Math 425, Section 7, Winter 2003: Introduction to Probability

First Mid-Term: Solutions

1. Five cards are selected at random from a standard deck of 52 playing cards.

(i) What is the probability of getting a "straight", *i.e.,* the five cards have *succesive denominations*, such as $3, 4, 5, 6, 7$?
(You can consider the denominations arranged in order from 2 to 14.)

***** To count the number of possible straights, notice that a straight starts at one denomination, and works "up" through five denominations. Therefore a straight has only 9 possible starting denominations: 2, 3, 4, 5, 6, 7, 8, 9, 10. However, once you start a suit, each denomination in the straight can be any one of four possible suits. Therefore there are 9×4^5 possible straights, and therefore the probability of a straight is

$$P(\text{straight}) = \frac{9 \times 4^5}{\binom{52}{5}}.$$

(ii) A "full house" is a hand with three cards of one denomination and two of another. Is a "full house" more likely than a "straight"?

***** There are 13 choices of denomination for the triplet in the full house, and four possible triplets of that denomination, and then 12 denominations and 6 (4 choose 2) possible pairs in that denomination. Altogether that gives you $13 \times 12 \times 4 \times 6 = 3744$ possibilities. There are 9216 possible straights, so a flush is rarer.

2. Suppose that we have 3 cards identical in form except that both sides of the first card are colored red, both sides of the second card are colored black, and one side of the third card is colored red and the other side black. The 3 cards are mixed up in a hat and 1 card is randomly selected, and a random side of that card is put down on the table, leaving the other side showing. *If* we see that the side of the card showing is red, what is the probability that the other side is black?

This is the example in the text in chapter three (we did the variant in class where the second face was supposed to be black). Let R be the even that the first side showing is red, and let RB be the event that the first face showing is red and the flip side is black. The problem asks for the *conditional probability* of RB, given R, that is,

$$P(RB \mid R) = \frac{P(RB \cap R)}{P(R)} = \frac{P(RB)}{P(R)}.$$

But $P(R) = \frac{1}{2}$ (think of six equally likely faces, three of them red), and $P(RB) = \frac{1}{6}$, since you have a $\frac{1}{3}$ possibility of picking the card with mixed colors, and a $\frac{1}{2}$ possibilityof showing the red side first. Altogether, this gives you $P(RB \mid R) = \frac{1}{3}$.

3. A forest contains 20 elk, 6 of which have been captured and tagged, then released. A certain time later 5 of the 20 are captured again. What is the probability that exactly two of these will be tagged and three not? State explicitly all the assumptions you are making.

***** This is familiar as the elk problem done in class with slightly different numbers. There are $\binom{20}{5}$ equally likely five-tuples of elk one can pick out of the forest (I am **ASSUMING** they are equally likely), and there are $\binom{6}{2}$ pairs of the six tagged elk and $\binom{14}{3}$ triples of untagged elks. I am assuming that both the tagged and untagged elk are **EQUALLY LIKELY** to be picked. Finally, under these assumptions, we get

$$P(2 \quad \text{tagged}, \, 3 \quad \text{untagged}) = \frac{\binom{6}{2} \cdot \binom{14}{3}}{\binom{20}{5}}.$$

4. Two players, A and B, play a game where a fair coin is flipped, and if a head H appears, B pays A one dollar. If a tail T is flipped, A pays B one dollar. After the \$1 is exchanged, they flip the coin again, following the same procedure. They continue in this way until one of the two players has no more money. If B has no money left we say A has won, and if A has no money left, we say B has won. Suppose that A starts out with M dollars, and B starts out with N dollars. The probability that A or B wins will depend on how much A or B has to begin with.

(a.) What is the probability that A wins, if both A and B start with \$1?

***** In this case, after the first flip, either A or B will have no money left, so the game will surely end after the first flip. If the coin is a fair coin, they each have probability $\frac{1}{2}$ of winning.

(b.) Show that if $P(A; M, N)$ is the probability that A wins, when A starts out with $\$M$, and B starts out with $\$N$, then

$$P(A; M, N) = \frac{1}{2} P(A; M+1, N-1) + \frac{1}{2} P(A; M-1, N+1).$$

***** This kind of problem is fruitfully analyzed by conditioning on the outcome of the first move of the game, here the first flip. Let H be the event, "the first flip showed heads", and T "the first flip showed tails". Conditoning tells us

$$P(A; M, N) = P(A; M, n \mid H) P(H) + P(A; M, N \mid T) P(T)$$

$$= \frac{1}{2} P(A; M, N \mid H) + \frac{1}{2} P(A; M, N \mid T).$$

But notice next that $P(A; M, N \mid H) = P(A; M+1, N-1)$, and $P(A; M, N \mid T) = P(A; M-1, N+1)$ (why?!). This settles the problem.
For more details, you can look up the *Gambler's Ruin Problem* in the text. The solution is similar to the one for the problem of the men throwing their hats into the ring and then seeing whether they can all pick different hats from their own.

5. (a.) A well-made (fair) die is thrown twice. Consider the events A = "the first throw gives a 2 or a 5", and B = "the sum of the two throws is at least 7". Are the events A and B independent or dependent? Explain your answer.

***** The event A has probability $\frac{1}{3}$, assuming the rolls are independent. Looking at the 6×6 table of possible outcomes of two independent rolls of a fair die, we see that $P(B) = \frac{21}{36}$. Now the event $A \cap B$ is the same as the event "the first roll was a 2 or a 5, while the second roll was a 5 or 6 (in the first case) or a 2, 3, 4, 5 or 6 (in the second case).". This event contains seven possible outcomes, {(2,5), (2,6), (5,2), (5,3), (5,4), (5,5), (5,6)} and hence, $P(A \cap B) = \frac{7}{36}$, while $P(A) \cdot P(B) = \frac{1}{3} \cdot \frac{21}{36} = \frac{7}{36}$, also. Thus the events are independent.

(b.) 8 toys are thrown randomly from a Mardi Gras parade truck at four friends watching the parade. What is the probability that each of the friends will get two toys? Make your assumptions explicit.

***** We can solve this similarly to the blackboards problem: we distribute 8 toys to four onlookers, where we are making no distinction between the eight (non-distinguished) toys. The number of such distributions where there are no conditions placed on the n, m is simply $\binom{8+3}{3}$. Since there is only one way to have each onlooker receive two toys, the probability of this occuring is simply

$$P(\text{"each gets two"}) = \frac{1}{\binom{11}{3}} = \frac{1}{165}.$$