POSITIVITY IN THE COHOMOLOGY OF FLAG BUNDLES (AFTER GRAHAM)

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In [Gr], Graham proves that the structure constants of the equivariant cohomology ring of a flag variety are positive combinations of monomials in the roots:

Theorem 1 ([Gr, Cor. 4.1])**.** *Let $X = G/B$ be the flag variety for a complex semisimple group G with maximal torus $T \subset B$, and let $\{\sigma_w \in H_T^*X \,|\, w \in W\}$ be the basis of (B-invariant) Schubert classes. Let $\{\alpha_i\}$ be the simple roots which are* negative *on B. Then in the expansion*

$$\sigma_u \cdot \sigma_v = \sum_w c_{uv}^w \, \sigma_w,$$

the coefficients c_{uv}^w are in $\mathbb{Z}_{\geq 0}[\alpha]$.

Graham deduces this from a more general result about varieties with finitely many unipotent orbits, which is proved using induction and a calculation in the rank-one case.

The goal of this note is to give a short, geometric proof of Graham's positivity theorem, based on a transversality argument. Here I only discuss type A, but other types work as well. (For a type-uniform version, a change of language is needed: one should replace vector bundles with corresponding principal G-bundles.)

Throughout, Fl denotes the variety of (complete) flags in $\mathbb{C}^n$, and if $V \to X$ is a vector bundle, $\mathbf{Fl}(V) \to X$ is the bundle of flags in V.

Recall that for $T' \cong (\mathbb{C}^*)^n$, we have $BT' = (\mathbb{P}^\infty)^{\times n}$ and $H_{T'}^* Fl = H^*(ET' \times^{T'} Fl) = H^*\mathbf{Fl}(E')$, where E' is the sum of the n tautological line bundles on BT'. The *effective* action on Fl is by $T \cong (\mathbb{C}^*)^n/\mathbb{C}^*$, and the classifying space for this torus is $BT = (\mathbb{P}^\infty)^{\times n-1}$. We will usually deal with the effective torus.

Let $\mathbb{P} = \mathbb{P}^m \times \cdots \times \mathbb{P}^m$ ($n-1$ factors), with $m \gg 0$, and write $H^*\mathbb{P} = \mathbb{Z}[\alpha_1, \ldots, \alpha_{n-1}]$. (We always assume that m is large enough so that there are no relations in the relevant degrees.) Let $M_i = p_1^*(\mathcal{O}(-1))$ be the tautological bundle on the ith factor, and let $\alpha_i = -c_1(M_i)$. Note that the class of any effective cycle in $H^*\mathbb{P}$ is a positive polynomial in the α's.

Let

$$L_i = M_1 \otimes \cdots \otimes M_{i-1}$$

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for $1 \leq i \leq n$ (so $L_1 = \mathcal{O}$ is the trivial line bundle), and let $E_i = L_1 \oplus \cdots \oplus L_i$. Thus we have a flag $E_\bullet$ in $E = E_n$. Let $\widetilde{E}_\bullet$ be the opposite flag, with $\widetilde{E}_i = L_n \oplus \cdots \oplus L_{n+1-i}$. In the flag bundle $p : \mathbf{Fl}(E) \to \mathbb{P}$, with universal quotient flags $Q_\bullet$, we have Schubert loci $\mathbf{\Omega}_w = \Omega_w(E_\bullet \to Q_\bullet)$, defined by

$$\mathbf{\Omega}_w = \{x \in \mathbf{Fl}(E) \mid \operatorname{rk}(E_p \to Q_q) \leq \#(i \leq q \mid w(i) \leq p)\}. \tag{1}$$

Opposite Schubert loci $\widetilde{\mathbf{\Omega}}_w = \Omega_w(\widetilde{E}_\bullet \to Q_\bullet)$ are defined similarly. We also have "Schubert cell bundles" $\mathbf{\Omega}_w^o$: these are affine bundles over $\mathbb{P}$ which are open in the corresponding loci $\mathbf{\Omega}_w$, and are defined by replacing the inequality in (1) with an equality.

The classes $[\mathbf{\Omega}_w]$ form a basis for $H^*\mathbf{Fl}(E)$ over $H^*\mathbb{P}$, as w ranges over S_n. Writing

$$[\mathbf{\Omega}_u] \cdot [\mathbf{\Omega}_v] = \sum_w c_{uv}^w [\mathbf{\Omega}_w]$$

with $c_{uv}^w \in H^*\mathbb{P}$, our main result is the following:

Proposition 2. *The polynomials c_{uv}^w are* positive*, that is, $c_{uv}^w \in \mathbb{Z}_{\geq 0}[\alpha_1, \ldots, \alpha_{n-1}]$.*

This implies Graham's positivity theorem (in this context), since $\mathbb{P}$ approximates BT for m sufficiently large, and $\mathbf{Fl}(E)$ approximates $ET \times^T Fl$, with $[\mathbf{\Omega}_w]$ corresponding to the equivariant class σ_w. (See [Fu2, §9].)

Proposition 2 is a consequence of a transversality statement:

Proposition 3. *For any $u, v, w \in S_n$, there is a translate $\mathbf{\Omega}'_v$ of $\mathbf{\Omega}_v$ by the action of a connected algebraic group such that $\mathbf{\Omega}'_v$ intersects $\mathbf{\Omega}_u$ and $\widetilde{\mathbf{\Omega}}_{w_0 w}$ properly and generically transversally.*

To deduce Proposition 2, first note that the intersection $\mathbf{\Omega}_u \cap \widetilde{\mathbf{\Omega}}_{w_0 w}$ is always proper and generically transverse. Thus Proposition 3 says that $\mathbf{\Omega}'_v \cap (\mathbf{\Omega}_u \cap \widetilde{\mathbf{\Omega}}_{w_0 w})$ is proper and generically transverse. By [Fu1, Ex. (8.1.11)], this says that

$$[\mathbf{\Omega}_v] \cdot [\mathbf{\Omega}_u] \cdot [\widetilde{\mathbf{\Omega}}_{w_0 w}] = [\mathbf{\Omega}'_v \cap \mathbf{\Omega}_u \cap \widetilde{\mathbf{\Omega}}_{w_0 w}].$$

(Since $\mathbf{\Omega}'_v = g \cdot \mathbf{\Omega}_v$ for some g in a connected algebraic group, $[\mathbf{\Omega}'_v] = [\mathbf{\Omega}_v]$.) Using relative Poincaré duality (see e.g. [Fu2, §A.6]), we have

$$c_{uv}^w = p_*([\mathbf{\Omega}_u] \cdot [\mathbf{\Omega}_v] \cdot [\widetilde{\mathbf{\Omega}}_{w_0 w}]) = p_*([\mathbf{\Omega}_u \cap \mathbf{\Omega}'_v \cap \widetilde{\mathbf{\Omega}}_{w_0 w}]).$$

This is an effective class in $H^*\mathbb{P}$, so Proposition 2 follows.

Proof of Proposition 3. This is essentially an application of Kleiman's theorem. The endomorphism bundle

$$\begin{aligned} \mathbf{End}(E) &= \bigoplus_{i,j} L_i^{-1} \otimes L_j \\ &= \left(\bigoplus_{i<j} M_i \otimes \cdots \otimes M_{j-1} \right) \oplus \mathcal{O}^{\oplus n} \oplus \left(\bigoplus_{i>j} M_j^{-1} \otimes \cdots \otimes M_{i-1}^{-1} \right) \end{aligned}$$

has global sections in lower-triangular matrices, so the group B of (invertible) lower-triangular matrices acts on $\mathbf{Fl}(E)$, fixing the flag $\widetilde{E}_\bullet$ and stabilizing $\widetilde{\mathbf{\Omega}}_{w_0\,w}$. (Note that the entries of a matrix in B are global sections of the line bundles $M_j^{-1} \otimes \cdots \otimes M_{i-1}^{-1}$, i.e., multi-homogeneous polynomials. This is a connected group over $\mathbb{C}$, acting on a fiber $p^{-1}(x) \subset \mathbf{Fl}(E)$ by first evaluating the sections at x.)

Now let $H = (GL_{m+1})^{\times(n-1)}$, and for $b \in B$, let b_x be the evaluation at $x \in \mathbb{P}$ (so the action of b on $p^{-1}(x)$ is by b_x). Consider the semidirect product $\Gamma = B \rtimes H$, given by $(h \cdot b \cdot h^{-1})_x = b_{h^{-1}\cdot x}$. (This action of H on B is just the usual action of H on global sections of the equivariant vector bundle $\mathbf{End}(E)$.)[1] As a semidirect product of connected groups, Γ is a connected algebraic group. We claim that the locus $\widetilde{\mathbf{\Omega}}^o_{w_0\,w}$ is homogeneous for the action of Γ. Indeed, B acts transitively on each fiber of $\widetilde{\mathbf{\Omega}}^o_{w_0\,w}$, and the action of H on $\mathbf{Fl}(E)$ induces a transitive action on the set of fibers of $\widetilde{\mathbf{\Omega}}^o_{w_0\,w}$. (The line bundles L_i are equivariant for H, so H preserves the flag $\widetilde{E}_\bullet$, and therefore acts on $\widetilde{\mathbf{\Omega}}_{w_0\,w}$.)

Finally, note that $\mathbf{\Omega}^o_u$ and $\widetilde{\mathbf{\Omega}}^o_{w_0\,w}$ intersect transversally, as do $\mathbf{\Omega}^o_v$ and $\widetilde{\mathbf{\Omega}}^o_{w_0\,w}$. The proposition follows from Lemma 4 below, taking $U = \mathbf{\Omega}_u$, $V = \mathbf{\Omega}_v$, and $W = \widetilde{\mathbf{\Omega}}_{w_0\,w}$, with their stratifications by Schubert loci. $\square$

Lemma 4. *Let X be a nonsingular variety over a field of characteristic 0, with an action of a connected algebraic group Γ. Let $U, V, W \subset X$ be subvarieties with stratifications*

$$\begin{gathered} U_0 \subset \cdots \subset U_\ell = U, \\ V_0 \subset \cdots \subset V_m = V, \\ W_0 \subset \cdots \subset W_n = W, \end{gathered}$$

with each stratum $U_i \smallsetminus U_{i-1}$ nonsingular. Assume also that Γ acts on W, with each stratum $W_i \smallsetminus W_{i-1}$ a disjoint union of homogeneous spaces.

If $U_i \smallsetminus U_{i-1}$ meets $W_k \smallsetminus W_{k-1}$ transversally for all i, k, and similarly for $V_j \smallsetminus V_{j-1}$ and $W_k \smallsetminus W_{k-1}$, then there is an element $g \in \Gamma$ such that $g \cdot V$ meets $U \cap W$ properly and generically transversally.

This can be deduced from results found in [Sp]; see also [Si] for a vast generalization. The proof of this version is quite short, so we give it here.

Proof. Applying Kleiman's theorem (cf. [Ha, III.10.8]) to the pairs $(U_i \smallsetminus U_{i-1} \cap W_k \smallsetminus W_{k-1})$ and $(V_j \smallsetminus V_{j-1} \cap W_k \smallsetminus W_{k-1})$ inside the homogeneous space $W_k \smallsetminus W_{k-1}$, we can choose $g \in \Gamma$ such that each intersection

$$\begin{aligned} &(U_i \smallsetminus U_{i-1} \cap W_k \smallsetminus W_{k-1}) \cap g \cdot (V_j \smallsetminus V_{j-1} \cap W_k \smallsetminus W_{k-1}) \\ &= (U_i \smallsetminus U_{i-1} \cap W_k \smallsetminus W_{k-1}) \cap (g \cdot V_j \smallsetminus g \cdot V_{j-1} \cap W_k \smallsetminus W_{k-1}) \end{aligned}$$

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[1] Alternatively, one could take Γ to be the subgroup of $\mathrm{Aut}(\mathbf{Fl}(E))$ generated by the images of B and G via the homomorphisms corresponding to their respective actions.

is transverse, so the intersection $U \cap W \cap g \cdot V$ is proper and generically transverse. □

Remark 5. All that is required in the proof of Proposition 3 are the facts that $\mathbb{P}$ is homogeneous for the action of an algebraic group H, and L_i are H-equivariant line bundles such that $L_i^{-1} \otimes L_j$ is globally generated for $i > j$.

Remark 6. To recover the result that for (type A) equivariant Schubert calculus, the structure constants c_{uv}^w are in $\mathbb{Z}_{\geq 0}[t_2 - t_1, \ldots, t_n - t_{n-1}]$, let $\mathbb{P}' = (\mathbb{P}^{m'})^{\times n}$ and choose a map $\varphi : \mathbb{P}' \to \mathbb{P}$ such that $\varphi^* M_i = M_i' \otimes (M_{i+1}')^{-1}$, where M_i' is the tautological bundle on the ith factor of $\mathbb{P}'$, with $t_i = c_1(M_i')$. (Note that φ will not be holomorphic!)

The T'-equivariant class of a Schubert variety (for $T' = (\mathbb{C}^*)^n$) can be identified with the class of the locus $\Omega_w(E'_\bullet \to Q_\bullet) \subset \mathbf{Fl}(E')$, where $E_i' = M_1' \oplus \cdots \oplus M_i'$ is a flag of bundles on $\mathbb{P}'$. Since this is $\varphi^{-1}\boldsymbol{\Omega}_w$, the equivariant structure constants are $\varphi^* c_{uv}^w$, which are positive in the variables $\varphi^* \alpha_i = t_{i+1} - t_i$.

Remark 7. The naive choice of flag, with $F_i = M_1 \oplus \cdots \oplus M_i$, does not work: The bundle $\mathbf{End}(F)$ has only diagonal global sections, so the corresponding loci $\boldsymbol{\Omega}_w^o$ are not homogeneous. This explains why one does not see positivity over $\mathbb{P}'$.

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References

[Fu1] W. Fulton, *Intersection Theory*, second edition, Springer, 1998.

[Fu2] W. Fulton, *Equivariant cohomology in algebraic geometry*, lectures at Columbia University, notes by D. Anderson, 2007. Available at `www.math.lsa.umich.edu/~dandersn/eilenberg`.

[Gr] W. Graham, "Positivity in equivariant Schubert calculus," *Duke Math. J.* **109** (2001), 599–614.

[Ha] R. Hartshorne, *Algebraic Geometry*, Springer, 1977.

[Si] S. Sierra, "A general homological Kleiman-Bertini theorem," `arXiv:0705.0055v2 [math.AG]`.

[Sp] R. Speiser, "Transverality theorems for families of maps," *Algebraic Geometry (Sundance 1986)*, 235–252, Lecture Notes in Math. **1311**, 1988.

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