lemma 2.5. *We have*

$$\sigma_\mu|_\mu = \prod (t_j - t_i),$$

where the product is over $i \in I(\mu)$, $j \in J(\mu)$, *with* $i < j$.

(This was also proved in §5.)

Lemma 2.6. *We have*

$$(\sigma_\square|_\nu - \sigma_\square|_\lambda) c_{\lambda\mu}^\nu = \sum_{\tilde{\lambda} \to \lambda} c_{\tilde{\lambda}\mu}^\nu - \sum_{\nu \to \tilde{\nu}} c_{\lambda\mu}^{\tilde{\nu}}.$$

This formula, due to Molev and Sagan, Okounkov, and others, is used to compute factorial Schur polynomials...

Proof. The key is to use associativity. We have

$$\begin{aligned} \sigma_\square \cdot (\sigma_\lambda \cdot \sigma_\mu) &= \sum_\rho \sigma_\square \cdot c_{\lambda\mu}^\rho \, \sigma_\rho \\ &= \sum_\rho c_{\lambda\mu}^\rho \left(\sum_{\tilde{\rho} \to \rho} \sigma_{\tilde{\rho}} + (\sigma_\square|_\rho) \, \sigma_\rho \right). \end{aligned}$$

On the other hand,

$$\begin{aligned}(\sigma_{\square}\cdot\sigma_\lambda)\cdot\sigma_\mu &= \left(\sum_{\tilde\lambda\to\lambda}\sigma_{\tilde\lambda} + (\sigma_{\square}|_\lambda)\,\sigma_\lambda\right)\cdot\sigma_\mu \\ &= \sum_{\tilde\lambda\to\lambda}\sum_\rho c^\rho_{\tilde\lambda\mu}\,\sigma_\rho + (\sigma_{\square}|_\lambda)\sum_\rho c^\rho_{\lambda\mu}\,\sigma_\rho.\end{aligned}$$

Equating coefficients of σ_ρ yields the formula. □

Lemma 2.7. *We have*

$$(\sigma_{\square}|_\lambda - \sigma_{\square}|_\mu)c^\lambda_{\lambda\mu} = \sum_{\tilde\mu\to\mu} c^\lambda_{\lambda\tilde\mu}.$$

Proof. By Lemma 2.4, the LHS is $(\sigma_{\square}|_\lambda - \sigma_{\square}|_\mu)\,\sigma_\mu|_\lambda$, and the RHS is $\sum_{\tilde\mu\to\mu}\sigma_{\tilde\mu}|_\lambda$. The statement then follows from the Pieri-Monk formula, Lemma 2.2. □

Lemma 2.8. *Letting λ' denote the partition conjugate to λ, the polynomial $c^{\nu'}_{\lambda'\mu'}$ is obtained from $c^\nu_{\lambda\mu}$ by the substitution $t_i \mapsto -t_{n+1-i}$.*

This is a basic property of the duality map $D : Gr(r, n) \to Gr(n-r, n)$.

The main theorem of [Knutson-Tao] is deduced from the following two statements:

Proposition 2.9. *The polynomials $c^\nu_{\lambda\mu}$ in $\Lambda = \mathbb{Z}[t_1, \ldots, t_n]$ satisfy and are uniquely determined by the following properties:*

(1) *For all λ,*

$$c^\lambda_{\lambda\lambda} = \prod(t_i - t_j),$$

the product over $i \in I(\lambda)$, $j \in J(\lambda)$ such that $i < j$.

(2) *For all λ, μ,*

$$(\sigma_{\square}|_\lambda - \sigma_{\square}|_\mu)c^\lambda_{\lambda\mu} = \sum_{\tilde\mu\to\mu} c^\lambda_{\lambda\tilde\mu}.$$

(3) *For all λ, μ, ν,*

$$(\sigma_{\square}|_\nu - \sigma_{\square}|_\lambda)c^\nu_{\lambda\mu} = \sum_{\tilde\lambda\to\lambda} c^\nu_{\tilde\lambda\mu} - \sum_{\nu\to\tilde\nu} c^{\tilde\nu}_{\lambda\mu}.$$

Theorem 2.10. *The polynomials $\sum_{puzzles}\prod_{eq.\ pieces}(t_j - t_i)$, i.e., the sums of weights of all puzzles of type $\Delta^\nu_{\lambda\mu}$, satisfy the three properties of Proposition 2.9.*

Remark 2.11. We have seen in Lemmas 2.1 – 2.8 that the $c^\nu_{\lambda\mu}$ satisfy the properties in Proposition 2.9. The key ingredient in proving these was the (equivariant) Pieri-Monk formula. Note that the classical formula for multiplication by $\sigma_\square$ does not determine the structure of the classical cohomology ring! (For this, one needs to describe multiplication by σ_k for all k.)

Proposition 2.9 and Theorem 2.10 together imply that the classical Littlewood-Richardson numbers are computed by the "classical" puzzles. Note, however, that the properties described in Proposition 2.9 are all vacuous in classical cohomology (with $t = 0$). This is an example of the utility of the additional structure of equivariant cohomology.

Exercise 2.12. State and prove an analogue of Proposition 2.9 for the flag variety $Fl(n)$. (In place of $\sigma_\square$, one needs all the divisor classes σ_{s_k}, for $1 \leq k \leq n-1$.)

Challenge: Find an analogue of Theorem 2.10 for $Fl(n)$!!

We now prove the uniqueness assertion of Proposition 2.9

Proof. First, we claim that $c^\lambda_{\lambda\mu} = \sigma_\mu|_\lambda$ (so $c^\lambda_{\lambda\mu} = 0$ unless $\lambda \supset \mu$). Indeed, this is true for $\lambda = \mu$ by (1) and Lemma 2.5. Now use induction on $|\lambda| - |\mu|$, together with (2). Note that the degrees of the polynomials decrease.

Next, we show that all the $c^\nu_{\lambda\mu}$ are determined. Indeed, $c^\nu_{\lambda\mu}$ is known if $\nu = \lambda$ by the previous paragraph.

`To be finished soon...` □

References

[Knutson-Tao] A. Knutson and T. Tao, "Puzzles and (equivariant) cohomology of Grassmannians," *Duke Math. J.* **119** (2003), no. 2, 221–260.

[Vakil] R. Vakil, "A geometric Littlewood-Richardson rule," preprint math.AG/0302294, *Annals of Math.*, to appear.