A Unified Framework for Pricing Credit and Equity Derivatives *†

Erhan Bayraktar ‡ Bo Yang §

Abstract

We propose a model which can be jointly calibrated to the corporate bond term structure and equity option volatility surface of the same company. Our purpose is to obtain explicit bond and equity option pricing formulas that can be calibrated to find a risk neutral model that matches a set of observed market prices. This risk neutral model can then be used to price more exotic, illiquid or over-the-counter derivatives. We observe that our model matches the equity option implied volatility surface well since we properly account for the default risk in the implied volatility surface. We demonstrate the importance of accounting for the default risk and stochastic interest rate in equity option pricing by comparing our results to Fouque et al. (2003), which only accounts for stochastic volatility.

Keywords: Defaultable Bond, Defaultable Stock, Equity Options, Stochastic Interest Rate, Implied Volatility, Multiscale Perturbation Method.

Contents

*This work is supported in part by the National Science Foundation, under grant DMS-060449.

†We would like to thank the two anonymous referees and the anonymous AE for their constructive comments, which helped us improve our paper.

‡Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA; e-mail:erhan@umich.edu.

§Morgan Stanley, 1585 Broadway, 3rd Floor, New York, NY 10036; e-mail: Bo.Yang@morganstanley.com.

1 Introduction

Our purpose is to build an intensity-based modeling framework that can be jointly calibrated to corporate bond prices and stock options, and can be used to price more exotic derivatives. The same company has stocks, stock options, bonds and several other derivatives. When this company defaults, the payoffs of all of these instruments are affected; therefore, their prices all contain information about the default risk of the company. In our framework we use the Vasicek model for the interest rate, and use doubly stochastic Poisson process to model default. We assume that the bonds have recovery of market value and that stocks become valueless at the time of default. Using the multi-scale modeling approach of Fouque et al. (2003) we obtain explicit bond pricing equation with three free parameters which we calibrate to the corporate bond term structure. On the other hand, stock option pricing formula contains seven parameters, three of which are common with the bond option pricing formula. (The common parameters are multiplied with the loss rate in the bond pricing formula.) The parameters that remain unknown after our calibration to the corporate yield curve are determined by calibrating them to the stock option implied volatility surface. The calibration results reveal that our hybrid model is able to produce implied volatility surfaces that match the data closely. We compare the implied volatility surfaces that our model produces to those of Fouque et al. (2003). We see that even for longer maturities our model has a prominent skew: compare Figures 2 and 3. Even when we ignore the stochastic volatility effects, our model fits the implied volatility of the Ford Motor Company well and performs better than the model of Fouque et al. (2003); see Figure 1. (Observe that when we ignore the stochastic volatility our model has one less parameter to calibrate than that of Fouque et al. (2003).) This points to the importance of accounting for the default risk for companies with low ratings.

Our model has three building blocks: (1) We model the default event using the multi-scale stochastic intensity model of Papageorgiou and Sircar (2008). We also model the interest rate using an Ornstein-Uhlenbeck process (Vasicek model). As it was demonstrated in Papageorgiou and Sircar (2008), these modeling assumptions are effective in capturing the corporate yield curve; (2) We assume the stock price process jumps to zero when the company defaults. This stock price model was considered in Bayraktar (2008). Our model specification for the stock price differs from the jump to default models for the stock price considered by Carr and Linetsky (2006) and Linetsky (2006), which take the default intensity to be functions of the stock price; (3) We also account for the stochastic volatility in the modeling of the stocks since even the index options (when there is no risk of default) possess implied volatility skew. We model the volatility using the fast scale stochastic volatility model of Fouque et al. (2000). We demonstrate on index options (when there is no risk of default) that we match the performance of the two time scale volatility model Fouque et al. (2003) (see Section 4.4). The latter model extends Fouque et al. (2000) by including a slow factor in the volatility to get a better fit to longer maturity options. We see from Section 4.4 that when one assumes the interest rate to be stochastic, the calibration performance of the stochastic volatility model with only the fast factor is as good as the two scale stochastic volatility model. This is why we choose the volatility to be driven by only the fast factor. Even though interest rate is stochastic we are able to obtain explicit asymptotic pricing formulas for stock options. Thanks to these explicit pricing formulas the inverse problem that we face in calibrating our model to the corporate bond and stock data can be solved with considerable ease. Our modeling framework can be thought of as a hybrid of the models of Fouque et al. (2000), which only considers pricing options in a stochastic volatility model with constant interest rate, and Papageorgiou and Sircar (2008), which only considers a framework for pricing derivatives on bonds. Neither of these models has the means to transfer default information from bond market to equity markets and vice versa, which we are set to do in this paper. We should note that our model also takes the treasury yield curve, historical stock prices, and historical spot rate data to estimate some of its parameters (see Section 4).

Our model extends Bayraktar (2008) by taking the interest rate process to be stochastic, which leads to a richer theory and more calibration parameters, and therefore, better fit to data: (i) When the interest rate is deterministic the corporate bond pricing formula turns out to be very crude and does not fit the bond term structure well (compare (2.57) in Bayraktar (2008) and (4.1)); (ii) With deterministic interest rates the bond pricing and the stock option pricing formulas share only one common term, "the average intensity of default" (this parameter is multiplied by the loss rate in the bond pricing equation, under our loss assumptions). Therefore, the effect of the default risk is not accounted for in the implied volatility surface as much as it should be. And our calibration analysis demonstrates that this has a significant impact. When the volatility is taken to be a constant, both our new model and the model in Bayraktar (2008) have three free parameters. The model in Bayraktar (2008) produces a below par fit to the implied volatility surface (see e.g. Figure 5 in that paper), whereas our model produces an excellent fit (see Section 4.3 and Figure 1).

The other defaultable stock models are those of Carr and Linetsky (2006), Linetsky (2006) and Carr and Wu (2006), which assume that the interest rate is deterministic. Carr and Linetsky (2006), Linetsky (2006) take the volatility and the intensity to be functions of the stock price and obtain a one-dimensional diffusion for the pre-default stock price evolution. Using the fact that the resolvents of particular Markov processes can be computed explicitly, they obtain pricing formulas for stock option prices. On the other hand Carr and Wu (2006) uses a CIR stochastic volatility model. This paper also models the intensity to be a function of the volatility and another endogenous CIR factor. The option prices in this framework are computed numerically using inverse the Fourier transform. We, on the other hand, use asymptotic expansions to provide explicit pricing formulas for stock options in a framework that combines a) the Vasicek interest rate model, b) fast-mean reverting stochastic volatility model, c) defaultable stock price model, d) multi-scale stochastic intensity model.

Our calibration exercise differs from that of Carr and Wu (2006) since they perform a time series analysis to obtain the parameters of the underlying factors (from the the stock option prices and credit default swap spread time series), whereas we calibrate our pricing parameters to the daily implied volatility surface and bond term structure data. Our purpose is to find a risk neutral model that matches a set of observed market prices. This risk neutral model can then be used to price more exotic, illiquid or over-the-counter derivatives. For further discussion of this calibration methodology we refer to Cont and Tankov (2004) (see Chapter 13), Fouque et al. (2000), Fouque et al. (2003) and Papageorgiou and Sircar (2008).

The rest of the paper is organized as follows: In Section 2, we introduce our modeling framework. We also describe how the CDS spread can be computed in our framework. In Section 3, we introduce the asymptotic expansion method. We obtain explicit (asymptotic) prices for bonds and equity options in Section 3.3. In Section 4, we describe the calibration of our parameters and discuss our empirical results. Figures, which show our calibration results, are located after the references.

2 A Framework for Pricing Equity and Credit Derivatives

2.1 The model

Let $(\Omega, \mathcal{H}, \mathbb{P})$ be a complete probability space supporting (i) correlated standard Brownian motions $\vec{W}_t = (W_t^0, W_t^1, W_t^2, W_t^3, W_t^4)$, $t \geq 0$, with

$$\mathbb{E}[W_t^0, W_t^i] = \rho_i t, \quad \mathbb{E}[W_t^i, W_t^j] = \rho_{ij} t, \quad i,j \in \{1,2,3,4\}, \; t \geq 0, \tag{2.1}$$

for some constants $\rho_i, \rho_{ij} \in (-1,1)$, and (ii) a Poisson process N independent of $\vec{W}$. Let us introduce the Cox process (time-changed Poisson process) $\tilde{N}_t \triangleq N(\int_0^t \lambda_s ds)$, $t \geq 0$, where

$$\begin{aligned}
\lambda_t &= f(Y_t, Z_t), \\
dY_t &= \frac{1}{\epsilon}(m - Y_t)dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}} dW_t^2, \quad Y_0 = y, \\
dZ_t &= \delta c(Z_t)dt + \sqrt{\delta} g(Z_t) dW_t^3, \quad Z_0 = z,
\end{aligned} \tag{2.2}$$

in which ϵ, δ are (small) positive constants and f is a strictly positive, bounded, smooth function. We also assume that the functions c and g satisfy Lipschitz continuity and growth conditions so that the diffusion process for Z_t has a unique strong solution. We model the time of default as

$$\tau = \inf\{t \geq 0 : \tilde{N}_t = 1\}. \tag{2.3}$$

We also take interest rate to be stochastic and model it as an Ornstein-Uhlenbeck process

$$dr_t = (\alpha - \beta r_t)dt + \eta dW_t^1, \quad r_0 = r, \tag{2.4}$$

for positive constants α, β, and η.

We model the stock price as the solution of the stochastic differential equation

$$d\bar{X}_t = \bar{X}_t \left(r_t dt + \sigma_t dW_t^0 - d\left(\tilde{N}_t - \int_0^{t\wedge\tau} \lambda_u du \right) \right), \quad \bar{X}_0 = x, \tag{2.5}$$

where the volatility is stochastic and is defined through

$$\sigma_t = \sigma\left(\widetilde{Y}_t\right); \quad d\widetilde{Y}_t = \left(\frac{1}{\epsilon}(\tilde{m} - \widetilde{Y}_t) - \frac{\tilde{\nu}\sqrt{2}}{\sqrt{\epsilon}} \Lambda(\widetilde{Y}_t) \right) dt + \frac{\tilde{\nu}\sqrt{2}}{\sqrt{\epsilon}} dW_t^4, \quad \widetilde{Y}_0 = \tilde{y}. \tag{2.6}$$

Here, Λ is a smooth, bounded function of one variable which represents the market price of volatility risk. The function σ is also a bounded, smooth function. Note that the discounted stock price is a martingale under the measure $\mathbb{P}$, and at the time of default, the stock price jumps down to zero. The pre-banktruptcy stock price coincides with the solution of

$$dX_t = (r_t + \lambda_t)X_t dt + \sigma_t X_t dW_t^0, \quad X_0 = x. \tag{2.7}$$

It will be useful to keep track of different flows of information. Let $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ be the natural filtration of $\vec{W}$. Denote the default indicator process by $I_t = 1_{\{\tau \leq t\}}$, $t \geq 0$, and let $\mathbb{I} = \{\mathcal{I}_t, t \geq 0\}$ be the filtration generated by I. Finally, let $\mathbb{G} = \{\mathcal{G}_t, t \geq 0\}$ be an enlargement of $\mathbb{F}$ such that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{I}_t$, $t \geq 0$.

Since we will take ϵ and δ to be small positive constants, the processes Y and $\widetilde{Y}$ are fast mean reverting, and Z evolves on a slower time scale. See Fouque et al. (2003) for an exposition and motivation of multi-scale modeling in the context of stochastic volatility models.

We note that our specification of the intensity of default coincides with that of Papageorgiou and Sircar (2008), who considered only a framework for pricing credit derivatives. Our stock price specification is similar to that of Linetsky (2006) and Carr and Linetsky (2006) who considered a framework for only pricing equity options on defaultable stocks. Our volatility specification, on the other hand, is in the spirit of Fouque et al. (2000).

Bayraktar (2008) considered a similar modeling framework to the one considered here, but the interest rate was taken to be deterministic. In this paper, by extending this modeling framework to incorporate stochastic interest rates, we are able to consistently price credit and equity derivatives and produce more realistic yield curves and implied volatility surfaces.

2.2 Equity and Credit Derivatives

In our framework, we will price European options and bonds of the same company in a consistent way.

1. The price of a European call option with maturity T and strike price K is given by

$$\begin{aligned} C(t;T,K) &= \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)(\bar{X}_T - K)^+ 1_{\{\tau > T\}} \middle| \mathcal{G}_t\right] \\ &= 1_{\{\tau > t\}} \mathbb{E}\left[\exp\left(-\int_t^T (r_s + \lambda_s) ds\right)(X_T - K)^+ \middle| \mathcal{F}_t\right], \end{aligned} \tag{2.8}$$

in which the equality follows from Lemma 5.1.2 of Bielecki and Rutkowski (2002). (This lemma, lets us write a conditional expectation with respect to $\mathcal{G}_t$ in terms of conditional expectations with respect to $\mathcal{F}_t$). Also, see Linetsky (2006) and Carr and Linetsky (2006) for a similar computation.

On the other hand, the price of a put option with the same maturity and strike price is

$$\begin{aligned}
\mathrm{Put}(t;T) &= \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)(K-X_T)^+ 1_{\{\tau>T\}}\Big|\mathcal{G}_t\right] + \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right) K 1_{\{\tau\le T\}}\big|\mathcal{G}_t\right] \\
&= 1_{\{\tau>t\}}\left(\mathbb{E}\left[\exp\left(-\int_t^T (r_s+\lambda_s)ds\right)(K-X_T)^+\Big|\mathcal{F}_t\right]\right. \\
&\left. + K\mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)\Big|\mathcal{F}_t\right] - K\mathbb{E}\left[\exp\left(-\int_t^T (r_s+\lambda_s)ds\right)\Big|\mathcal{F}_t\right]\right).
\end{aligned} \tag{2.9}$$

2. Consider a defaultable bond with maturity T and par value of 1 dollar. We assume the recovery of the market value, introduced by Duffie and Singleton (1999). In this model, if the issuer company defaults prior to maturity, the holder of the bond recovers a constant fraction $1-l$ of the pre-default value, with $l \in [0,1]$. The price of such a bond is

$$\begin{aligned}
B^c(t;T) &= \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)1_{\{\tau>T\}} + \exp\left(-\int_t^\tau r_s ds\right)1_{\{\tau\le T\}}(1-l)B^c(\tau-;T)\Big|\mathcal{G}_t\right] \\
&= \mathbb{E}\left[\exp\left(-\int_t^T (r_s + l\,\lambda_s)ds\right)\Big|\mathcal{F}_t\right],
\end{aligned} \tag{2.10}$$

on $\{\tau > t\}$, see Duffie and Singleton (1999) and Schönbucher (1998).

3. In the Section 3, we will obtain explicit pricing formulas for equity options and bonds. These formulas will be calibrated to the observed prices. Once our model is calibrated we can then determine the prices of more exotic derivatives. As an example, below we will show how a CDS contract can be priced in our framework.

Consider a credit default swap (CDS) written on B^c, which is a insurance against losses incurred upon default from holding a corporate bond. The protection buyer pays a fixed premium, the so-called CDS spread, to the protection seller. The premium is paid on fixed dates $\mathcal{T} = (T_1, \cdots, T_M)$, with T_M being the maturity of the CDS contract. We denote the CDS spread at time t by $c^{ds}(t;\mathcal{T})$. Our purpose is to determine a fair value for the CDS spread so that what the protection buyer buyer is expected to pay, the value of the premium leg of the contract, is equal to what the protection seller is expected to pay, the value of the protection leg of the contract. For a more detailed description of the CDS contract, see Bielecki and Rutkowski (2002) or Schönbucher (2003).

The present value of the premium leg of the contract is

$$
\begin{aligned}
\text{Premium}(t;\mathcal{T}) &= c^{ds}(t;\mathcal{T})\,\mathbb{E}\left[\sum_{m=1}^{M}\exp\left(-\int_t^{T_m} r_s ds\right)1_{\{\tau>T_m\}}\middle|\mathcal{G}_t\right] \\
&= 1_{\{\tau>t\}}c^{ds}(t;\mathcal{T})\sum_{m=1}^{M}\mathbb{E}\left[\exp\left(-\int_t^{T_m}(r_s+\lambda_s)ds\right)\middle|\mathcal{F}_t\right],
\end{aligned}
\tag{2.11}
$$

in which we assumed that $t < T_1$. The present value of the protection leg of the contract under our assumption of *recovery of market value* is (assuming $l \in [0,1)$)

$$
\begin{aligned}
\text{Protection}(t;\mathcal{T}) &= 1_{\{\tau>t\}}\mathbb{E}\left[\exp\left(-\int_t^{\tau} r_s ds\right)1_{\{\tau\le T_M\}}l\,B^c(\tau-;T_M)\middle|\mathcal{G}_t\right] \\
&= 1_{\{\tau>t\}}\left(\frac{l}{1-l}\right)\left(B^c(t;T_M)-\mathbb{E}\left[\exp\left(-\int_t^{T_M} r_s ds\right)1_{\{\tau>T_M\}}\middle|\mathcal{G}_t\right]\right) \\
&= 1_{\{\tau>t\}}\left(\frac{l}{1-l}\right)\left(B^c(t;T_M)-\mathbb{E}\left[\exp\left(-\int_t^{T_M}(r_s+\lambda_s)ds\right)\middle|\mathcal{F}_t\right]\right),
\end{aligned}
\tag{2.12}
$$

in which the second equality follow from (2.10). Now, the CDS spread can be determined, by setting $\text{Protection}(t;\mathcal{T}) = \text{Premium}(t;\mathcal{T})$ and using equations (2.11) and (2.12), as

$$
c^{ds}(t;\mathcal{T}) = 1_{\{\tau>t\}}\frac{l}{1-l}\frac{B^c(t;T_M)-\mathbb{E}\left[\exp\left(-\int_t^{T_M}(r_s+\lambda_s)ds\right)\middle|\mathcal{F}_t\right]}{\sum_{m=1}^{M}\mathbb{E}\left[\exp\left(-\int_t^{T_m}(r_s+\lambda_s)ds\right)\middle|\mathcal{F}_t\right]} \quad \text{when } l\in[0,1).
\tag{2.13}
$$

Note that when $l = 1$

$$
\text{Protection}(t;\mathcal{T}) = 1_{\{\tau>t\}}\mathbb{E}\left[\left(\int_t^{T_M}\lambda_u du\right)\exp\left(-\int_t^{T_M}(r_s+\lambda_s)ds\right)\right].
\tag{2.14}
$$

Observe that computing $c^{ds}(t;\mathcal{T})$ requires the value of l (an unobserved quantity) and the value of $B^c(t;T_M)$. This value may or may not be available from the bond price data. If $B^c(t;T_M)$ is not quoted, then one has to construct the yield curve to obtain this value. Moreover, to compute $c^{ds}(t;\mathcal{T})$ we also need $\mathbb{E}\left[\exp\left(-\int_t^{T_i}(r_s+\lambda_s)ds\right)\middle|\mathcal{F}_t\right]$, $i\in\{1,\cdots,M\}$, which are not available. (Since the loss rate may not be equal to 1 these values can not be recovered directly from the bond prices.)

In the next section, we will develop approximate pricing formulas for equity options and defaultable bonds. We will then calibrate these formulas to the option and bond data in Section 4, and obtain the value of the loss rate and other model parameters. If we let $\widetilde{B}^c(t,T;l)$ denote the approximation for the price at time t of a defaultable bond that matures at time T, and has loss rate l (see (4.1)), then the model implied CDS spead with maturity T_M can be obtained as

$$c^{ds}_{\text{model}}(t,T_M) = \frac{l}{1-l}\frac{\widetilde{B}^c(t,T_M;l) - \widetilde{B}^c(t,T_M;1)}{\sum_{m=1}^{M} \widetilde{B}^c(t,T_m;1)}. \tag{2.15}$$

Usually, to determine the CDS spread, one assumes that the bond has a recovery of face value. We, on the other hand, use the recovery of market value assumption on the bond to determine the value of the CDS spread. This is because we would like to first calibrate our model to the bond prices, for which we made a recovery of market value assumption. Also, the simplicity of the CDS spread formula under the recovery of market value assumption justifies our choice.

3 Explicit Pricing Formulas for Credit and Equity Derivatives

3.1 Pricing equation

Let $P^{\epsilon,\delta}$ denote

$$P^{\epsilon,\delta}(t,X_t,r_t,Y_t,\widetilde{Y}_t,Z_t) = \mathbb{E}\left[\exp\left(-\int_t^T (r_s + l\lambda_s)ds\right) h(X_T)\middle|\mathcal{F}_t\right]. \tag{3.1}$$

When $l=1$ and $h(X_T) = (X_T-K)^+$, $P^{\epsilon,\delta}$ is the price of a call option (on a defaultable stock). On the other hand, when $h(X_T)=1$, $P^{\epsilon,\delta}$ becomes the price of a defaultable bond.

Using the Feynman-Kac formula, we can characterize $P^{\epsilon,\delta}$ as the solution of

$$\begin{aligned} &\mathcal{L}^{\epsilon,\delta}P^{\epsilon,\delta}(t,x,r,y,\tilde{y},z) = 0, \\ &P^{\epsilon,\delta}(T,x,r,y,\tilde{y},z) = h(x), \end{aligned} \tag{3.2}$$

where the partial differential operator $\mathcal{L}^{\epsilon,\delta}$ is defined as

$$\mathcal{L}^{\epsilon,\delta} \triangleq \frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\delta}\mathcal{M}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\epsilon}}\mathcal{M}_3, \tag{3.3}$$

in which

$$\mathcal{L}_0 \triangleq \nu^2 \frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y} + \tilde{\nu}^2 \frac{\partial^2}{\partial \tilde{y}^2} + (\tilde{m}-\tilde{y})\frac{\partial}{\partial \tilde{y}} + 2\rho_{24} v\tilde{v}\frac{\partial^2}{\partial y \partial \tilde{y}},$$
$$\mathcal{L}_1 \triangleq \rho_2\sigma(\tilde{y})\nu\sqrt{2}x\frac{\partial^2}{\partial x\partial y} + \rho_{12}\eta\nu\sqrt{2}\frac{\partial^2}{\partial r\partial y} + \rho_4\sigma(\tilde{y})\tilde{\nu}\sqrt{2}x\frac{\partial^2}{\partial x\partial \tilde{y}} + \rho_{14}\eta\tilde{\nu}\sqrt{2}\frac{\partial^2}{\partial r\partial \tilde{y}} - \Lambda(\tilde{y})\tilde{\nu}\sqrt{2}\frac{\partial}{\partial \tilde{y}},$$
$$\mathcal{L}_2 \triangleq \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2(\tilde{y})x^2\frac{\partial^2}{\partial x^2} + (r+f(y,z))x\frac{\partial}{\partial x} + (\alpha-\beta r)\frac{\partial}{\partial r} + \sigma(\tilde{y})\eta\rho_1 x\frac{\partial^2}{\partial x\partial r} + \frac{1}{2}\eta^2\frac{\partial^2}{\partial r^2} - (r+l\,f(y,z))\cdot,$$

$$\mathcal{M}_1 \triangleq \sigma(\tilde{y})\rho_3 g(z)x\frac{\partial^2}{\partial x\partial z} + \eta\rho_{13}g(z)\frac{\partial^2}{\partial r\partial z}, \quad \mathcal{M}_2 \triangleq c(z)\frac{\partial}{\partial z} + \frac{1}{2}g^2(z)\frac{\partial^2}{\partial z^2},$$
$$\mathcal{M}_3 \triangleq \rho_{23}\nu\sqrt{2}g(z)\frac{\partial^2}{\partial y\partial z} + \rho_{34}\tilde{\nu}\sqrt{2}g(z)\frac{\partial^2}{\partial \tilde{y}\partial z}.$$

3.2 Asymptotic expansion

We construct an asymptotic expansion for $P^{\epsilon,\delta}$ as $\epsilon, \delta \to 0$. First, we consider an expansion of $P^{\epsilon,\delta}$ in powers of $\sqrt{\delta}$

$$P^{\epsilon,\delta} = P_0^\epsilon + \sqrt{\delta}P_1^\epsilon + \delta P_2^\epsilon + \cdots \tag{3.4}$$

By inserting (3.4) into (3.2) and comparing the δ^0 and δ terms, we obtain that P_0^ϵ satisfies

$$\left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)P_0^\epsilon = 0, \tag{3.5}$$
$$P_0^\epsilon(T,x,r,y,\tilde{y},z) = h(x),$$

and that P_1^ϵ satisfies

$$\left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)P_1^\epsilon = -\left(\mathcal{M}_1 + \frac{1}{\sqrt{\epsilon}}\mathcal{M}_3\right)P_0^\epsilon, \tag{3.6}$$
$$P_1^\epsilon(T,x,y,\tilde{y},z,r) = 0.$$

Next, we expand the solutions of (3.5) and (3.6) in powers of $\sqrt{\epsilon}$

$$P_0^\epsilon = P_0 + \sqrt{\epsilon}P_{1,0} + \epsilon P_{2,0} + \epsilon^{3/2}P_{3,0} + \cdots \tag{3.7}$$
$$P_1^\epsilon = P_{0,1} + \sqrt{\epsilon}P_{1,1} + \epsilon P_{2,1} + \epsilon^{3/2}P_{3,1} + \cdots \tag{3.8}$$

Inserting the expansion for P_0^ϵ into (3.5) and matching the $1/\epsilon$ terms gives $\mathcal{L}_0 P_0 = 0$. We choose P_0 not to depend on y and $\tilde{y}$ because the other solutions have exponential growth at infinity (see e.g. Fouque et al. (2003)). Similarly, by matching the $1/\sqrt{\epsilon}$ terms in (3.5) we obtain that $\mathcal{L}_0 P_{1,0} + \mathcal{L}_1 P_0 = 0$. Since $\mathcal{L}_1$ takes derivatives only with respect to y and $\tilde{y}$, we observe that $\mathcal{L}_0 P_{1,0} = 0$. We choose $P_{1,0}$ not to depend on y and $\tilde{y}$.

Now equating the order-one terms in the expansion of (3.5) and using the fact that $\mathcal{L}_1 P_{1,0} = 0$, we get that

$$\mathcal{L}_0 P_{2,0} + \mathcal{L}_2 P_0 = 0, \tag{3.9}$$

which is a Poisson equation for $P_{2,0}$ (see e.g. Fouque et al. (2000)). The solvability condition for this equation requires that

$$\langle \mathcal{L}_2 \rangle P_0 = 0, \tag{3.10}$$

where $\langle \cdot \rangle$ denotes the averaging with respect to the invariant distribution of $(Y_t, \widetilde{Y}_t)$, whose density is given by

$$\Psi(y,\tilde{y}) = \frac{1}{2\pi\nu\tilde{\nu}} \exp\left\{ -\frac{1}{2(1-\rho_{24}^2)} \left[\left(\frac{y-m}{\nu}\right)^2 + \left(\frac{\tilde{y}-\tilde{m}}{\tilde{\nu}}\right)^2 - 2\rho_{24}\frac{(y-m)(\tilde{y}-\tilde{m})}{\nu\tilde{\nu}} \right] \right\}. \tag{3.11}$$

Let us denote

$$\bar{\sigma}_1 \triangleq \langle \sigma(\tilde{y}) \rangle, \quad \bar{\sigma}_2^2 \triangleq \langle \sigma^2(\tilde{y}) \rangle, \quad \bar{\lambda}(z) = \langle f(y,z) \rangle. \tag{3.12}$$

To demonstrate the effect of averaging on $\mathcal{L}_2$, let us write

$$\langle \mathcal{L}_2 \rangle := \frac{\partial}{\partial t} + \frac{1}{2}\bar{\sigma}_2^2 x^2 \frac{\partial^2}{\partial x^2} + (r+\bar{\lambda}(z))x\frac{\partial}{\partial x} + (\alpha - \beta r)\frac{\partial}{\partial r} + \bar{\sigma}_1 \eta \rho_1 x \frac{\partial^2}{\partial x \partial r} + \frac{1}{2}\eta^2 \frac{\partial^2}{\partial r^2} - (r + l\,\bar{\lambda}(z))\cdot \tag{3.13}$$

Together with the terminal condition

$$P_0(T,x,r;z) = h(x), \tag{3.14}$$

equation (3.10) defines the leading order term P_0. On the other hand from (3.9), we can also deduce that

$$P_{2,0} = -\mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) P_0. \tag{3.15}$$

Matching the $\sqrt{\epsilon}$ order terms in the expansion of (3.5) yields

$$\mathcal{L}_0 P_{3,0} + \mathcal{L}_1 P_{2,0} + \mathcal{L}_2 P_{1,0} = 0, \tag{3.16}$$

which is a Poisson equation for $P_{3,0}$. The solvability condition for this equation requires that

$$\langle \mathcal{L}_2 P_{1,0} \rangle = -\langle \mathcal{L}_1 P_{2,0} \rangle = \langle \mathcal{L}_1 \mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) \rangle P_0, \tag{3.17}$$

which along with the terminal condition

$$P_{1,0}(T,x,r;z) = 0, \tag{3.18}$$

completely identifies the function $P_{1,0}$. To obtain the second equality in (3.17) we used (3.15).

Next, we will express the right-hand side of (3.17) more explicitly. To this end, let ψ, κ, and ϕ be the solutions of the Poisson equations

$$\mathcal{L}_0\psi(\tilde{y}) = \sigma(\tilde{y}) - \bar{\sigma}_1, \quad \mathcal{L}_0\kappa(\tilde{y}) = \sigma^2(\tilde{y}) - \bar{\sigma}_2^2, \quad \text{and} \quad \mathcal{L}_0\phi(y,z) = (f(y,z) - \bar{\lambda}(z)), \tag{3.19}$$

respectively. First observe that

$$(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle)P_0 = \frac{1}{2}(\sigma^2(\tilde{y}) - \bar{\sigma}_2^2)x^2\frac{\partial^2 P_0}{\partial x^2} + (\sigma(\tilde{y}) - \bar{\sigma}_1)\eta\rho_1 x\frac{\partial^2 P_0}{\partial x \partial r} + l\,(f(y,z) - \bar{\lambda}(z))\left(x\frac{\partial P_0}{\partial x} - P_0\right). \tag{3.20}$$

Now, along with (3.19), we can write

$$\mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle)P_0 = \frac{1}{2}\kappa(\tilde{y})x^2\frac{\partial^2 P_0}{\partial x^2} + \psi(\tilde{y})\eta\rho_1 x\frac{\partial^2 P_0}{\partial x \partial r} + l\,\phi(y,z)\left(x\frac{\partial P_0}{\partial x} - P_0\right). \tag{3.21}$$

Applying the differential operator $\mathcal{L}_1$ to the last expression yields

$$\begin{aligned}
\langle \mathcal{L}_1\mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle)\rangle P_0 &= l\,\rho_2\nu\sqrt{2}\langle \sigma\phi_y \rangle(z)x^2\frac{\partial P_0}{\partial x^2} + l\,\rho_{12}\eta\nu\sqrt{2}\langle \phi_y \rangle(z)\frac{\partial}{\partial r}\left(x\frac{\partial P_0}{\partial x} - P_0\right) \\
&+ \rho_4\tilde{\nu}\sqrt{2}\left(\frac{1}{2}\langle \sigma\kappa_{\tilde{y}} \rangle x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2 P_0}{\partial x^2}\right) + \langle \sigma\psi_{\tilde{y}} \rangle\eta\rho_1 x\frac{\partial}{\partial x}\left(x\frac{\partial^2 P_0}{\partial x \partial r}\right)\right) \\
&+ \rho_{14}\eta\tilde{\nu}\sqrt{2}\left(\frac{1}{2}\langle \kappa_{\tilde{y}} \rangle\frac{\partial}{\partial r}\left(x^2\frac{\partial^2 P_0}{\partial x^2}\right) + \langle \psi_{\tilde{y}} \rangle\eta\rho_1\frac{\partial}{\partial r}\left(x\frac{\partial^2 P_0}{\partial x \partial r}\right)\right) \\
&- \tilde{\nu}\sqrt{2}\left(\frac{1}{2}\langle \Lambda\kappa_{\tilde{y}} \rangle x^2\frac{\partial P_0}{\partial x^2} + \langle \Lambda\psi_{\tilde{y}} \rangle\eta\rho_1 x\frac{\partial^2 P_0}{\partial x \partial r}\right).
\end{aligned} \tag{3.22}$$

Finally, we insert the expression for P_1^ϵ in (3.8) into (3.6) and collect the terms with the same powers of ϵ. Arguing as before, we obtain that $P_{0,1}$ is independent of y and $\tilde{y}$ and satisfies:

$$\langle \mathcal{L}_2 \rangle P_{0,1} = -\langle \mathcal{M}_1 \rangle P_0, \quad P_{0,1}(T,x,r;z) = 0. \tag{3.23}$$

3.3 Explicit pricing formula

We approximate $P^{\epsilon,\delta}$ defined in (3.1) by

$$\widetilde{P}^{\epsilon,\delta} = P_0 + \sqrt{\epsilon}P_{1,0} + \sqrt{\delta}P_{0,1}. \tag{3.24}$$

Since the Vasicek interest rate process is unbounded, which implies that the potential term in $\mathcal{L}_2$ or the discounting term in (3.1) is unbounded, the arguments of Fouque et al. (2003) can not be directly used. However as in Cotton et al. (2004) and Papageorgiou and Sircar (2008), one can write

$$\begin{aligned} P^{\epsilon,\delta}(t, X_t, r_t, Y_t, \widetilde{Y}_t, Z_t) &= B(t,T)\mathbb{E}^T\left[\exp\left(-\int_t^T l\lambda_s ds\right) h(X_T)\middle|\mathcal{F}_t\right] \\ &=: B(t,T)F^{\epsilon,\delta}(t, X_t, r_t, Y_t, \widetilde{Y}_t, Z_t), \end{aligned} \tag{3.25}$$

in which

$$\frac{d\mathbb{P}^T}{d\mathbb{P}} = \frac{\exp\left(-\int_0^T r_s ds\right)}{B(0,T)}, \tag{3.26}$$

and

$$B(t,T) = \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)\middle|\mathcal{F}_t\right]. \tag{3.27}$$

Now, the analysis of Fouque et al. (2003) can be used to approximate $F^{\epsilon,\delta}(t,x,r,y,\tilde{y},z)$. As a result of this analysis for each $(t,x,r,y,\tilde{y},z)$, there exists a constant C such that $|P^{\epsilon,\delta} - \widetilde{P}^{\epsilon,\delta}| \leq C\cdot(\epsilon+\delta)$ when h is smooth, and $|P^{\epsilon,\delta} - \widetilde{P}^{\epsilon,\delta}| \leq C\cdot(\epsilon\log(\epsilon)+\delta+\sqrt{\epsilon\delta})$ when h is a put or a call pay-off. In what follows, we will obtain P_0, $P_{1,0}$ and $P_{0,1}$ explicitly.

Our first objective is to develop a closed-form expression for P_0, the solution of (3.10) and (3.14).

Proposition 3.1. *The leading order term P_0 in (3.24) is given by:*

$$P_0(t,x,r;z) = B_0^c(t,r;z,T,l)\int_{-\infty}^{\infty} h(\exp(u))\frac{1}{\sqrt{2\pi v_{t,T}}}\exp(-\frac{(u-m_{t,T})^2}{2v_{t,T}})du, \tag{3.28}$$

where

$$B_0^c(t,r;z,T,l) \triangleq \exp\big(-l\bar{\lambda}(z)(T-t) + a(T-t) - b(T-t)r\big), \tag{3.29}$$

in which the functions $a(s)$ and $b(s)$ are defined as:

$$a(s) = \left(\frac{\eta^2}{2\beta^2} - \frac{\alpha}{\beta}\right)s + \left(\frac{\eta^2}{\beta^3} - \frac{\alpha}{\beta^2}\right)(\exp(-\beta s) - 1) - \frac{\eta^2}{4\beta^3}(\exp(-2\beta s) - 1) \tag{3.30}$$

and $b(s) = (1-\exp(-\beta s))/\beta$. On the other hand,

$$\begin{aligned} v_{t,T} = &\left(\bar{\sigma}_2^2 + \frac{2\eta\rho_1\bar{\sigma}_1}{\beta} + \frac{\eta^2}{\beta^2}\right)(T-t) + \left(\frac{2\eta\rho_1\bar{\sigma}_1}{\beta^2} + \frac{2\eta^2}{\beta^3}\right)\exp(-\beta(T-t)) \\ &- \frac{\eta^2}{2\beta^3}\exp(-2\beta(T-t)) - \left(\frac{2\eta\rho_1\bar{\sigma}_1}{\beta^2} + \frac{3\eta^2}{2\beta^3}\right), \end{aligned} \tag{3.31}$$

and

$$m_{t,T} = \log(x) + \bar{\lambda}\cdot(T-t) - a(T-t) + b(T-t)r - \frac{1}{2}v_{t,T}. \tag{3.32}$$

Proof. By applying the Feynman-Kac theorem to (3.10) and (3.14) we have that

$$P_0(t,x,r;z) = \mathbb{E}\left[\exp\left(-\int_t^T (r_s + l\bar{\lambda}(z))ds\right) h(S_T)\middle| S_t = x, r_t = r\right], \tag{3.33}$$

where the dynamics of S is given by

$$dS_t = (r_t + \bar{\lambda}(z))S_t dt + \bar{\sigma}_2 S_t d\widetilde{W}_t^0, \tag{3.34}$$

in which $\widetilde{W}^0$ is a Wiener process whose correlation with W^1 is $\bar{\rho}_1 = \frac{\bar{\sigma}_1}{\bar{\sigma}_2}\rho_1$.

Let us define

$$\widetilde{P}_0(t,x,r) = \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right) h(\widetilde{S}_T)\middle| \widetilde{S}_t = x, r_t = r\right], \tag{3.35}$$

in which

$$d\widetilde{S}_t = r_t \widetilde{S}_t dt + \bar{\sigma}_2 \widetilde{S}_t d\widetilde{W}_t^0. \tag{3.36}$$

Then

$$P_0(t,x,r;z) = e^{-l\bar{\lambda}(z)\,(T-t)} \widetilde{P}_0(t, x\exp(\bar{\lambda}(z)(T-t)), r). \tag{3.37}$$

Now, by following Geman et al. (1995), we change the probability measure $\mathbb{P}$ to the forward measure $\mathbb{P}^T$ through the Radon-Nikodym derivative (3.26)

We can obtain the following representation of $\tilde{P}_0$ using the T-forward measure

$$\widetilde{P}_0(t,\widetilde{S}_t,r_t) = B(t,T)\mathbb{E}^T\left[h(\widetilde{S}_T)|\mathcal{F}_t\right] = B(t,T)\mathbb{E}^T\left[h(F_T)|\mathcal{F}_t\right], \tag{3.38}$$

in which

$$F_t \triangleq \frac{\widetilde{S}_t}{B(t,T)}, \tag{3.39}$$

which is a $\mathbb{P}^T$ martingale. Note that an explicit expression for $B(t,T)$ is available since r_t is a Vasicek model, and it is given in terms of the functions a and b:

$$B(t,T) = \exp(a(T-t) - b(T-t)r_t). \tag{3.40}$$

By applying Itô's formula to (3.39), we observe that the dynamics of F are

$$dF_t = F_t(\bar{\sigma}_1 d\widetilde{W}_t^0 + b(T-t)\eta d\widetilde{W}_t^1), \tag{3.41}$$

in which $\widetilde{W}^1$ is a $\mathbb{P}^T$ Brownian motion whose correlation with the $\widetilde{W}^0$ (which is still a Brownian motion under $\mathbb{P}^T$) is $\bar{\rho}_1$. Given X_t and $B(t,T)$, the random variable $\log F_T$ is normally distributed with variance

$$\begin{aligned} v_{t,T} &= \bar{\sigma}_2^2(T-t) + \eta^2 \int_t^T b^2(T-s)ds + 2\eta\bar{\rho}_1\bar{\sigma}_2 \int_t^T b(T-s)ds \\ &= \left(\bar{\sigma}_2^2 + \frac{2\eta\bar{\rho}_1\bar{\sigma}_2}{\beta} + \frac{\eta^2}{\beta^2}\right)(T-t) + \left(\frac{2\eta\bar{\rho}_1\bar{\sigma}_2}{\beta^2} + \frac{2\eta^2}{\beta^3}\right)\exp(-\beta(T-t)) \\ &- \frac{\eta^2}{2\beta^3}\exp(-2\beta(T-t)) - \left(\frac{2\eta\bar{\rho}_1\bar{\sigma}_2}{\beta^2} + \frac{3\eta^2}{2\beta^3}\right), \end{aligned} \tag{3.42}$$

and mean

$$m_{t,T} = \log F_t - \frac{1}{2}\int_t^T (\bar{\sigma}_2^2 + b^2(T-s)\eta^2 + \bar{\rho}_1\bar{\sigma}_2 b(T-s)\eta)ds = \log\left(\frac{\widetilde{S}_t}{B(t,T)}\right) - \frac{1}{2}v_{t,T}. \tag{3.43}$$

Now the result immediately follows. □

An immediate corollary of the last proposition is the following:

Corollary 3.1. *i) When $l = 1$, $h(x) = (x-K)^+$, then (3.28) becomes*

$$C_0(t,x,r;z) = xN(d_1) - KB_0^c(t,r;z,T,1)N(d_2), \tag{3.44}$$

in which N is the standard normal cumulative distribution function and

$$d_{1,2} = \frac{\log \frac{x}{KB_0^c(t,r;z,T,1)} \pm \frac{1}{2}v_{t,T}}{\sqrt{v_{t,T}}}. \tag{3.45}$$

ii) When $l = 1$, and $h(x) = (K-x)^+$, then (3.28) becomes

$$Put_0(t,x,r;z) = -x + xN(d_1) - KB_0^c(t,r;z,T,1)N(d_2) + KB_0^c(t,r;z,T,0). \tag{3.46}$$

iii) When $h(x) = 1$, then (3.28) coincides with (3.30) in Papageorgiou and Sircar (2008).

Proposition 3.2. *The correction term $\sqrt{\epsilon}P_{1,0}$ is given by*

$$\begin{aligned} \sqrt{\epsilon}P_{1,0} &= -(T-t)\left(V_1^\epsilon(z)x^2\frac{\partial^2 P_0}{\partial x^2} + V_2^\epsilon x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2 P_0}{\partial x^2}\right)\right) \\ &+ l\,V_3^\epsilon(z)\left(-x\frac{\partial^2 P_0}{\partial x\partial\alpha} - \frac{\partial P_0}{\partial\alpha}\right) + V_4^\epsilon x^2\frac{\partial^3 P_0}{\partial x^2\partial\alpha} + V_5^\epsilon x\frac{\partial^2 P_0}{\partial\eta\partial x} + V_6^\epsilon x\frac{\partial^2 P_0}{\partial x\partial\alpha}, \end{aligned} \tag{3.47}$$

in which

$$
\begin{aligned}
V_1^\epsilon(z) &= \sqrt{\epsilon}\left(l\,\rho_2\nu\sqrt{2}\langle\sigma\phi_y\rangle(z) - \tilde{\nu}\sqrt{2}\frac{1}{2}\langle\Lambda\kappa_{\tilde{y}}\rangle\right), \quad V_2^\epsilon = \frac{1}{2}\sqrt{\epsilon}\rho_4\tilde{\nu}\sqrt{2}\langle\sigma\kappa_{\tilde{y}}\rangle,\\
V_3^\epsilon(z) &= \sqrt{\epsilon}(\rho_{12}\eta\nu\sqrt{2}\langle\phi_y\rangle(z)),\\
V_4^\epsilon &= -\sqrt{\epsilon}\left(\frac{1}{2}\rho_{14}\eta\tilde{\nu}\sqrt{2}\langle\kappa_{\tilde{y}}\rangle - \rho_4\tilde{\nu}\sqrt{2}\langle\sigma\psi_{\tilde{y}}\rangle\eta\rho_1 + \rho_{14}\eta\tilde{\nu}\sqrt{2}\langle\psi_{\tilde{y}}\rangle\bar{\sigma}_1\rho_1^2\right),\\
V_5^\epsilon &= -\sqrt{\epsilon}(\rho_{14}\eta\tilde{\nu}\sqrt{2}\langle\psi_{\tilde{y}}\rangle\rho_1),\\
V_6^\epsilon &= \sqrt{\epsilon}(-\rho_4\tilde{\nu}\sqrt{2}\langle\sigma\psi_{\tilde{y}}\rangle\eta\rho_1 + \rho_{14}\eta\tilde{\nu}\sqrt{2}\langle\psi_{\tilde{y}}\rangle\bar{\sigma}_1\rho_1^2 - \tilde{\nu}\sqrt{2}\langle\Lambda\psi_{\tilde{y}}\rangle\eta\rho_1).
\end{aligned}
\tag{3.48}
$$

Observe that V_i^ϵ, $i \in \{1, \cdots, 6\}$ may be functions of the initial value of the slow factor Z. They do not depend on initial values of Y, $\widetilde{Y}$; the effect of the fast scale factors are *averaged out* (in the approximation formula). The V parameters depend on the fast factors through their mean reversion level, volatility and their correlation with the other state variables. These parameters also do not depend on r directly, but P_0 and $P_{1,0}$ are functions of this variable. Note that if we take the volatility, σ_t, to depend on a slow factor (besides the fast factor $\widetilde{Y}$), say $\widetilde{Z}$, then the parameters V_2^ϵ, V_4^ϵ, V_5^ϵ, V_6^ϵ will be functions of the initial value of $\widetilde{Z}$. On the other hand, V_1^ϵ will depend on both z and $\tilde{z}$, and V_3^ϵ will only depend on z. One should note that the above expressions for these parameters will still look the same.

Proof. Recall that $P_{1,0}$ is the solution of (3.17) and (3.18) and that the right-hand-side of (3.17) is given by (3.22). The result is a simple algebraic exercise given the following four observations:

1) $x^n \frac{\partial^n}{\partial x^n}$ commutes with $\langle\mathcal{L}_2\rangle$.

2) $-(T-t)(x^n \frac{\partial^n}{\partial x^n})P_0$ solves:

$$\langle\mathcal{L}_2\rangle u = \left(x^n \frac{\partial^n}{\partial x^n}\right)P_0, \quad u(T,x,r;z) = 0. \tag{3.49}$$

3) By differentiating (3.10) and (3.14) with respect to α, we see that $-\frac{\partial P_0}{\partial \alpha}$ also solves

$$\langle\mathcal{L}_2\rangle u = \frac{\partial P_0}{\partial r}, \quad u(T,x,r;z) = 0. \tag{3.50}$$

4) Using 1) and 2) above and the equation we obtain by differentiating (3.10) with respect to η, we can show that $1/\eta \cdot (\bar{\sigma}_1\rho_1 x \frac{\partial^2 P_0}{\partial x \partial \alpha} - \frac{\partial P_0}{\partial \eta})$ solves

$$\langle\mathcal{L}_2\rangle u = \frac{\partial^2 P_0}{\partial r^2}, \quad u(T,x,r;z) = 0. \tag{3.51}$$

□

Remark 3.1. *By differentiating (3.10) with respect to r, we obtain*

$$\langle \mathcal{L}_2 \rangle \frac{\partial P_0}{\partial r} = -x \frac{\partial}{\partial x} P_0 + \beta \frac{\partial P_0}{\partial r} + P_0. \tag{3.52}$$

Using observation 2 in the proof of Proposition 3.2, we see that $\frac{1}{\beta}\left(-(T-t)(x\frac{\partial P_0}{\partial x} - P_0) + \frac{\partial P_0}{\partial r}\right)$ *solves*

$$\langle \mathcal{L}_2 \rangle u = \frac{\partial P_0}{\partial r}, \quad u(T, x, r; z) = 0. \tag{3.53}$$

Now, it follows from observation 3 in the proof of Proposition 3.2 that

$$-\frac{\partial P_0}{\partial \alpha} = \frac{1}{\beta}\left(-(T-t)\left(x\frac{\partial P_0}{\partial x} - P_0\right) + \frac{\partial P_0}{\partial r}\right). \tag{3.54}$$

Using this identity, we can express (3.47) only in terms of the "Greeks".

Next, we obtain an explicit expression for $P_{0,1}$, the solution of (3.23). We need some preparation first. By differentiating (3.10) with respect to z, we see that $\frac{\partial P_0}{\partial z}$ solves

$$\langle \mathcal{L}_2 \rangle u = -\bar{\lambda}'(z) x \frac{\partial P_0}{\partial x} + l\, \bar{\lambda}'(z) P_0, \quad u(T, x, r; z) = 0. \tag{3.55}$$

As a result (see Observation 2 in the proof of Propostion 3.2)

$$\frac{\partial P_0}{\partial z} = (T-t)\bar{\lambda}'(z)\left(x\frac{\partial P_0}{\partial x} - l\, P_0\right), \tag{3.56}$$

from which it follows that $-\langle \mathcal{M}_1 \rangle P_0$ can be represented as

$$-\langle \mathcal{M}_1 \rangle P_0 = -(T-t)\bar{\lambda}'(z)\left(\bar{\sigma}_1 \rho_3 g(z)\left(x^2\frac{\partial^2 P_0}{\partial x^2} + (1-l)x\frac{\partial P_0}{\partial x}\right) + \eta\rho_{13} g(z)\left(x\frac{\partial^2 P_0}{\partial x \partial r} - l\frac{\partial P_0}{\partial r}\right)\right). \tag{3.57}$$

Proposition 3.3. *The correction term $\sqrt{\delta} P_{0,1}$ is given by*

$$\begin{aligned}\sqrt{\delta} P_{0,1} = V_1^\delta(z)\frac{(T-t)^2}{2}\left(x^2\frac{\partial^2 P_0}{\partial x^2} + (1-l)x\frac{\partial P_0}{\partial x}\right) + V_2^\delta(z)\frac{1}{\beta}\Bigg[x\frac{\partial^2 P_0}{\partial \alpha \partial x} - l\,\frac{\partial P_0}{\partial \alpha} \\ + \frac{(T-t)^2}{2}\left(x^2\frac{\partial^2 P_0}{\partial x^2} - l\, x\frac{\partial P_0}{\partial x} + l\, P_0\right) - (T-t)\left(x\frac{\partial^2 P_0}{\partial r \partial x} - l\,\frac{\partial P_0}{\partial r}\right)\Bigg],\end{aligned} \tag{3.58}$$

in which

$$V_1^\delta(z) = \sqrt{\delta}\bar{\lambda}'(z)\bar{\sigma}_1\rho_3 g(z), \quad V_2^\delta(z) = \sqrt{\delta}\bar{\lambda}'(z)\eta\rho_{13} g(z). \tag{3.59}$$

Proof. We construct the solution from the following observations and superposition since

$\langle \mathcal{L}_2 \rangle$ is linear:

1) We first observe that $\frac{(T-t)^2}{2}(x^n \frac{\partial^n}{\partial x^n})P_0$ solves

$$\langle \mathcal{L}_2 \rangle u = -(T-t) \left(x^n \frac{\partial^n}{\partial x^n} \right) P_0, \quad u(T, x, r; z) = 0. \tag{3.60}$$

2) Next, we apply $\langle \mathcal{L}_2 \rangle$ on $(T-t)\frac{\partial P_0}{\partial r}$ and obtain

$$\langle \mathcal{L}_2 \rangle \left((T-t) \frac{\partial P_0}{\partial r} \right) = -\frac{\partial P_0}{\partial r} + (T-t) \left(-x \frac{\partial P_0}{\partial x} + \beta \frac{\partial P_0}{\partial r} + P_0 \right), \tag{3.61}$$

as a result of which we see that

$$\frac{1}{\beta} \left[-\frac{\partial P_0}{\partial \alpha} - \frac{(T-t)^2}{2} \left(x \frac{\partial P_0}{\partial x} - P_0 \right) + (T-t) \frac{\partial P_0}{\partial r} \right] \tag{3.62}$$

solves

$$\langle \mathcal{L}_2 \rangle u = (T-t) \frac{\partial P_0}{\partial r}, \quad u(T, x, r; z) = 0. \tag{3.63}$$

□

4 Calibration of the Model

In this section, we will calibrate the loss rate l and the parameters

$$\{\bar{\lambda}(z), V_1^\epsilon(z), V_2^\epsilon, V_3^\epsilon(z), V_4^\epsilon, V_5^\epsilon, V_6^\epsilon, V_1^\delta(z), V_2^\delta(z)\},$$

which appear in the expressions (3.28), (3.47), and (3.58) on a daily basis (see, e.g., Fouque et al. (2003) and Papageorgiou and Sircar (2008) for similar calibration exercises carried out only for the option data or only for the bond data). We demonstrate this calibration on Ford Motor Company. Note that there are some common parameters between equity options and corporate bonds. Therefore, our model will be calibrated simultaneously to both of these data sets. We will also calibrate the parameters of the interest rate and stock models to the yield curve data, historical spot rate data and historical stock price data.

We look at how our model-implied volatility matches the real option implied volatility. We compare our results against those of Fouque et al. (2003). We see that even when we make the unrealistic assumption of constant volatility, our model is able to produce a very good fit.

Finally, in the context of index options (when $\lambda = 0$), using SPX 500 index options data, we show the importance of accounting for stochastic interest rates by comparing our model to that of Fouque et al. (2000, 2003).

4.1 Data description

- The daily closing stock price data is obtained from finance.yahoo.com.
- The stock option data is from OptionMetrics under WRDS database, which is the same database used in Carr and Wu (2006).
    - For index options, SPX 500 in our case, we use the data from their Volatility Surface file. The file contains information on standardized options, both calls and puts, with expirations of 30, 60, 91, 122, 152, 182, 273, 365, 547, and 730 calender days. Implied volatilities there are interpolated data using a methodology based on kernel smoothing algorithm. The interpolated implied volatilities are very close to real data because there are a great number of options each day for SPX 500 with different maturities and strikes. The calibration results for index options are presented in Figure 4 and only the data set on the June 8, 2007 is used.
    - On September 15, 2006 (Friday) Ford announced that it would not be paying dividends (see e.g., http://money.cnn.com/2006/09/15/news/companies/ford/index.htm). Therefore, call options on Ford after that date do not have early exercise premium starting from Sep 18, 2006. We use Ford's implied volatility surface on April 4, 2007 and June 8, 2007 to create Figures 1 and 4, respectively. We excluded the observations with zero trading volume or with maturity less than 9 days.

      We find that the results given by using interpolated implied volatilities in the Volatility Surface File and data implied volatilities differ. This may be due to the fact that there are a limited number of option prices available for individual companies; i.e., there may not be enough data points for the implied volatilities to be accurately interpolated. Therefore, we use the Option Price file, which contains the historical option price information, of the OptionMetrics database
- For both days (April 4, 2007 and June 8, 2007), we U.S government Treasury yield data with maturities: 1 month, 3 monts, 6 months, 1 year, 2 years, 3 years, 5 years, 7 years, 10 years, 20 years. This data set is available at: www.treasury.gov/offices/domestic-finance/debt-management/interest-rate/yield.shtml.
- Corporate bond data is obtained from Bloomberg. Number of available bond quotes and bond maturities vary. Typically there are around 15 data points, for example, on June 8th, we have the following maturities: 0.60278, 1.0222, 1.1861, 1.3139, 1.4083, 1.5944, 2.3889, 2.6028, 3.0194, 3.2694, 3.3972, 3.6472, 4.1722, 4.3806, 6.3139, 9.5194.

4.2 The parameter estimation

The following parameters can be directly estimated from the spot-rate and stock price historical data:

1. The parameters of the interest rate model $\{\alpha, \beta, \eta\}$ are obtained by a least-square fitting to the Treasury yield curve as in Papageorgiou and Sircar (2008).
2. $\bar{\rho}_1 = \frac{\bar{\sigma}_1}{\bar{\sigma}_2}\rho_1$, the "effective" correlation between risk-free spot rate r (we use the one-month treasury bonds as a proxy for r) and stock price in (3.34) is estimated from historical risk-free spot rate and stock price data.
3. $\bar{\sigma}_2$, the "effective" stock price volatility in (3.34) is estimated from the historical stock price data.

Now, we detail the calibration method for l, $\bar{\lambda}(z)$ and $V_1^\epsilon(z)$, V_2^ϵ, $V_3^\epsilon(z)$, V_4^ϵ, V_5^ϵ, V_6^ϵ, $V_1^\delta(z)$, $V_2^\delta(z)$. We will minimize the in-sample quadratic pricing error using non-linear least squares to calibrate these parameters on a daily basis. This way we find a risk neutral model that matches a set of observed market prices. This risk neutral model can then be used to price more exotic, illiquid or over-the-counter derivatives. This practice is commonly employed; and for further discussion of this calibration methodology we refer to Cont and Tankov (2004) (see Chapter 13 and the references therein).

Our calibration is carried out in two steps in tandem:

Step 1. Estimation of $l\bar{\lambda}(z)$ and $\{lV_3^\epsilon(z), lV_2^\delta(z)\}$ from the corporate bond price data. The approximate price formula in (3.24) for a defaultable bond is

$$\widetilde{B}^c = B_0^c + \sqrt{\epsilon}B_{1,0}^c + \sqrt{\delta}B_{0,1}^c, \tag{4.1}$$

in which B_0^c is given by (3.29) and

$$\begin{aligned} \sqrt{\epsilon}B_{1,0}^c &= lV_3^\epsilon(z)\frac{\partial B_0^c}{\partial \alpha}, \\ \sqrt{\delta}B_{0,1}^c &= lV_2^\delta(z)\frac{1}{\beta}\left[-\frac{\partial B_0^c}{\partial \alpha} + \frac{(T-t)^2}{2}B_0^c + (T-t)\frac{\partial B_0^c}{\partial r}\right]. \end{aligned} \tag{4.2}$$

We obtain $\{l\bar{\lambda}(z), lV_3^\epsilon(z), lV_2^\delta(z)\}$ from least-squares fitting, i.e. by minimizing

$$\sum_{i=1}^{n}(B_{\text{obs}}^c(t, S_i) - B_{\text{model}}^c(t, S_i; l\bar{\lambda}, lV_3^\epsilon(z), lV_2^\delta(z)))^2, \tag{4.3}$$

where $B^c_{\text{obs}}(t, S_i)$ is the observed market price of a bond that matures at time S_i and $B^c_{\text{model}}(t, S_i; l\bar{\lambda}, lV_3^\epsilon(z), lV_2^\delta(z))$ is the corresponding model price obtained from (4.1). Here, n is the number of bonds that are traded at time t. For a fixed value of $l\bar{\lambda}(z)$ it follows from (4.1) that $\{lV_3^\epsilon(z), lV_2^\delta(z)\}$ can be determined as the least squares solution of

$$\begin{pmatrix} \frac{\partial B_0^c}{\partial \alpha}(t, S_1), & \frac{1}{\beta}\left[-\frac{\partial B_0^c}{\partial \alpha} + \frac{(S_1-t)^2}{2} B_0^c + (S_1 - t)\frac{\partial B_0^c}{\partial r}\right] \\ \vdots & \vdots \\ \frac{\partial B_0^c}{\partial \alpha}(t, S_n), & \frac{1}{\beta}\left[-\frac{\partial B_0^c}{\partial \alpha} + \frac{(S_n-t)^2}{2} B_0^c + (S_n - t)\frac{\partial B_0^c}{\partial r}\right] \end{pmatrix} \begin{pmatrix} lV_3^\epsilon(z) \\ lV_2^\delta(z) \end{pmatrix} = \begin{pmatrix} B^c_{\text{obs}}(t, S_1) - B_0^c(t, S_1; l\bar{\lambda}) \\ \vdots \\ B^c_{\text{obs}}(t, S_n) - B_0^c(t, S_n; l\bar{\lambda}) \end{pmatrix}.$$

Now, we vary $l\bar{\lambda}(z) \in [0, M_1]$ and choose the point $\{l\bar{\lambda}, lV_3^\epsilon(z), lV_2^\delta\}(z)$ that minimizes (4.3). Here, we take $M_1 = 1$ guided by the results of Papageorgiou and Sircar (2008).

Step 2. Estimation of $\{l, V_1^\epsilon(z), V_2^\epsilon, V_4^\epsilon, V_5^\epsilon, V_6^\epsilon, V_1^\delta(z)\}$ from the equity option data: These parameters are calibrated from the stock options data by a least-squares fit to the observed implied volatility. We choose the parameters to minimize

$$\sum_{i=1}^{n} (I_{\text{obs}}(t, T_i, K_i) - I_{\text{model}}(t, T_i, K_i; \text{model parameters}))^2 \approx \sum_{i=1}^{n} \frac{(P_{\text{obs}}(t, T_i, K_i) - P_{\text{model}}(t, T_i, K_i; \text{model parameters}))^2}{\text{vega}^2(T_i, K_i)}, \tag{4.4}$$

in which $I_{\text{obs}}(t, T_i, K_i)$ and $I_{\text{model}}(t, T_i, K_i; \text{model parameters})$ are observed Black-Scholes implied volatility and model Black-Scholes implied volatility, respectively. The right hand side of (4.4) is from Cont and Tankov (2004), page 439. Here, $P_{\text{obs}}(t, T_i, K_i)$ is the market price of a European option (a put or a call) that matures at time T_i and with strike price K_i and $P_{\text{model}}(t, T_i, K_i; \text{model parameters})$ is the corresponding model price which is obtained from (3.24). As in Cont and Tankov (2004), vega(T_i, K_i) is the market implied Black-Scholes vega.

Let $P_0(t, T_i, K_i; \bar{\lambda}(z))$ be either of (3.44) and (3.46) with $K = K_i$ and $T = T_i$. Let us introduce the Greeks,

$$\begin{aligned} g_1 &= -(T-t)x^2 \frac{\partial^2 P_0}{\partial x^2}, \quad g_2 = -(T-t)x\frac{\partial}{\partial x}\left(x^2 \frac{\partial^2 P_0}{\partial x^2}\right), \quad g_3 = \frac{\partial}{\partial \alpha}\left(x\frac{\partial P_0}{\partial x} - P_0\right), \\ g_4 &= x^2 \frac{\partial^3 P_0}{\partial x^2 \partial \alpha}, \quad g_5 = x\frac{\partial^2 P_0}{\partial \eta \partial x}, \quad g_6 = x\frac{\partial^2 P_0}{\partial \alpha \partial x}, \quad g_7 = \frac{(T-t)^2}{2} x^2 \frac{\partial^2 P_0}{\partial x^2}, \\ g_8 &= \frac{1}{\beta}\left[x\frac{\partial^2 P_0}{\partial \alpha \partial x} - \frac{\partial P_0}{\partial \alpha} + \frac{(T-t)^2}{2}\left(x^2\frac{\partial^2 P_0}{\partial x^2} - x\frac{\partial P_0}{\partial x} + P_0\right) - (T-t)\left(x(\frac{\partial^2 P_0}{\partial r}\partial x) - \frac{\partial P_0}{\partial r}\right)\right], \end{aligned} \tag{4.5}$$

in which each term can be explicitly evaluated (see Appendix).

Now from (3.24) and the results of Section 3.3 (with $l = 1$), we can write

$$\begin{aligned} P_{\text{model}}(t, T_i, K_i) = P_0(t, T_i, K_i; \bar{\lambda}(z)) &+ V_1^\epsilon(z) g_1(T_i, K_i; \bar{\lambda}(z)) + V_2^\epsilon g_2(T_i, K_i; \bar{\lambda}(z)) \\ &+ V_3^\epsilon(z) g_3(T_i, K_i; \bar{\lambda}(z)) + V_4^\epsilon g_4(T_i, K_i; \bar{\lambda}(z)) + V_5^\epsilon g_5(T_i, K_i; \bar{\lambda}(z)) \\ &+ V_6^\epsilon g_6(T_i, K_i; \bar{\lambda}(z)) + V_1^\delta(z) g_7(T_i, K_i; \bar{\lambda}(z)) + V_2^\delta(z) g_8(T_i, K_i; \bar{\lambda}(z)). \end{aligned} \tag{4.6}$$

First, let us fix the value of l. Then, from Step 1, we can infer the values of $\{\bar{\lambda}(z), V_3^\epsilon(z), V_2^\delta(z)\}$. Now the fitting problem in (4.4) is a linear least squares problem for $\{V_1^\epsilon(z), V_2^\epsilon, V_4^\epsilon, V_5^\epsilon, V_6^\epsilon, V_1^\delta(z)\}$. Next, we vary $l \in [0, 1]$ and choose $\{l, V_1^\epsilon(z), V_2^\epsilon, V_4^\epsilon, V_5^\epsilon, V_6^\epsilon, V_1^\delta(z)\}$ so that (4.4) is minimized.

4.3 Fitting Ford's implied volatility

We will compare how well our model fits the implied volatility against the model of Fouque et al. (2003), which does not account for the default risk and for the randomness of the interest rates. Although, we only calibrate seven parameters (hence we refer to our model as the 7-parameter model) to the option prices (see the second step of the estimation in Section 4.2), we have many more parameters than the model of Fouque et al. (2003), which only has four parameters (we refer to this model as the 4-parameter model). Therefore, for a fair comparison, we also consider a model in which the volatility is a constant. In this case, as we shall see below, there are only three parameters to calibrate to the option prices, therefore we call it the 3-parameter model.

Constant Volatility Model. In this case, we take $\bar{\sigma}_1 = \bar{\sigma}_2 = \sigma$ in the expression for P_0 in Corollary 3.1 . The expression for $\sqrt{\delta}P_{0,1}$ remains the same as before. However, $\sqrt{\epsilon}P_{1,0}$ simplifies to

$$\sqrt{\epsilon}P_{1,0} = -(T-t)V_1^\epsilon(z)x^2 \frac{\partial^2 P_0}{\partial x^2} + V_3^\epsilon(z)\left(-x\frac{\partial^2 P_0}{\partial \alpha \partial x} + \frac{\partial P_0}{\partial \alpha}\right). \tag{4.7}$$

This model has only three parameters, $l, V_1^\epsilon(z), V_1^\delta(z)$ that need to be calibrated to the options prices, as opposed to the 4-parameter model of Fouque et al. (2003).

As it can be seen from Figure 1, as expected, our 7-parameter model outperforms the 4-parameter model of Fouque et al. (2003) and fits the implied volatility data well. But, what is surprising is that the 3-parameter model, which does not account for the volatility but accounts for the default risk and stochastic interest rate, has almost the same performance as the 7-parameter model.

The 7-parameter model has a very rich implied volatility surface structure, the surface has more curvature than that of the 4-parameter model of Fouque et al. (2003), whose volatility surface is more flat; see Figures 2 and 3. (The parameters to draw these figures are obtained by calibrating the models to the data implied volatility surface on June 8 2007.) The 7-parameter model has a recognizable skew even for longer maturities and has a much sharper skew for shorter maturities.

4.4 Fitting the implied volatility of the index options

The purpose of this section is to show the importance of accounting for stochastic interest rates in fitting the implied volatility surface. Interest rate changes should, indeed, be accounted for in pricing long maturity options. When we price index options, we set $\bar{\lambda} = 0$ and our approximation in (3.24) simplifies to

$$P^{\epsilon,\delta} \approx P_0 + \sqrt{\epsilon} P_{1,0}, \tag{4.8}$$

in which P_0 is given by Corollary 3.1 after settiing $\bar{\lambda}(z) = 0$, and

$$\sqrt{\epsilon} P_{1,0} = -(T-t) \left(V_1^{\epsilon} x^2 \frac{\partial^2 P_0}{\partial x^2} + V_2^{\epsilon} x \frac{\partial}{\partial x} \left(x^2 \frac{\partial^2 P_0}{\partial x^2} \right) \right) + V_4^{\epsilon} x^2 \frac{\partial^3 P_0}{\partial x^2 \partial \alpha} + V_5^{\epsilon} x \frac{\partial^2 P_0}{\partial \eta \partial x} + V_6^{\epsilon} x \frac{\partial^2 P_0}{\partial \alpha \partial x}. \tag{4.9}$$

Note that the difference of (4.8) with the model of Fouque et al. (2003) is that the latter allows for a slow evolving volatility factor to better match the implied volatility at the longer maturities. This was an improvement on the model of Fouque et al. (2000), which only has a fast scale component in the volatility model. We, on the other hand, by accounting for stochastic interest rates, capture the same performance by using only a fast scale volatility model.

From Figure 4, we see that both (4.8) and Fouque et al. (2003) outperform the model of Fouque et al. (2000), especially at the longer maturities ($T = 9$ months, 1 year, 1.5 years and 2 years), and that their performances are very similar. This observation emphasizes the importance of accounting for stochastic interest rates for long maturity contracts.

Appendix: Explicit formulae for the Greeks in (4.5)

When $h(x) = (x - K)^+$, we can explicitly express the Greeks in (4.5) in terms of $f(x) = \frac{1}{\sqrt{2\pi}}\exp(-x^2/2)$ as

$$
\begin{aligned}
& x^2\frac{\partial^2 C_0}{\partial x^2} = \frac{xf(d_1)}{\sqrt{v_{t,T}}}, \quad x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2 C_0}{\partial x^2}\right) = \frac{xf(d_1)}{\sqrt{v_{t,T}}}\left(1 - \frac{d_1}{\sqrt{v_{t,T}}}\right), \\
& \frac{\partial}{\partial \alpha}\left(x\frac{\partial C_0}{\partial x} - C_0\right) = -K\bar{B}^c(t,T)\left(\frac{T-t}{\beta} + \frac{\exp(-\beta(T-t)) - 1}{\beta^2}\right)\left(N(d_2) - \frac{f(d_2)}{\sqrt{v_{t,T}}}\right), \\
& \frac{\partial}{\partial \alpha}\left(x^2\frac{\partial^2 C_0}{\partial x^2}\right) = \frac{-xf(d_1)d_1}{v_{t,T}}\left(\frac{T-t}{\beta} + \frac{\exp(-\beta(T-t)) - 1}{\beta^2}\right), \\
& x\frac{\partial}{\partial x}\left(\frac{\partial C_0}{\partial \alpha}\right) = \frac{xf(d_1)}{\sqrt{v_{t,T}}}\left(\frac{T-t}{\beta} + \frac{\exp(-\beta(T-t)) - 1}{\beta^2}\right), \\
& \frac{\partial}{\partial r}\left(x\frac{\partial C_0}{\partial x} - C_0\right) = -K\bar{B}^c(t,T)\left(\frac{1 - \exp(-\beta(T-t))}{\beta}\right)\left(N(d_2) - \frac{f(d_2)}{\sqrt{v_{t,T}}}\right), \\
& x\frac{\partial}{\partial x}\left(\frac{\partial C_0}{\partial \eta}\right) = xf(d_1)\Bigg[-\frac{1}{\sqrt{v_{t,T}}}\left(\frac{\eta}{\beta^2}(T-t) + \frac{2\eta}{2\beta^3}(\exp(-\beta(T-t)) - 1\right) - \frac{\eta}{2\beta^3}(\exp(-2\beta(T-t)) - 1)) \\
& + \left(-\frac{1}{2}\log\left(\frac{x}{K\bar{B}^c_{t,T}}v_{t,T}^{-3/2} + \frac{1}{4\sqrt{v_{t,T}}}\right)\right)\times \\
& \left(\left(\frac{2\bar{\rho}_1\bar{\sigma}_2}{\beta} + \frac{2\eta}{\beta^2}\right)(T-t) + \left(\frac{2\bar{\rho}_1\bar{\sigma}_2}{\beta^2} + \frac{4\eta}{\beta^3}\right)\exp(-\beta(T-t)) - \frac{\eta}{\beta^3}\exp(-2\beta(T-t)) - \left(\frac{2\bar{\rho}_1\bar{\sigma}_2}{\beta^2} + \frac{3\eta}{\beta^3}\right)\right)\Bigg].
\end{aligned}
$$

References

Bayraktar, E. (2008). Pricing options on defaultable stocks, *Applied Mathematical Finance* **15 (3)**: 277–304.

Bielecki, T. R. and Rutkowski, M. (2002). *Credit Risk: Modeling, Valuation and Hedging*, Springer, New York.

Carr, P. and Linetsky, V. (2006). A jump to default extended CEV model: An application of Bessel processes, *Finance and Stochastics* **10**: 303–330.

Carr, P. and Wu, L. (2006). Stock options and credit default swaps: A joint framework for valuation and estimation, *Technical report*. Available at http://papers.ssrn.com/sol3/papers.cfm?abstract_id=748005.

Cont, R. and Tankov, P. (2004). *Financial Modeling with Jump Processes*, Chapman & Hall, Boca Raton, FL.

Cotton, P., Fouque, J.-P., Papanicolaou, G. and Sircar, R. (2004). Stochastic volatility corrections for interest rate derivatives, *Math. Finance* **14**(2): 173–200.

Duffie, D. and Singleton, K. (1999). Modeling term structure of defaultable bonds, *Review of Financial Studies* **12 (4)**: 687–720.

Fouque, J.-P., Papanicolaou, G. and Sircar, K. R. (2000). *Derivatives in Financial Markets with Stochastic Volatility*, Cambridge University Press, New York.

Fouque, J. P., Papanicolaou, G., Sircar, R. and Solna, K. (2003). Multiscale stochastic volatility asymptotics, *SIAM J. Multiscale Modeling and Simulation* **2 (1)**: 22–42.

Geman, H., Karoui, N. E. and Rochet, J. C. (1995). Changes of numéraire, changes of probability measures and option pricing, *Journal of Applied Probability* **32**: 443–458.

Linetsky, V. (2006). Pricing equity derivatives subject to bankruptcy, *Mathematical Finance* **16 (2)**: 255–282.

Papageorgiou, E. and Sircar, R. (2008). Multiscale intensity based models for single name credit derivatives, *Applied Mathematical Finance* **15 (1)**: 73–105.

Schönbucher, P. J. (1998). Term structure of defaultable bond prices, *Review of Derivatives Research* **2 (2/3)**: 161–192.

Schönbucher, P. J. (2003). *Credit Derivatives Pricing Models: Model, Pricing and Implementation*, Wiley, New York.

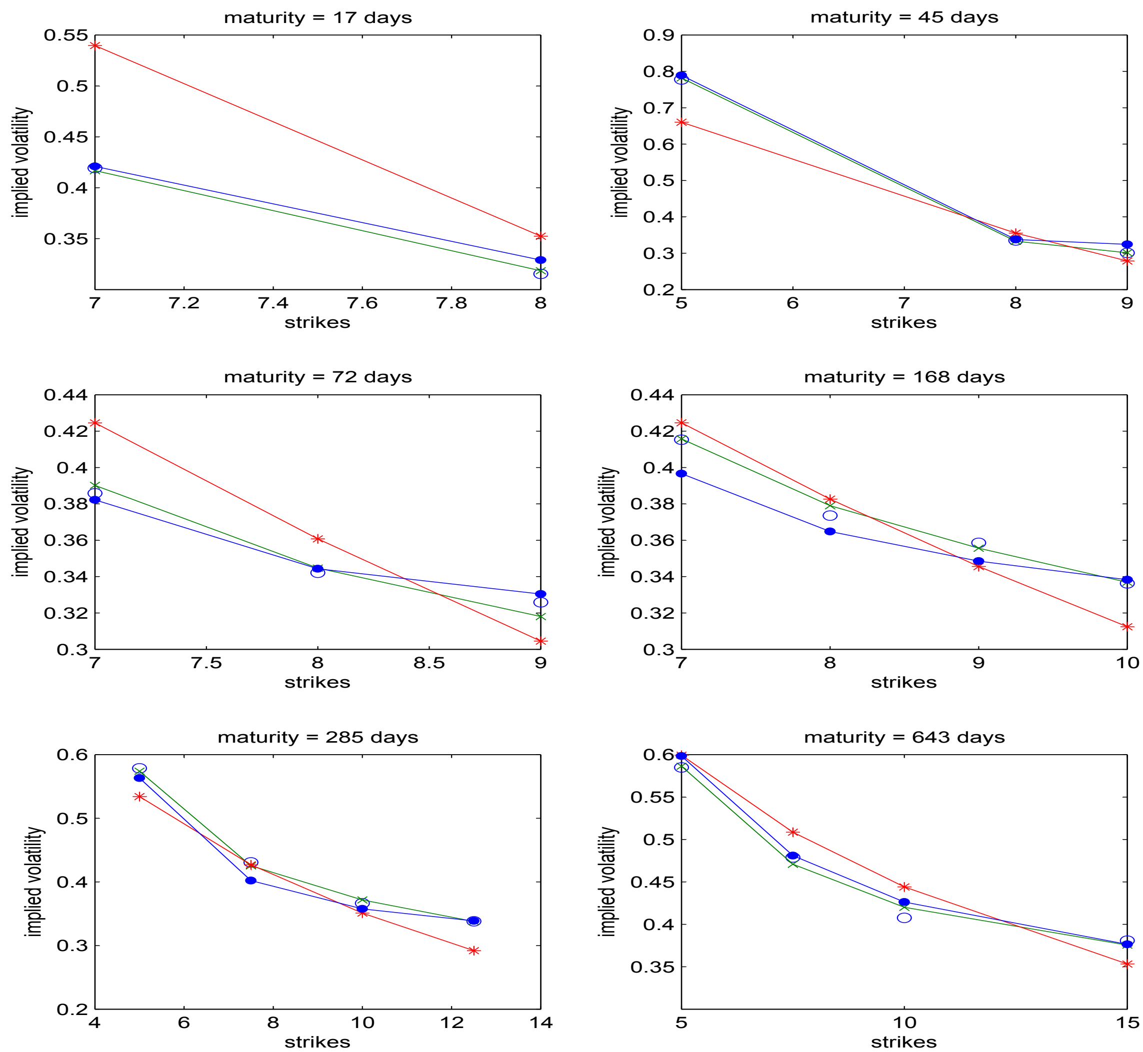

Figure 1: Implied volatility fit to the Ford call option data with maturities of [17, 45, 72, 168, 285, 643] calender days on April 4, 2007.
Model is calibrated aross all maturities but we plotted the implied volatilities for each maturity, separately. Here, stock price $x = 8.04$, historical volatility $\bar{\sigma}_2 = 0.3827$, one month treasury rate $r = 0.0516$, estimated correlation between risk-free spot rate(one month treasury) and stock price $\bar{\rho}_1 = -0.0327$. Also $\alpha = 0.0037$, $\beta = 0.0872$, $\eta = 0.0001$ which are obtained with a least-square fitting to the Treasury yield curve on the 4th of April.

Legend:

'o', empty circles = observed data;

'x', green = stochastic vol+stochastic hazard rate+stochastic interest rate = the 7-parameter model;

small full circle, blue = constant vol+stochastic hazard rate+ stochastic interest rate = the 3-parameter model

'*', red = The model of Fouque et al. (2003) which has constant interest rate+stochastic vol (slow and fast scales) = the 4 parameter model.

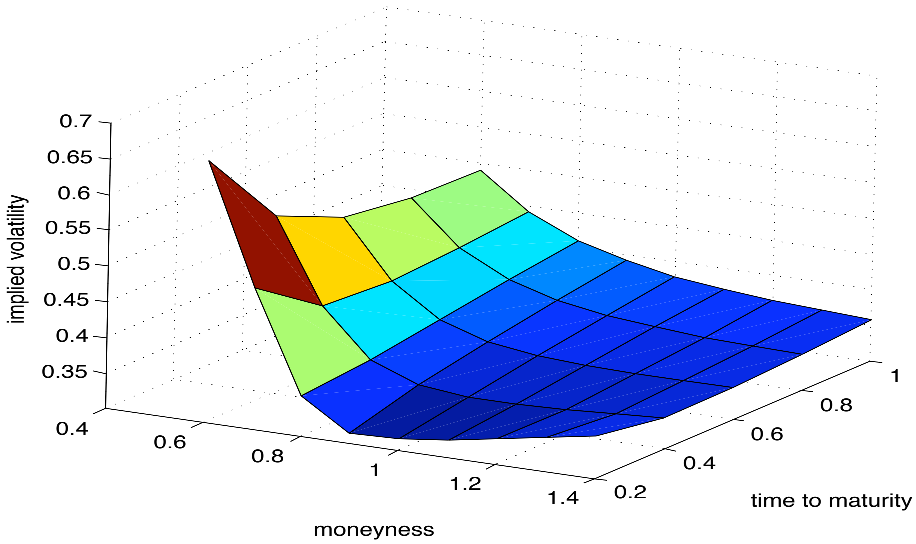

Figure 2: Implied volatility surface corresponding to (4.6), the 7-parameter model. Here, $\alpha = 0.0063$, $\beta = 0.1034$, $\eta = 0.012$, $r = 0.0476$ $\bar{\sigma}_2 = 0.2576$, $\bar{\lambda}(z) = 0.027$, $(V_1^{\epsilon}(z), V_2^{\epsilon}, V_3^{\epsilon}(z), V_4^{\epsilon}, V_5^{\epsilon}, V_6^{\epsilon}, V_1^{\delta}(z), V_2^{\delta}(z)) = (0.9960, -0.0014, 0.0009, 0.0104, -0.6514, 0.3340, -0.1837, -0.0001)$.

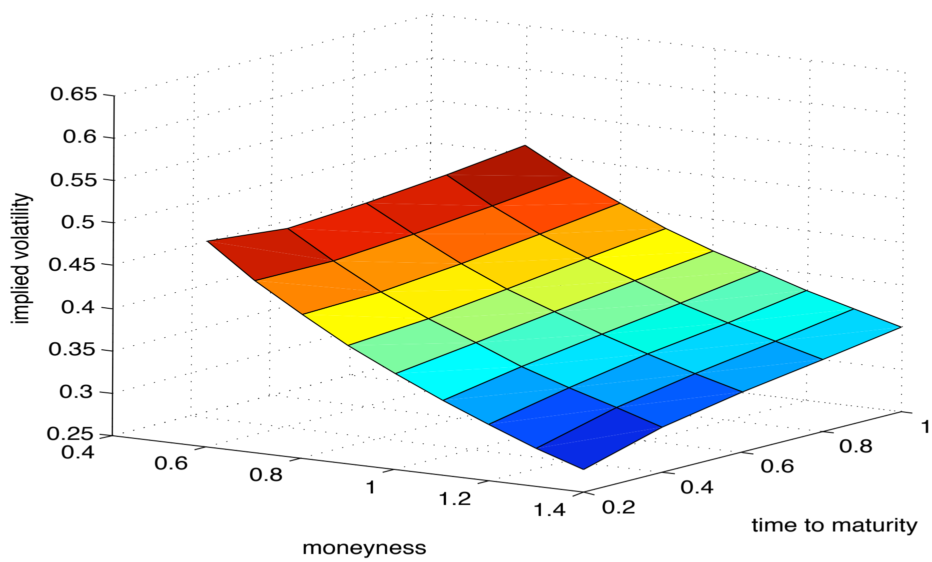

Figure 3: Implied Volatility Surface corresponding to the 4-parameter model of Fouque et al. (2003).
Here, $r = 0.046$, average volatility=0.2546, and the parameters in (4.3) of Fouque et al. (2003) are choosen to be $(V_2^{\epsilon}, V_3^{\epsilon}(z), V_0^{\delta}(z), V_1^{\delta}(z)) = (-0.0164, -0.1718, 0.0006, 0.0630)$. Note that the parameters here and Figure 2 are both obtained by calibrating the models to the data implied volatility surface of Ford Motor Company on June 8, 2007.

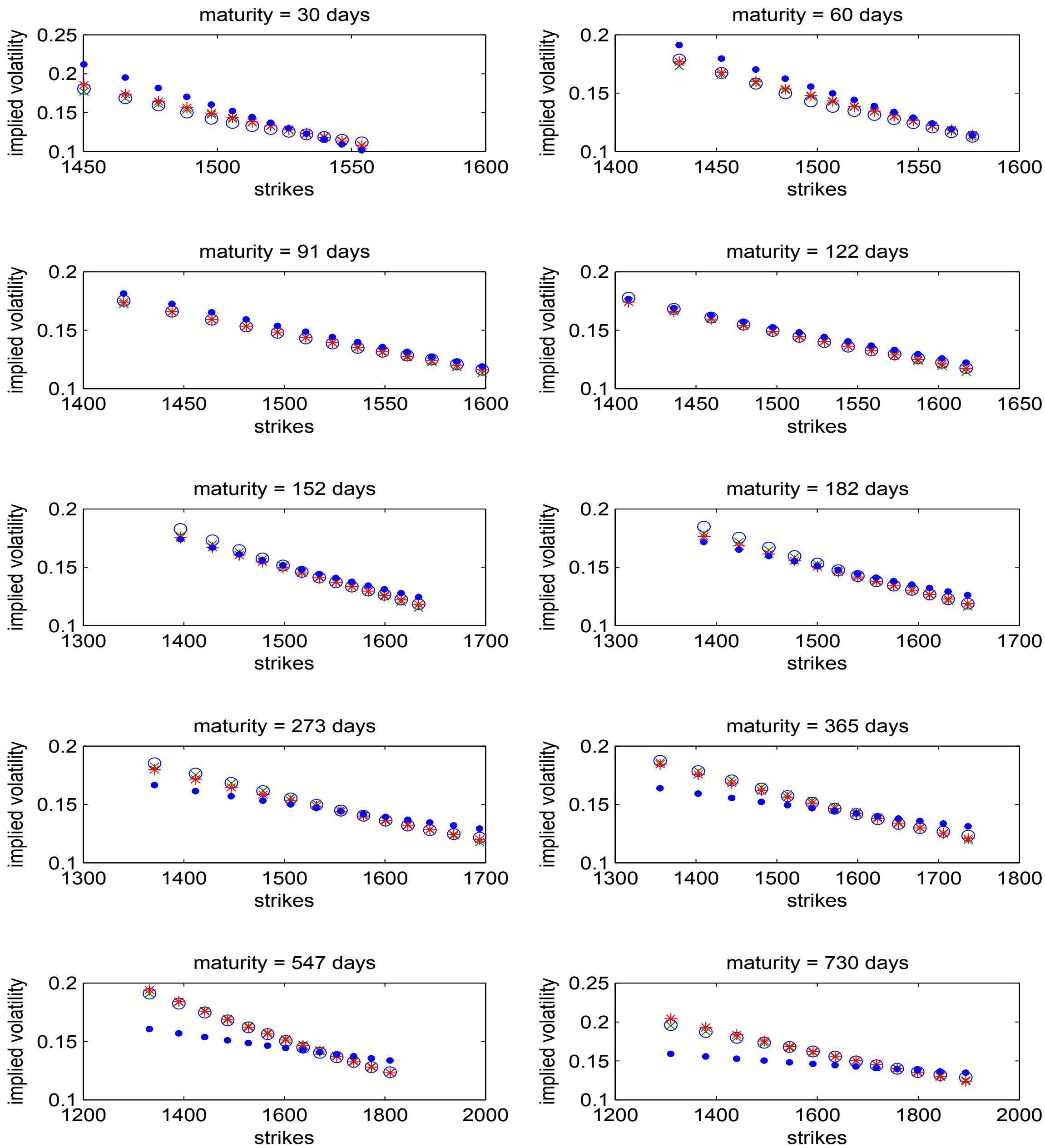

Figure 4: The fit to the Implied Volatility Surface of SPX on June 8, 2007 with maturities [30, 60, 91, 122, 152, 182, 273, 365, 547, 730] calender days. Recall from Section 4.1 that we use standardized options from the OptionMetrics. Models are calibrated aross all maturities, but we plot the implied volatility fits separately. The parameters are: stock price $x = 1507.67$, dividend rate $= 0.0190422$, historical volatility $\bar{\sigma}_2 = 0.1124$, one month treasury rate $r = 0.0476$, estimated correlation between risk-free spot rate(one month treasury) and stock price $\bar{\rho}_1 = 0.020454$. Also, $\alpha = 0.0078$, $\beta = 0.1173$, $\eta = 0.0241$, which are obtained from a least-square fitting to the Treasury yield curve.

Legend. 'o', empty cirles = observed data,; 'x", green = Implied volatility of (4.8), '*', red = Implied volatility of Fouque et al. (2003); small full circle, blue = Implied volatility of Fouque et al. (2000).