Third Group Problems: Solutions

Problem 1. Variable Bias Coins.

Suppose we have two coins, one with a bias $p = 1/3$ as probability for getting a head and the other with $p = 2/3$, and let us perform the following experiment with them. We pick one of the two coins at random (with equal likelihood for either coin), and then we flip that coin twice. Let $\mathbf{X}$ be the random variable which is 1 if the first coin flipped is a head and 0 if it comes up a tail, and let $\mathbf{Y}$ be the similar random variable for the second coin-flip. Show that $\mathbf{X}$ and $\mathbf{Y}$ are *not* independent.

Solution:

We can test for independence by computing the Covariance of $\mathbf{X}$ and $\mathbf{Y}$: if the Covariance is not 0 they are not independent. We need to compute first the expectation of $\mathbf{X}$, which we do by conditioning on the bias $\mathbf{P}$ of the coin:

$$E(\mathbf{X}) = E(E(\mathbf{X}|\mathbf{P})) = E(\mathbf{X}|\mathbf{P} = \frac{1}{3})\,P(\mathbf{P} = \frac{1}{3}) + E(\mathbf{X}|\mathbf{P} = \frac{2}{3})\,P(\mathbf{P} = \frac{2}{3}) = \frac{1}{3}\frac{1}{2} + \frac{2}{3}\frac{1}{2} = \frac{1}{2}.$$

The expectation of $\mathbf{Y}$ is also $\frac{1}{2}$, of course. To compute the Covariance, we compute the expectation of $\mathbf{XY}$ by conditioning in the same way:

$$E(\mathbf{XY}) = E(E(\mathbf{XY}|\mathbf{P})) = E(\mathbf{XY}|\mathbf{P} = \frac{1}{3})\,P(\mathbf{P} = \frac{1}{3}) + E(\mathbf{XY}|\mathbf{P} = \frac{2}{3})\,P(\mathbf{P} = \frac{2}{3})$$

$$= E(\mathbf{X}|\mathbf{P} = \frac{1}{3})\,E(\mathbf{Y}|\mathbf{P} = \frac{1}{3})\,P(\mathbf{P} = \frac{1}{3}) + E(\mathbf{X}|\mathbf{P} = \frac{2}{3})\,E(\mathbf{Y}|\mathbf{P} = \frac{2}{3})\,P(\mathbf{P} = \frac{2}{3})$$

$$= (\frac{1}{3})^2\,\frac{1}{2} + (\frac{2}{3})^2\,\frac{1}{2} = \frac{5}{18}.$$

Putting all this together, we get

$$\mathrm{Cov}(\mathbf{X}, \mathbf{Y}) = E(\mathbf{XY}) - E(\mathbf{X})\,E(\mathbf{Y}) = \frac{5}{18} - (\frac{1}{2})^2 = \frac{1}{36} \neq 0.$$

Hence, $\mathbf{X}$ and $\mathbf{Y}$ are not independent.

Problem 2. The Poisson Process and Customer Service.

This is a design question: you are trying to choose the right value of a parameter to minimize the cost of a certain (idealized) service operation. Customers arrive at a facility according to a Poisson process with rate parameter λ. There is a waiting cost of c per customer per unit time. The customers gather at the facility and are processed or dispatched in groups at fixed times $T, 2T, 3T, \ldots$. There is a dispatch cost of K for each dispatch, independent of the number of customers dispatched.

(a) What is the mean total customer waiting cost during the first cycle?

Solution:

The easier way to do this is to figure out what your expected waiting cost is over a small time interval, and then add these up over all the small time intervals. This will lead to an integral expression for the expected waiting costs over the interval from 0 to T. In the time interval $[t, t + \Delta t]$ we expect between λt and $\lambda(t + \Delta t)$ customers, by the basic property of the Poisson process. Hence the expected waiting costs in this interval would be between $c\,\lambda\,t$ and $c\,\lambda\,(t + \Delta t)$. Adding these up over a partition of the interval $[0, T]$, and letting the width Δt of the partition intervals go to zero, we get a *Riemann integral* for the answer:

$$\text{Expected waiting costs} = \int_0^T c\,\lambda\,t\,dt = \frac{1}{2}\,c\,\lambda\,T^2.$$

(b) What is the mean customer waiting cost plus dispatch cost per unit time during the first cycle?

Solution:

We simply add the mean customer waiting cost and the dispatch cost and divide by the length T of the time period:

$$\text{Mean total costs per unit time over the first interval} = \frac{\frac{1}{2}c\lambda T^2 + K}{T} = \frac{1}{2}\,c\,\lambda\,T + \frac{K}{T}.$$

(c) What value of T minimizes this mean cost per unit time?

Solution:

We simply take the derivative of the expression in part (b) and differentiate it with respect to T, set the derivative equal to 0 and solve for T:

$$\frac{1}{2}\,c\,\lambda - \frac{K}{T^2} = 0,$$

that is,

$$T = \sqrt{\frac{2K}{\lambda\,c}}.$$

Problem 3. The Monty Hall Problem.

On a TV game show, a contestant is confronted with three identical closed doors. A goat (which the contestant doesn't get to keep!) is behind each of two of the three doors, and a prize behind the third. The contestant is to choose one door. The game show host then opens one of the two unchosen doors, showing a goat. The contestant then has the option to switch the choice he or she had already made for the door concealing the prize. The door finally chosen is opened and the contestant wins the prize if this finally chosen door conceals the prize. Decide whether either of the two possible strategies of play is better for the contestant, *i.e.*, makes it more probable that the contestant will win the prize:

(a) When offered the opportunity to change the choice of door, the contestant doesn't change.

(b) When offered the opportunity to change the choice of door, the contestant changes his or her choice.

Solution:

This one is actually fairly elementary, and doesn't really use the techniques of the latter part of the course. First one notes that for strategy (a), the probability of getting the prize is simply the probability that the prize was picked at the beginning (Monty Hall's showing a goat the contestant didn't pick does not alter the probability that the contestant had picked the prize: this point is a bit like the "prisoner's dilemma", which you looked at earlier). Thus, the probability of winning with this strategy is $\frac{1}{3}$, since we will assume that there is an equal chance of the prize being behind any one of the three doors and that the contestant will pick any door with equal likelihood. On the other hand, the situation is exactly the opposite for strategy (b): the only possibility of losing when following strategy (b) is when the contestant has picked the prize with the original pick, and then changes to the second of the two goats he or she did not pick originally. Thus, the likelihood of winning following strategy (b) is $\frac{2}{3}$!

For those who still find this shift hard to believe, you can try a simple experiment with some playing cards and an accomplice. Choose, say, seven cards from a deck of playing cards, six of which are red. Shuffle the seven cards and try to pick the black card. Do not look at the card you have chosen but have your accomplice, "Monty Hall", offer to remove five red cards from the six that you did not choose. Then either stay with the card you chose or switch to the other one. This "accelerated Monty Hall" situation has probability 6/7 of a win if you switch, so it becomes very evident very quickly. Even doing the exact Monty Hall simulation, three cards of which one is black, etc., is rather fast in showing you the evidence in favor of the calculations made above.