

Math 425, Section 1, Fall 2003: Introduction to Probability

First Mid-Term: October 3, 2003.

Solutions.

1. (a.) My nephew Cole has some brightly colored plastic rings, good for teething on. They come with a post for him to stack them on. Three rings are red, four are blue and three are yellow. How many different color patterns can Cole make by stacking the rings on the pole?

**** Out of the ten positions from bottom to top, say, on the post, you have to specify three groups, *red, blue, yellow*, of size 3, 3 and 4, *respectively*. That is we need a multinomial coefficient

$$\binom{10}{3, 3, 4} = \frac{10!}{3! \cdot 3! \cdot 4!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{3! \cdot 3!} = 8400.$$

(b.) If four Americans, three Frenchmen and three Englishmen are to be seated in a row, how many arrangements are possible if all people of the same nationality must sit next to each other?

**** Since we must seat all Americans next to each other in one group, and all Frenchmen, etc., we get $3!$ possible arrangements of the nationalities. However, unlike the first half of the problem, these are people, and not teething rings we are ordering, and so we should keep track of the individuals: this means that for every arrangement of the nationalities, we also have to order the three Englishmen, the four Americans and the three French people as well. This gives us, finally,

$$\begin{aligned} & 3! (\text{arrangements of nationalities}) \cdot 4! (\text{orderings of the Americans}) \\ & \times 3! (\text{orderings of the French}) \cdot 3! (\text{orderings of the English}) = 5184. \end{aligned}$$

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2. Fred has just found out that there is a $1/3$ chance that he has contracted a viral infection the last time he went to the Twilight Zone. The only effect of the disease, if he has contracted it, is that if he has any children, they each have a $1/4$ chance of having six fingers on each hand.

Fred marries Martha the following year. What is the probability that their first two children will have the usual number of fingers on each hand? (You may assume that among the uninfected population, the probability of having six fingers on each hand is zero.) After Fred and Martha have two children, each with the usual number of fingers on each hand, what is the probability that their third child will have more than ten fingers?

**** This is a straight out Bayes's Theorem problem. Let F be the event "Fred got infected in the Twilight zone", and let S_1 be the event "the first child has six fingers on each hand", and S_2 the event "the second child has six fingers on each hand". Then the first part of the problem asks you to compute $P(S_1^c \cap S_2^c)$, the probability that neither of their first two children have six fingers on each hand.

One way to calculate this, which is shorter, is to use conditioning and independence in the following way:

$$\begin{aligned} P(S_1^c \cap S_2^c) &= P(S_1^c \cap S_2^c | F)P(F) + P(S_1^c \cap S_2^c | F^c)P(F^c) \\ &= P(S_1^c | F)P(S_2^c | F)P(F) + P(S_1^c | F^c)P(S_2^c | F^c)P(F^c) \\ &= \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{3} + 1 \cdot 1 \cdot \frac{2}{3} = \frac{41}{48} \end{aligned}$$

This is because if S_1 and S_2 are independent, given F , then so are S_1^c and S_2^c .

If you didn't realize that the complements are independent, we can compute by a longer method. (Incidentally, the following includes a hidden proof of the independence of the complements.) We compute:

$$\begin{aligned} P(S_1^c \cap S_2^c) &= 1 - P(S_1 \cup S_2) \text{ (why?)} \\ &= 1 - P(S_1) - P(S_2) + P(S_1 \cap S_2). \end{aligned}$$

Now, to compute $P(S_1)$, use conditioning:

$$P(S_1) = P(S_1 | F)P(F) + P(S_1 | F^c)P(F^c),$$

where, from the statement of the problem, we know $P(F) = 1/3$, $P(S_1 | F) = 1/4$, whereas $P(S_1 | F^c) = 0$. The same considerations apply to S_2 . Thus,

$$P(S_1) = P(S_2) = 1/4 \cdot 1/3 = 1/12.$$

Similarly,

$$\begin{aligned} P(S_1 \cap S_2) &= P(S_1 \cap S_2 | F)P(F) + P(S_1 \cap S_2 | F^c)P(F^c) \\ &= P(S_1 | F)P(S_2 | F)P(F) + 0 \\ &= \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{48}. \end{aligned}$$

Putting the pieces together, we have

$$P(S_1^c \cap S_2^c) = 1 - P(S_1) - P(S_2) + P(S_1 \cap S_2) = 1 - \frac{2}{12} + \frac{1}{48} = \frac{41}{48}.$$

For the second part, Let $N = S_1^c \cap S_2^c$ be the event "the first two children of Fred and Martha have five fingers on each hand", and S_3 the event "the third child has six fingers on each hand". We are asked to compute $P(S_3 | N)$. Use Bayes's theorem:

$$P(S_3 | N) = P(N | S_3) \frac{P(S_3)}{P(N)}.$$

This actually helps, since $P(S_3) = 1/12$, just as above, and $P(N) = \frac{43}{48}$ from part one, and

$$P(N | S_3) = P(N | F) = \frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16}.$$

Altogether,

$$P(S_3 | N) = \frac{9}{16} \cdot \frac{1/12}{41/48} = \frac{9}{16} \cdot \frac{48}{12 \cdot 41} = \frac{9}{164}.$$

3. There are two fair dice, one red and the other blue. Roll the two dice. Define the following events:

$$\begin{aligned} A &= \text{event that the red die is less than 4,} \\ B &= \text{event that the red die is 3, 4 or 5,} \\ C &= \text{event that the sum of the dice is 5.} \end{aligned}$$

Show that $P(A \cap B \cap C) = P(A)P(B)P(C)$. Explain why this is not enough to show that the events A, B, C are *mutually independent*. Finally, determine whether or not these three events are mutually independent.

**** The event $A \cap B \cap C$ is exactly the event that the red die came out a three, and the blue die came out a two. This has probability $\frac{1}{36}$. On the other hand,

$$P(A) = \frac{1}{2},$$

$$P(B) = \frac{1}{2},$$

$$P(C) = \frac{4}{36}.$$

Arithmetic finishes the first part. For the second part, one remarks that one must *also* show that A, B and C are pairwise independent. In particular,

$$P(A \cap B) = P(\text{“red die is 3”}) = \frac{1}{6} \neq \frac{1}{4} = P(A) \cdot P(B),$$

showing the three events are *not* mutually independent.

4. The Tigers and the Cubs (two baseball teams) play in the World Series. The series is won by the first team to win four games. The Tigers win each game with probability 0.6, independently of one another. What is the probability that the Tigers win the series? [You should work out a formula, but it is not necessary to evaluate your formula.]

**** Let T be the event that the Tigers win. It will help to break this as follows:

$$T = T_4 \cup T_5 \cup T_6 \cup T_7,$$

where T_i is the event “the Tigers win four out of i games”. They are mutually exclusive events, so

$$P(T) = P(T_4) + P(T_5) + P(T_6) + P(T_7).$$

For each of these cases, we have that

$$P(T_i) = (0.6)^4(0.4)^{i-4} \binom{i-1}{3}.$$

The first two factors gives the probability that the Tigers win exactly four of the i games, given that you know which games of the i they won or lost, and the last facctor gives the number of possible ways these games could be arranged. (Notice that the Tigers have to

win the last game, so we only have to know where the remaining 3 wins are among the $i - 1$ previous games.) Adding this up gives

$$P(T) = \sum_{i=4}^{i=7} (0.6)^4 (0.4)^{i-4} \binom{i-1}{3}.$$

You might have noticed that this is a special case of the “problem of the points” from the textbook.

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5. Two players, A and B , play a game where a fair coin is flipped, and if a head H appears, B pays A one dollar. If a tail T is flipped, A pays B one dollar. After the \$1 is exchanged, they flip the coin again, following the same procedure. They continue in this way until one of the two players has no more money. If B has no money left we say A has won, and if A has no money left, we say B has won. Suppose that A starts out with M dollars, and B starts out with N dollars. The probability that A or B wins will depend on how much A or B has to begin with.

(a.) What is the probability that A wins, if both A and B start with \$1?

**** Since one of the two players will lose on the first flip, this is just the probability that the coin comes up H on the first flip, that is, 0.5.

(b.) Show that if $P(A; M, N)$ is the probability that A wins, when A starts out with \$ M , and B starts out with \$ N , then

$$P(A; M, N) = \frac{1}{2}P(A; M + 1, N - 1) + \frac{1}{2}P(A; M - 1, N + 1).$$

**** This is just an example of conditioning on the outcome of the first move, here the first flip: let $A(M, N)$ be the events “ A wins, starting out with M dollars, while B starts with N dollars.” The problem asks one to begin calculating $P(A(M, N)) = P(A; M, N)$. We can condition on the outcome of the first flip, where H is the event that the first flip comes out a head, and T the event H^c . Then

$$\begin{aligned} P(A(M, N)) &= P(A(M, N)|H)P(H) + P(A(M, N)|T)P(T) \\ &= \frac{1}{2}P(A(M, N)|H) + \frac{1}{2}P(A(M, N)|T). \end{aligned}$$

But, if we know the first flip is a head, then $P(A(M, N)|H) = P(A(M + 1, N - 1))$ and similarly $P(A(M, N)|T) = P(A(M - 1, N + 1))$, since after the first flip, if the first flip is a head, the continuing game looks as though one had just started, but with A having one more dollar, and B having one less, and similarly for the case the first flip is a tail. So,

$$\begin{aligned} P(A(M, N)) &= \frac{1}{2}P(A(M, N)|H) + \frac{1}{2}P(A(M, N)|T) \\ &= \frac{1}{2}P(A; M + 1, N - 1) + \frac{1}{2}P(A; M - 1, N + 1), \end{aligned}$$

which is what was required.

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