# Math/Stats 425, Sec. 1, F 2003: Introduction to Probability

## Second Mid-Term: Solutions

**1.** Consider the following graph of the joint distribution of two random variables $\mathbf{X_1}, \mathbf{X_2}$.

| $\mathbf{X_1}$ \ $\mathbf{X_2}$ | 0 | 1 | $p_{\mathbf{X_1}}(x_1)$ |
|---|---|---|---|
| 0 | $\frac{10}{39}$ | $\frac{5}{39}$ | $? = \frac{15}{39}$ |
| 1 | $? = \frac{14}{39}$ | $\frac{10}{39}$ | $? = \frac{24}{39}$ |
| $p_{\mathbf{X_2}}(x_2)$ | $? = \frac{24}{39}$ | $? = \frac{15}{39}$ | 1 |

(i) Fill out the table by finding the values of each of the question marks.
• Marked above. The main square must add up to 1, and the marginals are obtained by adding the columns or rows.

(ii) Are the random variables $\mathbf{X_1}$ and $\mathbf{X_2}$ independent?
• No: $P(\mathbf{X_1} = 1, \mathbf{X_2} = 1) \neq P(\mathbf{X_1} = 1)P(\mathbf{X_2} = 1)$, for example.

(iii) What are the *marginal distributions* here?
• The marginal distributions are those for $\mathbf{X_1}$ and $\mathbf{X_2}$.

(iv) Which random variables here are *indicator* random variables?
• Indicator RV's have possible values just 0 and 1, so both $\mathbf{X_1}$ and $\mathbf{X_2}$ are indicator RV's.

(iv) What is a situation for which this distribution would be a good probability model? Why?
• Suppose we had 5 red balls and 8 white balls in a bowl, and $\mathbf{X_1}$ is the RV which gives 1 if we pick a red ball on the first draw, 0 if a white, and $\mathbf{X_2}$ is similar for the second draw, where one does NOT replace after the first draw. Then the table describes exactly the probabilities for this situation.

**2.** A fair die is to be rolled 1000 times. What is the approximate probability that a six will be rolled at least 200 times?

We can treat this as a coin problem: if we consider rolling a six as getting a "head", and rolling anything else as a "tail", then we are modeling flipping a coin with bias $p = 1/6$, and with $n = 1000$ flips. Let $\mathbf{X}$ be the binomial random variable with $p = 1/6$ and $n = 1000$. We want to estimate $P(\mathbf{X} \geq 200)$, and we will use DeMoivre's Theorem to do this. To use our edge correction method, we note that $P(\mathbf{X} \geq 200) = P(\mathbf{X} > 199.5)$, and that $E(\mathbf{X}) = \frac{1000}{6}$ and $Var(\mathbf{X}) = 1000 \cdot \frac{1}{6} \cdot \frac{5}{6}$. So, we get

$$\mathbf{X} > 199.5 \qquad \text{if and only if}$$

$$\frac{\mathbf{X}-\frac{1000}{6}}{\sqrt{1000\cdot\frac{1}{6}\cdot\frac{5}{6}}} > \frac{199.5-\frac{1000}{6}}{\sqrt{1000\cdot\frac{1}{6}\cdot\frac{5}{6}}},$$

and so

$$P\left(\frac{\mathbf{X}-\frac{1000}{6}}{\sqrt{1000\cdot\frac{1}{6}\cdot\frac{5}{6}}} > \frac{199.5-\frac{1000}{6}}{\sqrt{1000\cdot\frac{1}{6}\cdot\frac{5}{6}}}\right) \sim P\left(\mathbf{Z} > \frac{199.5-\frac{1000}{6}}{\sqrt{1000\cdot\frac{1}{6}\cdot\frac{5}{6}}}\right) = 1-\Phi\left(\frac{199.5-\frac{1000}{6}}{\sqrt{1000\cdot\frac{1}{6}\cdot\frac{5}{6}}}\right).$$

To carry out the arithmetic, we first get

$$\frac{199.5-\frac{1000}{6}}{\sqrt{1000\cdot\frac{1}{6}\cdot\frac{5}{6}}} = \frac{6\cdot 199.5-1000}{\sqrt{5000}} = \frac{19.7}{\sqrt{50}} = \frac{3.94}{\sqrt{2}} \sim 2.79,$$

and then look up $\Phi(2.79) = 0.9974$, or

$$P(\mathbf{X} \geq 200) \sim 1-\Phi(2.79) = 0.0026,$$

which is *not very probable!*

**3.** a) Let $\mathbf{X}$ be a continuous random variable with probability density function $f_{\mathbf{X}}(x)$. What is the probability density function $f_{\mathbf{Y}}$ of the random variable $\mathbf{Y}$, where $\mathbf{Y} = a\mathbf{X}+b$? Here $a > 0$ and $b$ are constants (real numbers).

• One can apply directly the formula for the probability density function for $\mathbf{Y} = g(\mathbf{X})$, given that for $\mathbf{X}$, where our case is $g(x) = ax+b$. The formula would be

$$f_{\mathbf{Y}}(y) = \frac{f_{\mathbf{X}}(g^{-1}(y))}{|g'(g^{-1}(y))|}.$$

In our case, $g^{-1}(y) = \frac{1}{a}(y-b)$, and $g'(x) \equiv a$, so

$$f_{\mathbf{Y}}(y) = \frac{1}{a}\cdot f_{\mathbf{X}}\left(\frac{1}{a}(y-b)\right).$$

b) Now suppose $\mathbf{X}$ is uniformly distributed over the interval $[0,2]$. What are the possible values $w$ of the random variable $\mathbf{W} \equiv e^{\mathbf{X}}$? What is the probability density function $f_{\mathbf{W}}(w)$ of $\mathbf{W}$?

• We can use the same method here, but in this example we know

$$f_{\mathbf{X}}(x) = \begin{cases} \frac{1}{2} & \text{if } x \in [0,2], \\ 0 & \text{otherwise.} \end{cases}$$

This time $g(x) = e^x$, and so $g^{-1}(y) = \log(y)$, and $g'(x) = e^x$. The range of possible values of $\mathbf{Y} = e^{\mathbf{X}}$ is from $e^0 = 1$ to $e^2$, and so the function $f_{\mathbf{Y}}(y)$ becomes

$$f_{\mathbf{Y}}(y) = \begin{cases} \frac{1}{2}\frac{1}{e^{\log(y)}} = \frac{1}{2y} & y \in [1,e^2], \\ 0 & \text{otherwise.} \end{cases}$$

**4.** Let $\mathbf{X}$ be a random variable with an exponential distribution with parameter $\lambda$. where $\lambda > 0$.

a) What is the cumulative distribution function $F_{\mathbf{X}}(x)$ of $\mathbf{X}$?

• The cumulative distribution function is just given by $F_{\mathbf{X}}(x) = P(\mathbf{X} < x) = \int_{-\infty}^{x} f_{\mathbf{X}}(s)ds$. Here we know that

$$f_{\mathbf{X}}(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, since $\int_0^x \lambda e^{-\lambda s} ds = -e^{-\lambda x} \mid_0^x = 1 - e^{-\lambda x}$, we get

$$F_{\mathbf{X}}(x) = \begin{cases} 0 & x \leq 0, \\ 1 - e^{-\lambda x} & x > 0. \end{cases}$$

b) What is the hazard rate function $\lambda(x)$ of $\mathbf{X}$? (Calculate this.)

• The hazard rate function is given by $\lambda(x) = \frac{f_{\mathbf{X}}(x)}{1-F_{\mathbf{X}}(x)}$. Here, for $x > 0$, the numerator is just $\lambda e^{-\lambda x}$ and the denominator is $1 - \{1 - e^{-\lambda x}\} = e^{-\lambda x}$, so that we get altogether:

$$\lambda(x) = \begin{cases} 0 & x \leq 0, \\ \lambda & x > 0. \end{cases}$$

c) Calculate $P(\mathbf{X} \geq t + 10 \mid \mathbf{X} \geq t)$. Notice that it does not depend on $t$: this is the *memoryless property* again. Interpret this.

•$P(\mathbf{X} > x) = 1 - F_{\mathbf{X}}(x) = e^{-\lambda x}$. Therefore, we can compute the conditional probability as follows:

$$\begin{aligned} P(\mathbf{X} > t + 10 \mid \mathbf{X} > t) &= \frac{P(\{\mathbf{X}>t+10\} \cap \{\mathbf{X}>t\})}{P(\mathbf{X}>t)} \\ &= \frac{P(\mathbf{X}>t+10)}{P(\mathbf{X}>t)} \\ &= \frac{e^{-\lambda(t+10)}}{e^{-\lambda t}} \\ &= e^{-\lambda 10}, \text{ by the law of exponents} \end{aligned}$$

This means that, if we think of $\mathbf{X}$ as describing the survival time, e.g., of an individual, the individual who has survived to a given instant has the same probability of surviving 10 more units of time which is independent of how long he or she has survived up to that instant.

d) Discuss the interpretation of exponential distributions to model waiting times. How does one know which $\lambda$ to use in such a problem?

• It makes sense to use the exponential distribution to model waiting times when the process being modeled appears to have the memoryless property. The parameter $\lambda$ is set by using the equation for the expectation of $\mathbf{X}$, namely, $E(\mathbf{X}) = \frac{1}{\lambda}$, if one knows the average or expected waiting time. That is, one would set

$$\lambda = \frac{1}{\text{expected waiting time}}.$$

**5.** A department store has a grand opening sale: they offer to give a prize to every person who makes the 1000th purchase since the last prize (so there is a prize every thousand purchases). They are expecting sales to be steady and average 5 per minute over the course of the afternoon. What is the probability that they will have at least two winners in the afternoon (between 1 PM and 5 PM)? Be sure to state what probability model you are using, and say how you found the parameters for the model.

• This is a situation where we're trying to count the number of occurrences of a random event. The random event in this problem is "the next 1000 sales". This suggests modeling by a Poisson distribution, with parameter $\lambda$. Recall that $\lambda$ is equal to the expectation of the Poisson-$\lambda$ random variable. Here, we expect to have $5 \cdot 4 \cdot 60 = 1200$ sales in the afternoon, which is 1.2 thousands of sales. So, we model $\mathbf{X}$, the random variable "the number of times a prize is given this afternoon" by a Poisson distribution with $\lambda = 1.2$. Now we can compute

$$P(\mathbf{X} \geq 2) = 1 - P(\mathbf{X} = 0) - P(\mathbf{X} = 1) = 1 - e^{-1.2}(1 + 1.2) = 1 - 2.2e^{-1.2} = 0.337,$$

using that $P(\mathbf{X} = k) = e^{-\lambda} \frac{\lambda^k}{k!}$.

**Quotations: Physicists on Probability.**

*I will never believe that God plays dice with the universe.*

Albert Einstein

`http://www.hawking.org.uk/lectures/dice.html`.
(This is a lecture by Stephen Hawking on Einstein's idea.)

*The conception of chance enters in the very first steps of scientific activity in virtue of the fact that no observation is absolutely correct. I think chance is a more fundamental conception that causality; for whether in a concrete case, a cause-effect relation holds or not can only be judged by applying the laws of chance to the observation.*

Max Born

(Sorry, don't have a reference for this one.)