Math 425, Section 6, Fall 1999: Introduction to Probability

Final Examination Solutions

1. a.) Let $\mathbf{X}$ be a standard normal random variable, that is, with mean 0 and variance 1. Let $\mathbf{Y}$ be the random variable $e^{\mathbf{X}}$. What is the probability density function of $\mathbf{Y}$?

Solution:
Use the formula on page 229, theorem 7.1, with $y = g(x) = e^x$, and thus $x = g^{-1}(y) = \log y$. Or, as in class, set $F_{\mathbf{Y}}(y) = P(\mathbf{Y} \leq y)$ and note that $F'_{\mathbf{Y}}(y) = f_{\mathbf{Y}}(y)$. Since

$$F_{\mathbf{Y}}(y) = P(e^{\mathbf{X}} \leq y) = P(\mathbf{X} \leq \log y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\log y} e^{-\frac{x^2}{2}} dx, \ y > 0.$$

Notice that, for $y \leq 0$, $f_{\mathbf{Y}}(y) = 0$. Take the derivative , using the fundamental theorem of calculus and the chain rule, to get

$$f_{\mathbf{Y}}(y) = F'_{\mathbf{Y}}(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\log y^2}{2}} \cdot \frac{1}{y}.$$

The factor $\frac{1}{y}$ appears because $\frac{d}{dy} \log y = \frac{1}{y}$.

b.) Recall from basic calculus that the return on \$1 invested for one year at annual rate r, compounded *continuously*, is \$ e^r. Given this, what is a possible interpretation for the random variable $e^{r+\mathbf{X}}$?

Solution:
One possible interpretation is that this random variable represents the return on an investment of \$ 1 compounded continuously for one year at the rate of interest $r + \mathbf{X}$, i.e., at a random rate of interest where the expected rate is r and $\mathbf{X}$ represents normal fluctuations about this mean. This kind of model actually is used at times in the analysis of financial transactions.

2. A coin is chosen to have a random bias, *i.e.,* the coin has probability $\mathbf{P}$ of coming up heads, where $\mathbf{P}$ is a random variable uniformly distributed over the interval $[0, 1]$. Then two players each flip the coin once. Let $\mathbf{X}$ be the random variable which is 1 if the first player flips a head and 0 if he flips a tail. Let $\mathbf{Y}$ be a similar random variable for the second player.

(a) Show that the probability that the first player flips a head is $\frac{1}{2}$.

Solution:
Do this by conditioning on the bias of the coin:

$$P(\mathbf{X} = 1) = \int_0^1 P(\mathbf{X} = 1 \mid \mathbf{P} = p) dp = \int_0^1 p dp = \frac{1}{2}.$$

(b) Show that the probability that both players flip heads is $\frac{1}{3}$.

Solution:
Again, compute this by conditioning on the bias $\mathbf{P}$:

$$P(\mathbf{X}=1,\mathbf{Y}=1)=\int_0^1 P(\mathbf{X}=1,\mathbf{Y}=1\mid\mathbf{P}=p)dp=\int_0^1 p^2dp=\frac{1}{3}.$$

(c) Are the random variables $\mathbf{X}$ and $\mathbf{Y}$ independent? Explain your answer.

Solution:
They are NOT independent since

$$\frac{1}{3}=P(\mathbf{X}=1,\mathbf{Y}=1)\neq\frac{1}{4}=P(\mathbf{X}=1)P(\mathbf{Y}=1).$$

You should recognize this problem as a continuous analogue of problem 1 on the group homework set number three.

3. Suppose that $\mathbf{X}$ and $\mathbf{Y}$ are independent geometric random variables with the same parameter p. What is the conditional probability mass function $P(\mathbf{X}=i\,|\,\mathbf{X}+\mathbf{Y}=n)$? Give an intuitive argument as well for this result.

Solution:
You can use the fact that $\mathbf{X}$ and $\mathbf{Y}$ represent waiting times until the first head is flipped, and their sum therefore represents the waiting time until the second head is flipped in a series of coin tosses, in this case of a coin with bias p. Thus, the distribution of the sum $\mathbf{X}+\mathbf{Y}$ is negative binomial, with parameters 2, p, and we can take the distribution directly from the chart provided you for the exam. Thus, the probability mass function is given by

$$p_{\mathbf{X}+\mathbf{Y}}(n)=\binom{n-1}{2-1}p^2(1-p)^{n-2},\ n\geq 2.$$

If you didn't see this interpretation, you could still find this mass function by the formulas we developed for the mass function of a sum of two independent random variables, as in the text, page 265 (the formula is given for continuous distributions there, but the same method works for discrete random variables as well, with the integral replaced by a sum). Then,

$$p_{\mathbf{X}|\mathbf{X}+\mathbf{Y}=n}(j)=\frac{P(\mathbf{X}=j\mid\mathbf{X}+\mathbf{Y}=n)}{P(\mathbf{X}+\mathbf{Y}=n)}=\frac{P(\mathbf{X}=j,\mathbf{Y}=n-j)}{P(\mathbf{X}+\mathbf{Y}=n)}$$
$$=\frac{P(\mathbf{X}=j)P(\mathbf{Y}=n-j)}{P(\mathbf{X}+\mathbf{Y}=n)}=\frac{p(1-p)^{j-1}p(1-p)^{n-j-1}}{(n-1)p^2(1-p)^{n-2}}=\frac{1}{n-1}.$$

This last distribution is constant, that is, any of the $n-1$ flips $1,2,...,n-1$ before the nth flip is equally likely to be the flip when the first head occurs. In other words, once we know when the second flip occurs, there seems to be no way to prefer any one of the previous $n-1$ flips as the flip when the first head arrives.

4. Five hundred independent rolls of a fair die will be made. What is the approximate probability that five will occur at least 100 times?

Solution:

This is a direct application of the Central Limit Theorem, or actually the special case known as the deMoivre-Laplace theorem, text page 212. Basically, we approximate by a standard normal random variable. In detail, we let $\mathbf{X_i}, i = 1, ..., 500$, be *iid*'s corresponding to Bernoulli trials with $p = \frac{1}{6}$, and set $\mathbf{X} = \sum_{i=1}^{500} \mathbf{X_i}$. Then,

$$P(\mathbf{X} \geq 100) = P(\mathbf{X} \geq 99.5) = P(\frac{\mathbf{X} - 500\frac{1}{6}}{\sqrt{500\frac{1}{6}\frac{5}{6}}} \geq \frac{99.5 - 500\frac{1}{6}}{\sqrt{500\frac{1}{6}\frac{5}{6}}}),$$

this last taking account of the fact that the *expectation* of $\mathbf{X}$ is $500\frac{1}{6}$, while the *variance* of $\mathbf{X}$ is $500\frac{1}{6}\frac{5}{6}$. Recall also that one usually passes from, e.g., 100 to 99.5, as above, to get a better approximating the discrete binomial distribution by a continuous normal distribution. Thus,

$$P(\mathbf{X} \geq 100) \approx P(\mathbf{Z} \geq \frac{99.5 - 500\frac{1}{6}}{\sqrt{500\frac{1}{6}\frac{5}{6}}}),$$

where $\mathbf{Z}$ is a standard normal random variable. But

$$P(\mathbf{Z} \geq \frac{99.5 - 500\frac{1}{6}}{\sqrt{500\frac{1}{6}\frac{5}{6}}}) \approx 1 - \Phi(1.94) = 1 - 0.9738 = 0.0262,$$

where Φ is as in the table you were provided, that is, the cumulative distribution function of the standard normal random variable. The arithmetic is even neater if one ignores the correction for replacing a discrete random variable by a continuous one, i.e., if one doesn't replace 100 by 99.5 above. The one gets $1 - \Phi(2) = 0.0228$. This is an underestimate of the probability compared to the answer given above.

5. a.) Let $\mathbf{X_1}$, $\mathbf{X_2}$ and $\mathbf{X_3}$ be independent random variables, each of which is uniformly distributed over the interval $[0, 1]$. What is the probability that $\mathbf{X_1} < \mathbf{X_2}$? What is the probability that $\mathbf{X_1} < \mathbf{X_2} < \mathbf{X_3}$?

Solution:
There are two ways to do these, at least. One way is to note that there are two possible orderings of the two random numbers $\mathbf{X_1}$ and $\mathbf{X_2}$, namely, $\mathbf{X_1} < \mathbf{X_2}$ and $\mathbf{X_1} > \mathbf{X_2}$. As these are equally likely, by symmetry, we conclude $P(\mathbf{X_1} < \mathbf{X_2}) = \frac{1}{2}$. there is a fine point here: you should note that there is a third possibility, $\mathbf{X_1} = \mathbf{X_2}$, but this has probability 0 of happening. Similarly, the probability $P(\mathbf{X_1} < \mathbf{X_2} < \mathbf{X_3}) = \frac{1}{6}$, because there are six equiprobable orderings of the three random variables.
Another way to do these is to compute the probabilities directly by integration. For example, the second probability is the volume of the region $\{x_1 < x_2 < x_3\}$ inside the unit cube $\{0 \leq x_i \leq 1, \ i = 1, 2, 3\}$. Thus, we must compute

$$\int_0^1 \int_0^{x_3} \int_0^{x_2} dx_1 \, dx_2 \, dx_3 = \int_0^1 \int_0^{x_3} x_2 \, dx_2 \, dx_3 = \int_0^1 \frac{x_3^2}{2} \, dx_3 = \frac{1}{6}.$$

The hard part this way is to recognize the limits of integration which describe the region $\{x_1 < x_2 < x_3\}$.

b.) Suppose that n points are independently chosen at random on the perimeter of a circle, and we want the probability that they all lie in some semicircle. (That is, we want the probability that there is a line passing through the center of the circle such that all the points lie on one side of the line.) Let $P_1, ..., P_n$ denote the points. Let A denote the event that all the points are on the same side of a line, and A_i denote the event that all the points lie in the semicircle beginning at P_i and going 180 degress clockwise starting from P_i, $i = 1, 2, ..., n$.

i.) Express A in terms of the A_i.

Solution:
As in the first part of this question, we are dealing here with a geometric probability question. The set of outcomes of our experiment now is the set of configurations of n points on the circle, and the subset A is the event "all n points lie within a semi-circle". Our arguments will have to be about these configurations of n points and how they can possible lie in a semi-circle.

A glance at a picture will convince you that any configuration of n points on the circle which all lie in a semi-circle will have to lie in a particular semi-circle of the form described. More detailedly, assume that the n points lie in a semi-circle. Draw a diameter such that the points all lie "to the right" of this diameter. The diameter is imagined oriented, i.e., we can think of it as having a head and a tail, as with vectors. Then, "to the right" means on the side gotten from the head of the diameter by rotating clockwise around the circle. Now take this diameter and rotate it clockwise until the first time you encounter one of the n points, say P_i. Then the n points will lie in the semi-circle defined by the diameter through the origin and P_i. This says the particular configuration of n points we are looking at are an outcome in the event A_i, that is, that the event A is the union of the various events A_i. In symbols, we have shown

$$A = \bigcup_{i=1}^{i=n} A_i.$$

ii.) Are the A_i mutually exclusive?

Solution:
Again, pictorially, the solution is obvious: yes, they are mutually exclusive. If the n points on the circle are in the semi-circle swept out in the clockwise direction from P_i, as above, and P_j is any other of the n points, then if we continue rotating the radius through the origin and P_i clockwise until the head of the radius comes to P_j, then we will have rotated the radius less than 180^o, and so the original P_i will be to the counterclockwise side of the diameter with P_j at its head. Thus, the configuration, if it is in the event A_i, cannot be in the event A_j, and so the events are mutually exclusive.

There is a fine point point here, however, which I have to mention. Strictly speaking, the events are not mutually exclusive. In the argument above, we have assumed tacitly that the n points are all *distinct*. It could happen that the points P_i and P_j above are one and the same point, since they were, after all, just chosen at random. The reason we do not have to worry about this is the same reason as in the first part of this question: the event "two of the n points coincide" has probability zero, so for the purposes of probability, we can ignore it.

iii.) Find $P(A)$.

Solution:
Now we can write

$$P(A) = P(\bigcup_{i=1}^{i=n} A_i) = \sum_{i=1}^{i=n} P(A_i)$$

, and it is easy to see that $P(A_i)$ is the same for each i, from geometric symmetry, so that

$$P(A) = nP(A_1) >$$

Now to calculate $P(A_1)$, once we have chosen the point P_1 at random, we have determined the semi-circle for the event A_1. We have to draw $n-1$ more points at random, and hope that they all come out on the clockwise side of the diameter through P_1. But this is the same as the experiment of flipping a fair coin $n-1$ times, and the probability of the $n-1$ points all being on the correct side for A_1 is $(\frac{1}{2})^{n-1}$. Altogether, then, we get

$$P(A) = n\,(\frac{1}{2})^{n-1}.$$

6. If 10 married couples are randomly seated at a round table, what is the expectation of the random variable $\mathbf{N}$ equal to the number of wives sitting next to their husbands? What is the variance of $\mathbf{N}$?

Solution:
This is the expected question about indicator random variables! Suppose that $\mathbf{I_i}, i = 1, ..., 10$, are the indicator variables where $\mathbf{I_i} = 1$ if couple i sits together, and 0 if they do not. Then the random variable $\mathbf{N}$ just counts the number of couples sitting together, so we have

$$\mathbf{N} = \sum_{i=1}^{i=10} \mathbf{I_i}.$$

By the fundamental formula about expectations,

$$E(\mathbf{N}) = \sum_{i=1}^{i=10} E(\mathbf{I_i}).$$

Now since the $\mathbf{I_i}$ are indicator variables, $E(\mathbf{I_i}) = P(\mathbf{I_i} = 1)$, while it is obvious that $P(\mathbf{I_i} = 1)$ is independent of i, that is, of which couple we consider. To compute $P(\mathbf{I_1} = 1)$, assume that husband # 1 is seated first, then of the remaining 19 seats which are available at random to wife # 1, only two will lead to sitting together, so $P(\mathbf{I_i} = 1) = \frac{2}{19}$. Altogether, this gives us

$$E(\mathbf{N}) = 10\,P(\mathbf{I_1} = 1) = 10\frac{2}{19} = \frac{20}{19}.$$

To compute the variance of $\mathbf{N}$, we use the formula

$$Var(\mathbf{N}) = E(\mathbf{N}^2) - E(\mathbf{N})^2,$$

and compute

$$E(\mathbf{N}^2) = E(\sum_{i,j=1}^{10} \mathbf{I_i I_j}) = \sum_{i,j=1}^{10} E(\mathbf{I_i I_j}) = \sum_{i=1}^{10} E(\mathbf{I_i}^2) + \sum_{i \neq j} E(\mathbf{I_i I_j}).$$

By symmetry, this gives

$$E(\mathbf{N}^2) = 10\,E(\mathbf{I_1}) + 10(10-1)E(\mathbf{I_1}\mathbf{I_2}).$$

Note that we used ${\mathbf{I_1}}^2 = \mathbf{I_1}$, since $\mathbf{I_1}$ is an indicator variable. So we really have to figure out $E(\mathbf{I_1}\mathbf{I_2}) = P(\mathbf{I_1} = 1, \mathbf{I_2} = 1)$. But $P(\mathbf{I_1} = 1, \mathbf{I_2} = 1) = P(\mathbf{I_2} = 1 \mid \mathbf{I_1} = 1)\,P(\mathbf{I_1} = 1)$. We already know $P(\mathbf{I_1} = 1)$, so we have to compute $P(\mathbf{I_2} = 1 \mid \mathbf{I_1}) = 1)$. But this is the same as having a line of 18 chairs in a row, for the 2nd couple to choose from randomly. There are 17 ways to seat them next to each other, and this out of $\binom{18}{2} = 9\cdot 17$ possible pairs of seats where they could be seated. Thus,

$$P(\mathbf{I_2} = 1 \mid \mathbf{I_1}) = \frac{1}{9}.$$

This gives

$$E(\mathbf{N}^2) = 10\,\frac{2}{19} + 90\,\frac{1}{9}\,\frac{2}{19} = \frac{40}{19},$$

and therefore

$$Var(\mathbf{N}) = \frac{40}{19} - \frac{20}{19}^2 = \frac{360}{361}.$$

BONUS: Let $\mathbf{X}$ and $\mathbf{Y}$ be independent random variables with exponential λ distributions. What is the conditional probability density function $f_{\mathbf{X}|\mathbf{X}+\mathbf{Y}=z}(x)$? Give an interpretation and intuitive reason for this. (See question 3. above.)

Solution:
This problem is the continuous version of question 3. above. Exponential-λ random variables represent waiting time until the first arrival in a Poisson process with intensity λ, so the sum $\mathbf{X}+\mathbf{Y}$ is a Gamma(2, λ) random variable, and the p.d.f. can be read from the table:

$$f_{\mathbf{X}+\mathbf{Y}}(x) = \frac{e^{-\lambda x}\lambda^2 x}{\Gamma(2)} = e^{-\lambda x}\lambda^2 x.$$

Hence, the conditional density function

$$f_{\mathbf{X}|\mathbf{X}+\mathbf{Y}=z}(x) = \frac{f_{\mathbf{X},\mathbf{Y}}(x, z-x)}{f_{\mathbf{X}+\mathbf{Y}}(z)} = \frac{\lambda e^{-\lambda x}\cdot \lambda e^{-\lambda(z-x)}}{\lambda^2\, z\, e^{-\lambda z}} = \frac{1}{z} < 0 \le x \le z,$$

and) elsewhere. This result is completely analogous to question 3. above: if we know the second arival time for a Poisson process, the first arrival time is uniformly distributed over the interval from 0 to the time of the second arrival.