EQUIVARIANT INTERSECTION THEORY
§8: EQUIVARIANT COHOMOLOGY OF TORIC VARIETIES

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1. Basic facts about toric varieties

Let $T \cong (\mathbb{C}^*)^n$ be a torus, with its lattice of characters M. Write $\Lambda = \mathrm{Sym}^\bullet_{\mathbb{Z}} M$ and $N = M^\vee = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$.

A *rational polyhedral pointed cone* in the vector space $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ is a set $\sigma = \{v = \sum t_i v_i \mid t_i \geq 0\}$ for some finite set of vectors $v_i \in N$, such that for nonzero v, $v \in \sigma$ implies $-v \notin \sigma$. (Thus the origin is the vertex of the cone.) Write $\sigma = \langle v_1, \ldots, v_r \rangle$, and say σ is generated by the v_i. A *face* of a cone σ is an intersection of σ with some hyperplane; equivalently, it is a cone generated by some subset of $\{v_i\}$. Write $\tau \leq \sigma$ if τ is a face of σ. A *fan* Δ in N is a finite collection of rational polyhedral pointed cones in $N_{\mathbb{R}}$, such that each face of a cone in Δ is also in Δ, and the intersection of two cones in Δ is a face of each. The *support* of Δ is $|\Delta| = \bigcup_{\sigma \in \Delta} \sigma$.

Such a fan gives rise to a toric variety $X = X(\Delta)$. This is obtained by glueing affine varieties

$$U_\sigma = \mathrm{Spec}(\mathbb{C}[M \cap \sigma^\vee]),$$

where $\sigma^\vee$ is the dual cone:

$$\sigma^\vee = \{u \in M_{\mathbb{R}} \mid \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma\}.$$

One checks that U_τ is an open subset of U_σ when $\tau \leq \sigma$, so $X(\Delta)$ can be described by glueing maximal cones of Δ along their common faces.

These toric varieties are all normal and Cohen-Macaulay.

For $\sigma = \{0\}$, $U_{\{0\}} = \mathrm{Spec}\, \mathbb{C}[M] = T$ is a dense open subset. This torus acts on $X(\Delta)$, extending the action $T = U_{\{0\}}$. The variety $X(\Delta)$ is *complete* (compact in the classical topology) if and only if $|\Delta| = N_{\mathbb{R}}$.

$X(\Delta)$ is nonsingular if and only if each $\sigma \in \Delta$ is generated by part of a basis for N. A toric variety is called *simplicial* if each cone $\sigma \in \Delta$ is simplicial, i.e., σ is generated by part of a basis for $N_{\mathbb{R}}$.

For each $\sigma \in \Delta$, there is a T-invariant subvariety $V(\sigma) \subset X$, with $\mathrm{codim}_{\mathbb{C}}(V(\sigma), X) = \dim_{\mathbb{R}}(\sigma)$. In fact, these are all the T-invariant subvarieties. We will show that when $X(\Delta)$ is complete and nonsingular, the

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classes $[V(\sigma)]$ generate $H_*(X(\Delta))$. (In fact, this is true for arbitrary toric varieties.)

2. Cohomology

Our goal is to compute H_T^*X (and H^*X) for complete nonsingular toric varieties $X = X(\Delta)$.

The torus $T = \operatorname{Spec} \mathbb{C}[M]$ acts on X, taking each open set U_σ to itself. The map $T \times U_\sigma \to U_\sigma$ defining the action corresponds to the map of $\mathbb{C}$-algebras

$$\begin{array}{rcl} \mathbb{C}[M \cap \sigma^\vee] & \to & \mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[M \cap \sigma^\vee] \\ \chi^u & \mapsto & \chi^u \otimes \chi^u. \end{array}$$

Each cone σ of dimension k corresponds to a T-invariant subvariety $V(\sigma)$ of codimension k, described locally as follows. In U_τ, the ideal of $V(\sigma) \cap U_\tau$ in $\mathbb{C}[M \cap \tau^\vee]$ is generated by

$$\{\chi^u \mid u \in M \cap \tau^\vee, \text{ such that } \langle u, v \rangle > 0 \text{ for } v \in \operatorname{rel.int}(\sigma)\}.$$

Thus there is a class $[V(\sigma)]^T \in H_T^*X$.

In particular, let $\rho_1, \dots, \rho_d$ be th *rays* of Δ, i.e., the one-dimensional cones. Let $D_1, \dots, D_d$ be the corresponding divisors ($D_i = V(\rho_i)$), and let $v_1, \dots, v_d$ be the *primitive generators* of the rays – that is, $v_i \in N$ is the lattice point on ρ_i nearest to the origin. If $\sigma = \langle v_{i_1}, \dots, v_{i_k} \rangle$, then $V(\sigma) = D_{i_1} \cap \cdots \cap D_{i_k}$, so $[V(\sigma)]^T = [D_{i_1}]^T \cdots [D_{i_k}]^T$.

Claim: The classes $[V(\sigma)]$ generate $H_*X = H^*X$, and so the classes $[V(\sigma)]^T$ generate H_T^*X over $\Lambda = \operatorname{Sym}^\bullet M$.

(Challenge: Is there an “easy” proof of this fact?)

Remark 2.1. If X is projective, one can find an ordering of the maximal cones $\sigma_1, \dots, \sigma_m$ such that the classes $[V(\tau_i)]$ form a basis, for certain $\tau_i \leq \sigma_i$. (Very roughly, the property one needs is *shellability* of the complex corresponding to the fan Δ. When X is projective, Δ arises from a polytope, and one can choose a height function to produce an appropriate ordering of the faces of this polytope. See [Fulton, §5.2].)

Question: For general X (not necessarily projective), does some subset of the classes $[V(\sigma)]$ give a basis for H^*X?

The T-fixed points are $p_\sigma = V(\sigma)$ for maximal cones σ. (So $\dim \sigma = \dim X = n$.) Let a_k be the number of k-dimensional cones. One expects H^*X is free on a_n generators, since H_T^*X is free on a_n generators, injecting into $H_T^*X^T = \Lambda^{\oplus a_n}$.

Proposition 2.2. $H^*X \cong \mathbb{Z}[X_1, \dots, X_d]/(I + J)$ *via* $X_i \mapsto [D_i]$. *The ideal* I *is generated by the monomials* $X_{i_1} \cdots X_{i_r}$ *such that* $v_{i_1}, \dots, v_{i_r}$ *are not contained in a single cone. (Equivalently,* $D_{i_1} \cap \cdots \cap D_{i_r} = \emptyset$*.)* J *is generated by the elements* $\sum_{i=1}^d \langle u, v_i \rangle X_i$*, for all* $u \in M$*.*

Remark 2.3. The ring $\mathbb{Z}[X]/I$ depends only on the combinatorial type of the fan; it is called the *Stanley-Reisner ring* of X (or of Δ). In fact, we will see $H_T^*X \cong \mathbb{Z}[X]/I$.

For $u \in M$, χ^u is a rational function on X, and $\mathrm{Div}(\chi^u) = \sum_{i=1}^d \langle u, v_i\rangle D_i$. Thus there is a well-defined map $\mathbb{Z}[X]/(I+J) \to H^*X$.

Let A be any commutative ring. (We will be most interested in the cases $A = \mathbb{Z}$ and $A = \Lambda$; if X is allowed to be simplicial, we could take $A = \mathbb{Q}$.) Let $\tau = \langle v_{i_1}, \dots, v_{i_k}\rangle$ be a cone in Δ of dimension k, and define $A[\tau] := A[X_{i_1}, \dots, X_{i_k}]$, a free A-algebra. Note that this is a quotient of $A[X]/I_A$, where $I_A \subset A[X]$ has the same generators as the ideal I above, the monomials $X_{i_1} \cdots X_{i_r}$ such that $v_{i_1}, \dots, v_{i_r}$ are not contained in any cone; indeed, there are no relations on $X_{i_1}, \dots, X_{i_k}$, and the map is given by

$$X_i \mapsto \begin{cases} X_i & \text{if } i = i_a \text{ for } 1 \le a \le k; \\ 0 & \text{otherwise.} \end{cases}$$

Define $C_k = C_k(A) = \bigoplus_{\dim \tau = k} A[\tau]$.

We will need a basic lemma:

Lemma 2.4. *With notation as above, there is an exact sequence of $A[X]/I$-modules*

$$0 \to A[X]/I \to C_n(A) \xrightarrow{d_n} C_{n-1}(A) \to \cdots \to C_1(A) \xrightarrow{d_1} C_0(A) \to 0.$$

For cones τ and γ of respective dimensions k and $k-1$, the differential d_k maps $A[\tau]$ to $A[\gamma]$ by 0 if $\gamma \not\leq \tau$. If $\tau = \langle v_{i_1}, \dots, v_{i_k}\rangle$ and $\gamma = \langle v_{i_1}, \dots, \widehat{v_{i_p}}, \dots, v_{i_k}\rangle$, then d_k is $(-1)^p\, \varphi$, where φ is the natural map $A[X_{i_1}, \dots, X_{i_k}] \to A[X_{i_1}, \dots, \widehat{X_{i_p}}, \dots, X_{i_k}]$.

Proof. (Rough sketch, to be filled in later.) First, verify that $d_{k-1} \circ d_k = 0$ – this is the usual computation for complexes of free modules. Next, show exactness for each graded piece of the complex, using the grading by $\mathbb{N}^d$. (A monomial $X_1^{m_1} \cdots X_d^{m_d}$ has degree $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{N}^d$.)

One sees that all graded pieces are 0 except for those where the set of v_i such that $m_i > 0$ spans a cone τ. Say $\dim \tau = j$; then $C_k(A)_{\mathbf{m}}$ is free (over A) on all k-dimensional cones γ containing τ. The complex $C_\bullet(A)_{\mathbf{m}}$ is the chain complex computing the reduced cohomology $\tilde{H}^*(S^{n-j-1})$, for the unit sphere $S^{n-j-1} \subset N_{\mathbb{R}}/\mathbb{R} \cdot \tau$. □

Corollary 2.5.

$$\sum_{m=0}^{\infty} \mathrm{rk}_A (A[X]/I)_m\, t^m \quad = \quad \sum_{i=0}^{n} \frac{(-1)^{n-i} a_i}{(1-t)^i}.$$

Exercise 2.6. Prove this. [Hint: Note for $C_i(A)$, one gets $a_i \binom{m+i-1}{i-1}$.]

Remark 2.7. Note that $A[X]/I$ is projective over A, so it is free for $A = \mathbb{Z}$ (a PID), and by the Quillen-Suslin theorem, also for $A = \Lambda$. (Recall that the Quillen-Suslin theorem states that if A is a principal ideal ring, then any projective module over $A[X]$ is free.)

For $u \in M$, write $z(u) = \sum \langle u, v_i \rangle X_i \in \mathbb{Z}[X]$. Thus J is the ideal $(z(u) \,|\, u \in M)$. (It clearly suffices to take u in a basis for M.) Let $z'(u) = z(u) - u \in \Lambda[X]$, and let $J' = (z'(u) \,|\, u \in M) \subset \Lambda[X]$. Write $I' = I\,\Lambda[X]$.

Our goal is still to construct isomorphisms

$$\begin{aligned} \mathbb{Z}[X]/(I+J) &\xrightarrow{\sim} H^*X && (1)\\ X_i &\mapsto [D_i] && (2) \end{aligned}$$

and

$$\begin{aligned} \Lambda[X]/(I'+J') &\xrightarrow{\sim} H^*_T X && (3)\\ X_i &\mapsto [D_i]^T. && (4) \end{aligned}$$

We have already seen that the map of (1) is well-defined. For the second map, note that I' maps to 0 for the same reason: $[D_{i_1}]^T \cdots [D_{i_k}]^T = 0$ if $D_{i_1} \cap \cdots \cap D_{i_k} = \emptyset$. Also, for $u \in M$ there is an equivariant line bundle L_u on a point, with $c_1^T(L_u) = u$. Pull L_u back to X; then χ^u is a section of L_u. Since $\mathrm{Div}(\chi^u) = \sum \langle u, v_i \rangle D_i$ (equivariantly), $u = c_1^T(L_u) = \sum \langle u, v_i \rangle [D_i]^T$ in $H^*_T X$. Therefore J' maps to 0, and the map of (3) is well-defined.

Note that there is also the map $H^*_T X \to H^* X$ given by $[D_i]^T \mapsto [D_i]$.

The fact that $H^*_T X$ is the Stanley-Reisner ring $\mathbb{Z}[X]/I$ will follow from (3) and the following lemma:

Lemma 2.8. *There is a canonical isomorphism*

$$\mathbb{Z}[X]/I \xrightarrow{\sim} \Lambda[X]/(I'+J').$$

Proof. Take a basis $u_1, \dots, u_n$ for M, so $\Lambda = \mathbb{Z}[u_1, \dots, u_n]$, and

$$\begin{aligned} \Lambda[X]/(I'+J') &= \mathbb{Z}[X_1, \dots, X_d, u_1, \dots, u_n]/(I'+J')\\ &= (\mathbb{Z}[X_1, \dots, X_d]/I)\,[u_1, \dots, u_n]/(\sum_i \langle u_j, v_i \rangle X_i - u_j)_{j=1}^n\\ &= \mathbb{Z}[X_1, \dots, X_d]/I. \end{aligned}$$

Regarding $\mathbb{Z}[X]/I$ as a Λ-module, this is a Λ-algebra isomorphism, via $u \mapsto z(u) = \sum \langle u, v_i \rangle X_i$. □

Note that $u_1 - z(u_1), \dots, u_n - z(u_n)$ is a regular sequence in $\Lambda[X]/I$. For $A = \mathbb{Z}$, the complex of Lemma 2.4 is an exact sequence of Λ-modules, where the module structure comes from $\Lambda \to \mathbb{Z}[X]/I$. Since we have the Hilbert series in Corollary 2.5, we know the ranks of these modules, for Λ regarded as degree 0, or with its usual grading ($\deg(u_i) = 1$).

Proposition 2.9. *$\Lambda[X]/(I'+J')$ is a finitely generated free Λ-module of rank a_n.*

Proof. The rank count comes from the above discussion. Indeed, since X is nonsingular, the module $C_n(\mathbb{Z})$ is isomorphic to $\Lambda^{\oplus a_n}$. (For maximal cones σ, $\mathbb{Z}[\sigma] \cong \Lambda$.) Now $\Lambda[X]/(I'+J') = \mathbb{Z}[X]/I$, and the exact sequence

$$0 \to \mathbb{Z}[X]/I \to \mathbb{C}_n(\mathbb{Z}) \xrightarrow{d_n} \mathbb{C}_{n-1}(\mathbb{Z}) \to \cdots$$

shows $\mathrm{rk}_\Lambda(\mathbb{Z}[X]/I) = \mathrm{rk}_\Lambda(C_n)$, since the image of d_n is a torsion Λ-module.

For freeness, we will show $\mathbb{Z}[X]/I$ is a projective Λ-module, and apply the Quillen-Suslin theorem. Let τ be a k-dimensional cone, and note that $\mathbb{Z}[\tau] \cong \Lambda/(u_{k+1},\dots,u_n) \cong \mathbb{Z}[u_1,\dots,u_k]$ as Λ-modules, where $\tau = \langle v_1,\dots,v_k\rangle$, and $u_i = v_i^*$ is the dual basis. This has projective dimension $n-k$ over Λ, since the complex of Lemma 2.4 is a Koszul resolution. Thus the projective dimension of $C_k(\mathbb{Z})$ is also $n-k$. Looking at this complex, we see the projective dimension of $\mathbb{Z}[X]/I$ is 0, i.e., it is a projective Λ-module. □

Corollary 2.10. *$\mathbb{Z}[X]/(I+J)$ is a finitely-generated free module of rank a_n over $\mathbb{Z}$.*

Proof. Indeed, $\mathbb{Z}[X]/(I+J) = (\Lambda[X]/(I'+J')) \otimes_\Lambda \mathbb{Z}$. □

Theorem 2.11. *There are isomorphisms*

$$\Lambda[X]/(I'+J') \quad\cong\quad H_T^*X$$

and

$$\mathbb{Z}[X]/(I+J) \quad\cong\quad H^*X.$$

Proof. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}[X]/I \arrow[r] \arrow[dl] \arrow[dd] & C_n(\mathbb{Z}) = \bigoplus\limits_{\dim\sigma=n} \mathbb{Z}[\sigma] \arrow[r, "d_n"] \arrow[dd, "\varphi"'] & C_{n-1}(\mathbb{Z}) \\
\Lambda[X]/(I'+J') \arrow[dr] & & & \\
& H_T^*X \arrow[r] & H_T^*X^T = \bigoplus\limits_{\dim\sigma=n} H_T^*(p_\sigma) \;\cong\; \Lambda^{\oplus a_n}. &
\end{tikzcd}$$

(As before, for n-dimensional cones σ, $\mathbb{Z}[\sigma] = \mathbb{Z}[X_{i_1},\dots,X_{i_n}] \xrightarrow{\sim} \Lambda$ via $X_{i_j} \mapsto u_{i_j}$, where $u_{i_j} = v_{i_j}^*$.)

Define the map φ by $X_{i_j} \mapsto u_{i_j}$ on the factor $\mathbb{Z}[\sigma] \to H_T^*(p_\sigma)$. This is an isomorphism.

We first show that the diagram commutes. Going counterclockwise around the square and projecting on the factor $H_T^*(p_\sigma)$, we have $X_i \mapsto [D_i]^T \mapsto [D_i]^T|_{p_\sigma}$; going clockwise, we have $X_i \mapsto X_i \mapsto u_i$. Thus the claim is that $[D_{i_j}]^T \mapsto u_{i_j}$ under the map $H_T^*X \to H_T^*(p_\sigma)$.

To see this, consider the open neighborhood $U_\sigma = \mathrm{Spec}(\mathbb{C}[u_{i_1},\dots,u_{i_n}]) \cong \mathbb{C}^n$ of p_σ. Factor the restriction map:

$$H_T^*X \to H_T^*(U_\sigma) = H_T^*\mathbb{C}^n \xrightarrow{\sim} H_T^*(p_\sigma).$$

In U_σ, the divisor D_{i_j} is defined by the equation $u_{i_j} = 0$, so $[D_{i_j}]^T \mapsto u_{i_j} \in H_T^*(U_\sigma) \cong H_T^*(p_\sigma)$.

Thus in the above diagram, the vertical arrow $\mathbb{Z}[X]/I \to H_T^*X$ is injective.

Next, we show that the map $\mathbb{Z}[X]/I \xrightarrow{\sim} \Lambda[X]/(I' + J') \to H_T^*X$ is surjective. That is, we claim H_T^*X is generated as a Λ-module by the classes $[V(\tau)]^T$. To see this, take a birational toric morphism $\pi : X' \to X$, with X' nonsingular and projective. (To construct X', subdivide the fan Δ to obtain a fan Δ' which is the normal fan of some polytope.) In the projective case, we know this (Remark 2.1). In fact, there is a basis of classes $[V(\sigma')]$ for H^*X', so the corresponding equivariant classes $[V(\sigma')]$ form a basis for H_T^*X'.

The map $\pi_* : H_T^*X' \to H_T^*X$ is surjective, since $\pi_*(\pi^*c) = c \cdot \pi_*(1) = c$ for any $c \in H_T^*X$. Finally,

$$\pi_*[V(\sigma')]^T = \begin{cases} [V(\sigma)]^T & \text{if } \pi(\sigma') \subset \sigma, \text{ with } \dim(\sigma') = \dim(\sigma); \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, in the first case, π maps $V(\sigma')$ birationally to $V(\sigma)$, while in the second case, the image has strictly smaller dimension. (Here we are using the same notation π for the map of varieties and the corresponding map of fans.) □

Remark 2.12. (Mention something about toric resolution of singularities?)

Corollary 2.13. *$\mathbb{Z}[X]/(I + J) \xrightarrow{\sim} H^*X$.*

Proof. This is just $(\Lambda[X]/(I' + J')) \otimes_\Lambda \mathbb{Z} \xrightarrow{\sim} (H_T^*X) \otimes_\Lambda \mathbb{Z}$. □

Corollary 2.14. *The map $H_T^*X \to H_T^*X^T$ is injective, and its image is the set of tuples (f_σ) such that $\dim(\sigma) = n$ and $f_\sigma - f_{\sigma'}$ is divisible by a generator of $M \cap \tau^\perp$ if $\tau = \sigma \cap \sigma'$ has dimension $n - 1$.*

Proof. This follows from the top exact sequence of the diagram in the proof of Theorem 2.11. □

Remark 2.15. The exact sequence in the above diagram, with $\mathbb{Z}$ coefficients, was originally used in [Danilov] to compute the (classical) cohomology ring.

Corollary 2.16. *(*[Brion-Vergne]*) H_T^*X is the ring of continuous functions f on $N_\mathbb{R} = |\Delta|$ such that for all n-dimensional cones σ, $f|_\sigma$ is given by an element of Λ. (That is, H_T^* is the ring of piecewise-polynomial functions on Δ.)*

Remark 2.17. For an arbitrary fan Δ, not necessarily complete, this ring of continuous piecewise-polynomial functions is isomorphic to the "equivariant operational Chow ring" $A_T^*(X(\Delta))$.

Remark 2.18. The same theorems hold when $X(\Delta)$ is simplicial (but not necessarily nonsingular) if one takes coefficients in $\mathbb{Q}$ instead of $\mathbb{Z}$. For a general toric variety, there is a spectral sequence constructed by Brylinski and Zhang. (Reference?)

Remark 2.19. We only needed the exact sequence of Lemma 2.4 for $A = \mathbb{Z}$. We saw that $\mathbb{Z}[X]/I$ is free over Λ, where the module structure is by $\Lambda \to \mathbb{Z}[X]$, $u \mapsto \sum \langle u, v_i \rangle X_i$. By a standard argument, $\mathbb{Z}[X]/I$ has a basis of homogeneous elements.

If there are b_j elements degree j in the basis, the Hilbert series of $\mathbb{Z}[X]/I$ is

$$\sum \frac{b_j\, t^j}{(1-t)^n} = \sum (-1)^{n-i} \frac{a_i}{(1-t)^i}.$$

Equivalently,

$$\sum b_j\, t^j = \sum_{i=0}^{n} a_i\, (t-1)^{n-i}. \tag{5}$$

This shows that there are a finite number of nonzero b_j's. Putting $t = 1$, we get $\sum b_j = a_n$. Thus $\Lambda[X]/I$ is free over Λ of rank a_n.

The numbers b_j are the even Betti numbers; the relation (5) gives a correspondence between the numbers b_j and a_i. (In fact, the Euler characteristic is $\chi(X) = a_n$.)

For more on the Stanley-Reisner ring $\mathbb{Z}[X]/I$ (e.g., it is Cohen-Macaulay), see [Stanley].

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