Math/Stats 425, Sec. 1, F 2003: Introduction to Probability

Final Examination. December 17, 2003: Solutions.

1. a) An urn contains N green balls and M blue balls. You draw a set of n balls out of the urn. Let $\mathbf{X}$ be the random variable "the number of green balls you drew". What kind of random variable is $\mathbf{X}$? What is its probability mass function?

b) Suppose we have the same urn as in part a) above, and suppose we draw a ball from the urn, record its color and replace it n times. Let $\mathbf{Y}$ be the random variable "the number of green balls drawn" (notice that this is a different random variable: the question is the same, but the experiment is different). What kind of random variable is $\mathbf{Y}$ and what is its probability mass function?

c) Finally, suppose the same urn as in a) and b), but suppose we do not know its proportion of green balls or blue balls. (That is, we do not know $\frac{N}{M+N}$.) Which experiment should we use, that in a) or that in b), to estimate the proportion of balls in the urn which are green? Why?

• It is perhaps best to solve all three of these parts together, since they are very closely related. The issues are "replacement and non-replacement" in sampling, and variance. The first represents drawing balls from an urn without replacement. This is modeled by a hypergeometric random variable. The hypergeometric has three parameters N, M, n exactly as in the problem, where n is a sample size, and N, M are the size of types within the total population $N+M$. The probability mass function would be that for $\mathbf{X}$, that is, $P(\mathbf{X}=i) = \frac{\binom{M}{i}\binom{N}{n-i}}{\binom{M+N}{n}}$, with mean $n\frac{M}{M+N}$, and variance $V(\mathbf{X}) = \frac{M+N-n}{M+N-1} n \frac{M}{M+N}\frac{N}{M+N}$. On the other hand, $\mathbf{Y}$ is a binomial random variable, since replacing the balls makes the experiment repeated Bernoulli trials (n of them) with parameters $n, p = \frac{M}{M+N}$. The mean is also $n\frac{M}{N+M}$, while the variance $V(\mathbf{Y}) = n\frac{M}{M+N}\frac{N}{M+N}$. Finally, for c), if we don't know the proportion $\frac{M}{M+N}$, then we can run the experiment for $\mathbf{X}$ or for $\mathbf{Y}$ to get an estimate for this ratio. The ratio $\frac{M}{M+N}$ is $\frac{1}{n}$ times the expectation of $\mathbf{X}$ or $\mathbf{Y}$. If we do this, it would be better to choose the experiment with the lesser variance. Comparing the two, we see that

$$V(\mathbf{Y}) = \frac{M+N-n}{M+N-1} V(\mathbf{X}),$$

and since $\frac{M+N-n}{M+N-1} \leq 1$ (you get equality when n, the draw size, is just one), we should choose the experiment represented by $\mathbf{Y}$ as more likely to give results close to the mean.

[25]

2. a) Let $\mathbf{X}$ be a continuous random variable with probability density function $f_{\mathbf{X}}(x)$. What is the probability density function $f_{\mathbf{Y}}$ of the random variable $\mathbf{Y}$, where $\mathbf{Y} = a\mathbf{X}+b$? Here $a>0$ and b are constants (real numbers).

• This is a repeat from the second mid-term, and is the simplest change of variable formula. The relationship between possible values of x, y, of $\mathbf{X}, \mathbf{Y}$, respectively, is given by $y = ax+b$. Thus $x = \frac{y-b}{a}$, $\frac{dy}{dx} = a$ and the density function $f_{\mathbf{Y}}$ is given by

$$f_{\mathbf{Y}} = \frac{f_x(\frac{y-b}{a})}{|a|} = \frac{f_x(\frac{y-b}{a})}{a},$$

since $a > 0$.

b) Now suppose $\mathbf{Z}$ is a normal random variable with mean 0 and variance 1. What are the possible values w of the random variable $\mathbf{W} \equiv e^{\mathbf{X}}$? What is the probability density function $f_{\mathbf{W}}(w)$ of $\mathbf{W}$? [This called the *log-normal distribution.*]

• We have looked at this one before, too. Here we have the relationship of possible values $w = e^z$, so $w > 0$, and $z = \log w$, $\frac{dw}{dz} = e^z$, and we get

$$f_{\mathbf{W}}(w) = \frac{1}{e^z} f_{\mathbf{Z}}(z)|_{z=\log w} = \frac{1}{w} f_{\mathbf{Z}}(\log w).$$

Substituting what we know for $f_{\mathbf{Z}}$, we get

$$f_{\mathbf{W}} = \frac{1}{\sqrt{2\pi} w} e^{-\frac{1}{2}\{\log w\}^2}, w > 0.$$

[25]

3. We have a box containing five coins. The i-th coin has bias $p = \frac{i}{5}$. One of the coins is drawn from the box and a head is flipped. What is the probability that the i-th coin was the coin chosen?

• This is an exercise in Bayes's theorem and calculating probability by conditioning. Let H be the event the flip was a head, and $C = i$ the event the i-th coin was selected (i.e., the one with $p = \frac{i}{5}$. Then we are trying to compute the conditional probability $P(C = i|H)$. By Bayes and conditioning, we have:

$$P(C = i|H) = \frac{P(H|C = i)P(C = i)}{P(H)}$$

$$= \frac{P(H|C = i)P(C = i)}{P(H|C = 1)P(C = 1) + \ldots + P(H|C = 5)P(C = 5)}.$$

To evaluate the numerator, use $P(C = i) = \frac{1}{5}$ and $P(H|C = i) = \frac{i}{5}$. For the numerator, we have to sum up five such terms:

$$P(H|C = 1)P(C = 1) + \ldots + P(H|C = 5)P(C = 5) = (\frac{1}{5})^2 \sum_{i=1}^{5} i = \frac{1}{25} \cdot \frac{5 \cdot 6}{2} = \frac{3}{5}.$$

Putting these two together gives

$$P(C = i|H) = \frac{\frac{i}{25}}{\frac{3}{5}} = \frac{i}{15}.$$

[25]

4. On tax returns, one rounds off the tax owed to the nearest dollar. The error from the true tax is a random variable $\mathbf{U}$ which is uniformly distributed over the interval $[-\frac{1}{2}, \frac{1}{2}]$. If we have 100 million taxpayers, the total error in the tax levy by allowing this practice could be as much as \$50 million. Estimate the probability that the government makes \$5000 by this practice. What assumptions are you making?

• This is a CLT problem. We have 100,000,000 taxpayers, each with their own round-off random variable $\mathbf{U}_n, n = 1, \ldots, 100,000,000$. Then the total amount of error across all taxpayers would be $\mathbf{S} = \sum_1^{100,000,000} \mathbf{U}_n$. We are asked to compute $P(\mathbf{S} \geq 5000)$. The mean of $\mathbf{U}$ is 0 and the variance is $\frac{1}{12}$, so that the central limit theorem would give

$$P(\mathbf{S} \geq 5000) = P\left(\frac{\mathbf{S}}{\sqrt{100,000,000 \cdot \frac{1}{12}}} \geq \frac{5000}{\sqrt{100,000,000 \cdot \frac{1}{12}}}\right) \approx P(\mathbf{Z} \geq \frac{\sqrt{12}}{2}).$$

But

$$P(\mathbf{Z} \geq \frac{\sqrt{12}}{2}) \approx P(\mathbf{Z} \geq 1.732) = 1 - \Phi(1.732) = 1 - 0.9582 = 0.042 = 4.2\%.$$

Of course, they stand the same risk of losing as much!

We have assumed the random variables $\mathbf{U}_n$ are independent of one another, and identically distributed.

[25]

5. a) Let $\mathbf{X}$ be a random variable with mean 10 and variance σ^2. Estimate the probability that $|\mathbf{X} - 10| > 2\sigma$. (This is referred to as the probability that $\mathbf{X}$ is *at least two standard deviations away from its mean.*)

• This is a straightforward Markov or Chebyshev estimation:

$$P(|\mathbf{X} - 10| > 2\sigma) = P(|\mathbf{X} - 10|^2 > 4\sigma^2) < \frac{E(|\mathbf{X} - 10|^2)}{4\sigma^2} = \frac{\sigma^2}{4\sigma^2} = \frac{1}{4} = 0.25.$$

b) What is the probability that the standard unit normal random variable $\mathbf{Z}$ is more than two standard deviations away from its mean?

• When you have a specific random variable you can say something more precise: recall that the mean of $\mathbf{Z}$ is 0 and the variance is 1, so we are trying to calculate

$$P(|\mathbf{Z}| \geq 2) = 2(1 - \Phi(2)) = 2(1 - 0.9772) = 2 \cdot 0.0228 = 0.0456 = 4.56\%.$$

[25]

6. We have three coins, a fair one, one with bias $p = \frac{1}{3}$ and one with bias $p = \frac{2}{3}$. Now we first flip the fair coin. If we get heads, we next flip the coin with bias $p = \frac{1}{3}$ *twice* and if we flip tails, we next flip the coin with bias $p = \frac{2}{3}$ *twice*. Let $\mathbf{X}_i, i = 1, 2$, be the random variables "the number of heads on the i-th flip of the biased coin chosen". Are $\mathbf{X}_1$ and $\mathbf{X}_2$ independent? Why was this to be expected?

• We can use conditional expectation to calculate the covariance of $\mathbf{X}_1$ and $\mathbf{X}_2$, $E(\mathbf{X}_1\mathbf{X}_2) - E(\mathbf{X}_1)E(\mathbf{X}_2)$. First, let $\mathbf{P}$ be the random variable, "the bias of the coin chosen and flipped twice". Then

$$E(\mathbf{X}_1) = E(E(\mathbf{X}_1|\mathbf{P})) = E(\mathbf{X}_1|\mathbf{P} = \frac{1}{3})P(\mathbf{P} = \frac{1}{3}) + E(\mathbf{X}_2|\mathbf{P} = \frac{2}{3})P(\mathbf{P} = \frac{2}{3}).$$

This last is

$$E(\mathbf{X}_1) = \frac{1}{3} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{2}.$$

The same calculation works for $E(\mathbf{X}_2)$ as well. For the covariance, we have to compute

$$E(\mathbf{X}_1\mathbf{X}_2) = E(\mathbf{X}_1\mathbf{X}_2|\mathbf{P} = \frac{1}{3})P(\mathbf{P} = \frac{1}{3}) + E(\mathbf{X}_1\mathbf{X}_2|\mathbf{P} = \frac{2}{3})P(\mathbf{P} = \frac{2}{3})$$

$$= (\frac{1}{3})^2\frac{1}{2} + (\frac{2}{3})^2\frac{1}{2} = \frac{1}{18} + \frac{4}{18} = \frac{5}{18}.$$

The covariance is then

$$Cov(\mathbf{X}_1, \mathbf{X}_2) = \frac{5}{18} - (\frac{1}{2})^2 = \frac{1}{36} \neq 0.$$

Since the covariance is not 0, the random variables cannot be independent. One should have expected this: if the first flip were heads, that could be considered evidence that the biased coin that was chosen favored heads, i.e., had $p = \frac{2}{3}$, and therefore that heads would be favored on the second flip of the chosen coin. Similarly for tails. This would lead one to say that they are positively correlated, "positive" because when the one goes up (more heads), the other does too.

[25]

7. An elevator starts at the basement floor with 8 people (not including an elevator operator), and discharges them all by the time it reaches the top floor (number 6). In how many ways could the operator have perceived the people leaving the elevator if he cannot distinguish between people, but can distinguish between floors? What if the people consisted of five men and three women and the operator could tell men from women?

• The frist version of the problem is a problem of putting 8 "objects" (the indistinguishable people) into six "urns" (i.e., floors), as in Ross, Chap. 1.6. Here we allow a floor to have no passengers get out (that is "empty urns", or non-negative solutions as opposed to positive solutions in sec 1.6). The number of ways of doing this is

$$\binom{8+6-1}{6-1} = \binom{13}{5} = \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{120} = 13 \cdot 11 \cdot 9 = 1287.$$

For the second half, we get the product of the previous problem for 5 men, and for 3 women, since now we can distinguish the men and the women, and they are independent of each other. So, we get:

$$\binom{5+6-1}{6-1} \cdot \binom{3+6-1}{6-1} = \binom{10}{5} \cdot \binom{8}{5} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{120} \cdot \frac{8 \cdot 7 \cdot 6}{3!} = 9 \cdot 7 \cdot 4 \cdot 8 \cdot 7 = 14112.$$

[25]

8. The number of people who enter an elevator on the ground floor is a Poisson random variable with mean 10. If there are 6 floors above the ground floor, and passengers get off at each floor with equal likelihood, independently of one another, what is the expected number of stops the elevator makes before discharging all its passengers? [You are to assume that no further passengers get on at the higher floors.]

• This one was already solved on the web page for the general case of N floors. the exam problem was teh same problem with $N = 6$. The original URL was
`http://www.math.lsa.umich.edu/~dburns/f03_425rossprobch6_7.pdf`.
I reprint here the relevant problem:

Chap. 7, prob. 56. Elevators.

This is a little easier since we don't have a so-called "random sum" as in #55. So there are N floors above the ground floor, and the number of riders $\mathbf{R}$ in the elevator entering at the ground level is random, but with a Poisson[10] distribution. Each of the passengers is equally likely to get out at any of the N possible floors, and this independently of what all the other passengers do. Compute the expected number of stops the elevator makes before discharging all its passengers. Note that this is going to be a lot like the birthdays problem above, except the number of people will be random. This makes it worth trying to compute the expectation by conditioning. First we introduce an indicator RV for each floor: $\mathbf{F}_i = 1$ if somebody gets out at floor i, and 0 otherwise. The number $\mathbf{S}$ of stops is given by $\mathbf{S} = \sum_{i=1}^{i=N} \mathbf{F}_i$.

$$E(\mathbf{S}) = NE(\mathbf{F}_1).$$

It is in computing $E(\mathbf{F}_1)$ that we will use conditioning.

$$E(\mathbf{F}_1) = E(E(\mathbf{F}_1|\mathbf{R} = r)).$$

However, $E(\mathbf{F}_1|\mathbf{R} = r)$ is not so hard to compute:

$$E(\mathbf{F}_1|\mathbf{R} = r) = Prob(\mathbf{F}_1 = 1|\mathbf{R} = r) = 1 - (\frac{N-1}{N})^r,$$

giving

$$\begin{aligned} E(\mathbf{S}) &= NE(\mathbf{F}_1) \\ &= \textstyle\sum_{r=0}^{r=\infty} N \cdot \{1 - (\frac{N-1}{N})^r\} \cdot e^{-10} \cdot \frac{10^r}{r!} \\ &= N\{1 - e^{-10} \cdot \textstyle\sum_{r=0}^{r=\infty} \frac{(\frac{10(N-1)}{N})^r}{r!}\} \\ &= N\{1 - e^{-10} e^{\frac{10(N-1)}{N}}\} \\ &= N(1 - e^{-\frac{10}{N}}). \end{aligned}$$

In our specific case on the final, with $N = 6$, we get $E(\mathbf{S}) = 6(1 - e^{-\frac{10}{6}}) = 4.87$.

[25]