The Poisson Process

This handout describes in very elementary terms the Poisson process which comes up in problem 2 of the third group assignment, actually gathering together in one place some of the remarks we have encountered about it throughout the second half of the term.

We will start from the Poisson distribution Poisson[λ], where a random variable X has the Poisson[λ] distribution if

$$P(X = n) = e^{-\lambda}\frac{\lambda^n}{n!}.$$

We have interpreted this distribution to describe the probability of "rare events" occuring, provided we had an idea of what the expected number of these events should be, given in this case by the parameter $\lambda = E(X)$.

In the *Poisson process* we introduce an additional parameter of time, which we will denote t. Thus we are counting rare events, which we will call *arrivals* when talking about the Poisson process, within a given interval of time. We still have a parameter λ, but now this refers to the expected number of arrivals *in a unit time interval.* One of the basic assumptions made about the Poisson process is that the probability distribution for arrivals in t units of time is given by the Poisson[$t\lambda$] distribution. Notice that this scales the expected number of arrivals to $t\lambda$: the number of expected arrivals is proportional to the length of time considered and the scale factor is λ, called the *intensity* in this context.

For your problem, the main thing you need to be aware of is the random variable $N(t)$, defined to be the number of arrivals between initial time 0 and time t. In light of what has already been said, the distribution of $N(t)$ is Poisson[$t\lambda$]. In problem 2, you need to consider what are your expected costs coming from waiting customers instantaneously at time t, which will be proportional, according to the problem, to the expected number of customers at time t.

There are several other properties which are interesting to know, but which aren't strictly necessary for the problem. In particular, in a Poisson process, we have what is called the independence of intervals, much like the "memoryless property" of the exponential distribution:

$$P(N(t+s) = n+m \mid N(t) = n) = P(N(s) = m).$$

This is not too surprising, since it turns out that the waiting time between arrivals for a Poisson process with intensity parameter λ is a random variable with the Exponential[λ] distribution. In fact, one can construct the Poisson process starting from this point of view, but that is a topic for a later day, even a later course.