Math 425, Section 1, Winter, '97: Introduction to Probability

First Mid-Term: Solutions

1. (a) When are two events E and F in the sample space S independent? When are three events E, F, G in S independent?

*** When $P(E \cap F) = P(E) \cdot P(F)$; when $P(E \cap G) = P(E) \cdot P(G), P(G \cap F) = P(G) \cdot P(F), P(E \cap F) = P(E) \cdot P(F)$ AND $P(E \cap F \cap G) = P(E) \cdot P(F) \cdot P(G)$.

(b) Let us flip a fair coin three times, and let A_{12} be the event that the first and second flips come out the same (*i.e.,* both heads or both tails). Similarly for events A_{13}, A_{23}. Are A_{12} and A_{13} independent? Are A_{12}, A_{13}, A_{23} all independent?

*** The event A_{12} represents four possibilities out of eight equally likely possibilities, so $P(A_{12}) = 1/2$; similarly, $P(A_{13}) = P(A_{23}) = 1/2$. The event $A_{12} \cap A_{13}$ is "all three tosses are the same", so $P(A_{12} \cap A_{13}) = 1/4 = P(A_{12}) \cdot P(A_{13})$, so they are independent; similarly for the other pairs of events. However, the event $A_{12} \cap A_{13} \cap A_{23}$ is the same event "all three tosses are the same", so $P(A_{12} \cap A_{13} \cap A_{23}) = 1/4 \neq P(A_{12}) \cdot P(A_{13}) \cdot P(A_{23}) = 1/8$. Thus the three events are NOT independent.

2. A trained flea sits on the real line at $x = 4$, and his master begins flipping a fair coin. Each time the coin shows a head, the flea hops one unit to the right, each time a tail shows he hops one unit to the left. Let $\mathbf{X}_n$ be the random variable "the flea's position after n flips". So, for example, if the first flip is a head, then $\mathbf{X}_1 = 4 + 1 = 5$.

(a) What is the probability mass function of $\mathbf{X}_3$?

*** $p_{\mathbf{X_3}}(i) = 0$, unless $i = 1, 3, 5, 7$, and $p_{\mathbf{X_3}}(1) = 1/8 = p_{\mathbf{X_3}}(7)$, $p_{\mathbf{X_3}}(3) = p_{\mathbf{X_3}}(5) = 3/8$.

(b) What is the probability that $\mathbf{X}_2 = 3$?

*** 0; $\mathbf{X_2}$ only takes on the values $2, 4, 6$.

(c) On average, where do you expect the flea to be after four coin flips?

*** This is asking you to find the *expectation* of $\mathbf{X_4}$. You can use the formula to conclude it is 4, or just reason that the coin is equally likely to come out heads as tails, so on average the flea shouldn't be moving either up or down the line. That is a good idea for motivation, but your answer should show the work done with the formula, especially at this early stage of our study of probability.

Bonus [5 pts.] What is the probability that the flea will reach the origin $x = 0$ before he reaches $x = 10$? Why?

*** This is the same as the gambler's ruin problem with a fair coin. Consider the distance units (10) between 0 and 10 on the line to be the N of the G.R. problem, and A to be 0, B to be 10. A ruins B if it wins all 10 units away from B, and A gets one more unit each time a tail is flipped. Thus, A is starting with 6 out of 10 distance units from B, whereas B starts with 4 from A. Thus, as shown in class, the probability that the flea will reach 0 before he reaches 10 is $i/N = 3/5$.

3. Ten weight lifters are competing in a team weight-lifting contest. Of the lifters, 3 are from the US, 4 are from Russia, 2 are from China, and 1 is from Canada. If the scoring takes account of the countries that the lifters represent, but not their individual identities, how many different outcomes are possible from the point of view of scores? How many different outcomes correspond to results in which the US has 1 competitor in the top three and 2 in the bottom three?

*** If we were just interested in the number of possible lists of the ten contestants in their order of finishing, we would have 10! possible lists. However, this overcounts, since we don't keep track of, *e.g.*, individual Americans. Thus we have to divide out by the different orderings of the various teams, and the final answer becomes $10!/3! \cdot 4! \cdot 2! \cdot 1! = 12600$. If we take the extra conditions into account, we can break the count down into three steps. First, there are 3 ways to put one American in the first three slots. Then there are 3 ways to put two Americans into the last three slots. Finally, for each of the $3 \cdot 3 = 9$ possibilities just listed, we have to place the remaining 7 competitors in order in the remaining 7 slots, but we still don't want to distinguish the 4 Russians or 2 Chinese. This gives us altogether $9 \cdot 7!/(4! \cdot 2! \cdot 1!) = 945$.

4. We have two boxes, A and B, with five marbles in each. We know one box contains 3 black marbles and 2 red ones, while the other contains 2 black ones and 3 red ones, but we do not know which is which. We draw a marble from box A, note the color and replace it. Then we draw another marble from box A.

(a) What is the probability of picking a black marble the first draw?

*** Let U_1 be the event "A is the urn with 3 black marbles" and U_2 be the event "A is the other urn". $P(U_1) = P(U_2) = 1/2$. Let B_1 be the event " a black marble was picked on the first draw", and B_2 be the event "a black marble was picked on the second draw". The first part asks for $P(B_1)$. We can figure this out by conditioning on the choice of urn: $P(B_1) = P(B_1 \mid U_1) \cdot P(U_1) + P(B_1 \mid U_2) \cdot P(U_2) = (3/5) \cdot (1/2) + (2/5) \cdot (1/2) = 1/2$.

(b) If the first marble is black, what is the probability that A is the box with three black marbles?

*** This asks us to find $P(U_1 \mid B_1)$. By Bayes' formula, $P(U_1 \mid B_1) = P(B_1 \mid U_1) \cdot P(U_1)/P(B_1) = (3/5) \cdot (1/2)/(1/2) = 3/5$.

(c) What is the probabilty that the second marble drawn is black, if the first one is black?

*** You want $P(B_2 \mid B_1)$. By the definition of conditional probability: $P(B_2 \mid B_1) = P(B_1 \cap B_2)/P(B_1) = \{P(B_1 \cap B_2 \mid U_1) \cdot P(U_1) + P(B_1 \cap B_2 \mid U_2) \cdot P(U_2)\}/P(B_1) = \{(3/5)^2 \cdot (1/2) + (2/5)^2 \cdot (1/2)\}/(1/2) = 13/25$.