HEDGING LIFE INSURANCE WITH PURE ENDOWMENTS [*]

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Abstract

We extend the work of Milevsky et al. (2005) and Young (2006) by pricing life insurance and pure endowments together. We assume that the company issuing the life insurance and pure endowment contracts requires compensation for their mortality risk in the form of a pre-specified instantaneous Sharpe ratio. We show that the price $P^{m,n}$ for m life insurances and n pure endowments is less than the sum of the price $P^{m,0}$ for m life insurances and the price $P^{0,n}$ for n pure endowments. Thereby, pure endowment contracts serve as a hedge against the (stochastic) mortality risk inherent in life insurance, and vice versa.

1 Introduction

Milevsky et al. (2005) present a framework for valuing (stochastic) mortality risk. They do this by assuming that the company issuing a mortality-contingent claim requires compensation for this risk in the form of a pre-specified instantaneous Sharpe ratio. They apply their method to price pure endowment contracts and show that the resulting price satisfies many desirable properties. Young (2006) applies the same method to price life insurance that pays at the moment of death of the insured.

We extend their work by pricing life insurance and pure endowments together. We show that the price $P^{m,n}$ for m life insurances and n pure endowments is less than the sum of the price $P^{m,0}$ for m life insurances and the price $P^{0,n}$ for n pure endowments. Thereby, pure endowment contracts serve as a hedge against the (stochastic) mortality risk inherent in life insurance, and vice versa. In our framework the price of an additional life insurance contract can be defined as the marginal price, $P^{m+1,n} - P^{m,n}$, when the insurer already holds m life insurances and n pure endowments. Similarly, one can marginally price an additional pure endowment. Defining a price in this way is related to work of Stoikov (2005), in which he uses the principle of equivalent utility to find the indifference price of an additional financial contract.

The remainder of the paper is organized as follows: In Section 2, we present our modeling framework and then show how to price a portfolio of life insurance and pure endowment contracts via the instantaneous Sharpe ratio, the method proposed by Milevsky et al. (2005). In Section 3, we show that the price $P^{m,n}$ for m life insurances and n pure endowments is monotone in m and n. We also show that $P^{m,n} \leq P^{m_1,n_1} + P^{m_2,n_2}$, in which $m_1 + m_2 = m$ and $n_1 + n_2 = n$ (subadditivity). Thus, life insurance and pure endowments are hedges for each other. As a corollary of the subadditivity, we show that the marginal price of a life insurance and pure endowment contract satisfy are in fact reasonable prices. Section 4 concludes the paper.

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[*] *Key Words.* Stochastic mortality, Sharpe ratio, life insurance, pure endowments, non-linear partial differential equations.

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2 Instantaneous Sharpe Ratio

2.1 Stochastic Mortality Model and Financial Market

In this section, we assume that an insurer has issued m life insurance and n pure endowment contracts, for $m, n \in \mathbb{N}$. In the setting of this paper, a life insurance contract is an obligation to pay \$1 at time T if the insured dies before that time, and a pure endowment is an obligation to pay \$1 at time T if its holder is alive then. The insurer also trades in the bond market to hedge its interest rate risk. First, we obtain the hedging strategy for the insurance company and then describe how to use the instantaneous Sharpe ratio to find the price of a basket of m life insurance and n pure endowment contracts.

We use the stochastic mortality modeling framework introduced in Milevsky et al. (2005). We assume that the mortality of the individuals purchasing the contracts are independent given the hazard rate, and we assume that the (common) hazard rate λ follows

$$d\lambda_t = \mu(\lambda_t, t)(\lambda_t - \underline{\lambda})dt + \sigma(t)(\lambda_t - \underline{\lambda})dW_t. \tag{2.1}$$

Here, $\underline{\lambda} > 0$ and W is a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$. We need some technical assumptions on the coefficients of this diffusion.

Assumption 2.1 *In the rest of the paper we assume that*

1. *The volatility function σ is either identically zero, or it is a continuous function of t such that $\sigma(t) > \kappa > 0$, $t \in [0, T]$.*
2. *The drift function μ is Lipschitz continuous function of λ and t and there exits $\varepsilon > 0$ such that if $0 < \lambda - \underline{\lambda} < \varepsilon$, then $\mu(\lambda, t) > 0$, $t \in [0, T]$.*

Under these assumptions a unique strong solution of (2.1) exists, and if the process starts at $\lambda_0 > \underline{\lambda}$, then $\lambda_t > \underline{\lambda}$ for all $t \in [0, T]$.

To determine the value of the basket of m life insurances and n pure endowments, we create a portfolio composed of the obligation to pay these the pure endowments and life insurances and of default-free zero-coupon bonds that pay \$1 at time T. This requires a model for the bond prices, and we use a short rate model. The dynamics of the short rate r, the rate at which the money market grows, are given by

$$dr_t = a(r_t, t)dt + b(r_t, t)dB_t, \tag{2.2}$$

in which $a, b \geq 0$ are deterministic functions, and B is a Brownian motion with respect to the probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$, that is independent of W, the Brownian motion driving the hazard rate in (2.1).

In this framework, the price of a default-free zero-coupon bond (at time t) that pays \$1 at time T is given by

$$F(r, t; T) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s ds\right)\Big| r_t = r\right], \tag{2.3}$$

in which $\mathbb{Q}$ is the risk-neutral pricing measure in the bond market. If the bond market's price of risk q is a deterministic function of the short rate and time, say $q_t = q(r_t, t)$, then $\mathbb{Q}$ is defined by its Radon-Nikodym derivative $d\mathbb{Q}/d\mathbb{P} = \exp\left(-\int_0^T q_s dB_s - \frac{1}{2}\int_0^T q_s^2 ds\right)$. Under this new measure $\mathbb{Q}$, $B_t^{\mathbb{Q}} \triangleq B_t + \int_0^t q_s ds$ is a standard Brownian motion.

¿From the Feynman-Kac representation (see e.g. Karatzas and Shreve (1991)), it follows that the price of the T-bond satisfies the following linear partial differential equation (PDE):

$$F_t + a^{\mathbb{Q}}(r, t)F_r + \frac{1}{2}b^2(r, t)F_r r - rF = 0, \quad F(r, t; T) = 1, \tag{2.4}$$

where $a^{\mathbb{Q}} = a - qb$. By using this PDE, we can obtain the dynamics of $F(r_s, s)$, $t \leq s \leq T$, given $F(r_t, t) = F(r, t)$ as

$$dF(r_s, s) = (r_s F(r_s, s) + q_s b(r_s, s)F_r(r_s, s))ds + b(r_s, s)F_r(r_s, s)dB_s. \tag{2.5}$$

2.2 Pricing using the Instantaneous Sharpe Ratio

Denote the value (price) of m life insurances and n pure endowments by $P^{m,n} = P^{m,n}(r, \lambda, t)$, in which we explicitly recognize that the price of the basket of life insurances and pure endowments depends on the short rate r and hazard rate λ at time t. Suppose the insurer creates a portfolio Π that contains the obligation to underwrite the basket of contracts and holds π_t T-bonds. Therefore, the value of the portfolio at time t is given by $\Pi_t = -P^{m,n}(r_t, \lambda_t, t) + \pi_t F(r_t, t)$. The insurer, by choosing the appropriate portfolio, can hedge against the interest rate risk; however, the mortality risk cannot be hedged. Note that there are two sources of mortality risk, the jump risk itself and the risk from the intensity of the jumps being stochastic. (The death of the individual is modeled as the first jump time of a doubly-stochastic Poisson process.)

Assume that $m, n \geq 1$. By using Itô's formula (see e.g. Cont and Tankov (2004)) we obtain

$$\begin{aligned}
\Pi_{t+h} = \Pi_t &- \int_t^{t+h} \mathcal{D}^a P^{m,n}(r_s, \lambda_s, s) ds + \int_t^{t+h} b(r_s, s)(\pi_s F_r(r_s, s) - P_r^{m,n}(r_s, \lambda_s, s)) dB_s \\
&- \int_t^{t+h} \sigma(s)(\lambda_s - \underline{\lambda}) P_\lambda^{m,n}(r_s, \lambda_s, s) dW_s + \int_t^{t+h} \pi_s (r_s F(r_s, s) + q_s b(r_s, s) F_r(r_s, s)) ds \\
&+ \int_t^{t+h} \left(P^{m,n}(r_s, \lambda_s, s) - P^{m-1,n}(r_s, \lambda_s, s) - F(r_s, s)\right) (dN_s^1 - m\lambda_s ds) \\
&+ \int_t^{t+h} \left(P^{m,n}(r_s, \lambda_s, s) - P^{m,n-1}(r_s, \lambda_s, s)\right) (dN_s^2 - n\lambda_s ds),
\end{aligned} \tag{2.6}$$

in which $\mathcal{D}^c$ is the operator defined on $\mathbb{R}_+ \times (\underline{\lambda}, \infty) \times [0, T]$ by its action on a test function f as

$$\begin{aligned}
\mathcal{D}^c f = f_t + c f_r + \frac{1}{2} b^2 f_{rr} + \mu(\lambda - \underline{\lambda}) f_\lambda + \frac{1}{2} \sigma^2 (\lambda - \underline{\lambda})^2 f_{\lambda\lambda} \\
- m\lambda(P^{m,n} - P^{m-1,n} - F) - n\lambda(P^{m,n} - P^{m,n-1}),
\end{aligned} \tag{2.7}$$

for any $c = c(r, t)$ that is a deterministic function of the short rate and time. In (2.6), N^1 and N^2 denote two independent Poisson process with intensities $m\lambda$ and $n\lambda$, respectively.

Next, we calculate the expectation and variance of Π_{t+h} conditional on the information available at time, namely $\mathcal{F}_t$:

$$\begin{aligned}
\mathbb{E}(\Pi_{t+h} | \mathcal{F}_t) = \Pi_t &- \mathbb{E}\left[\int_t^{t+h} \mathcal{D}^a P^{m,n}(r_s, \lambda_s, s) ds \Big| r_t, \lambda_t \right] \\
&+ \mathbb{E}\left[\int_t^{t+h} \pi_s (r_s F(r_s, s) + q_s b(r_s, s) F_r(r_s, s)) ds \Big| r_t, \lambda_t \right],
\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
\mathbf{Var}(\Pi_{t+h} | \mathcal{F}_t) = &\mathbb{E}\left[(Y_h - \mathbb{E}Y_h)^2 + \int_t^{t+h} b^2(r_s, s)(\pi_s F(r_s, s) - P_r^{m,n}(r_s, \lambda_s, s))^2 ds \Big| r_t, \lambda_t \right] \\
&+ \mathbb{E}\left[\int_t^{t+h} \sigma^2(s)(\lambda_s - \underline{\lambda})^2 \left(P_\lambda^{m,n}(r_s, \lambda_s, s)\right)^2 ds + \Big| r_t, \lambda_t \right] \\
&+ \mathbb{E}\left[\int_t^{t+h} m\lambda_s (P^{m,n}(r_s, \lambda_s, s) - P^{m-1,n}(r_s, \lambda_s, s) - F(r_s, s))^2 ds \Big| r_t, \lambda_t \right] \\
&+ \mathbb{E}\left[\int_t^{t+h} n\lambda_s (P^{m,n}(r_s, \lambda_s, s) - P^{m,n-1}(r_s, \lambda_s, s))^2 ds \Big| r_t, \lambda_t \right],
\end{aligned} \tag{2.9}$$

in which Y_h is defined by

$$Y_h = \Pi_t - \int_t^{t+h} \mathcal{D}^a P^{m,n}(r_s, \lambda_s, s)ds + \int_t^{t+h} \pi_s(r_s F(r_s, s) + q_s b(r_s, s) F_r(r_s, s))ds. \quad (2.10)$$

We choose π_t in order to minimize the local variance $\lim_{h\to 0} \mathbf{Var}(\Pi_{t+h}|\mathcal{F}_t)/h$ and thereby obtain $\pi_t = P^{m,n}(r_t, \lambda_t, t)/F_r(r_t, t)$. With this choice, the drift and local variance of the portfolio Π, respectively, become

$$\lim_{h\to 0} \frac{1}{h}(\mathbb{E}[\Pi_{t+h}|\mathcal{F}_t] - \Pi_t) = -\mathcal{D}^{a^{\mathbb{Q}}} P^{m,n}(r, \lambda, t) + r P_r^{m,n}(r, \lambda, t) \frac{F(r,t)}{F_r(r,t)}, \quad (2.11)$$

and

$$\lim_{h\to 0} \frac{1}{h}\mathbf{Var}(\Pi_{t+h}|\mathcal{F}_t) = \sigma^2(\lambda - \underline{\lambda})^2 \left(P_\lambda^{m,n}\right)^2 + m\lambda(P^{m,n} - P^{m-1,n} - F) + n\lambda(P^{m,n} - P^{m,n-1}). \quad (2.12)$$

Now, we introduce the pricing mechanism via instantaneous Sharpe ratio. Because the local variance in (2.12) is positive, the insurer is unable to completely hedge the mortality risk in the basket of contracts. When we want to determine a price, the preferences of the insurance company come into play. We specify these preferences in terms of a Sharpe ratio. The *instantaneous Sharpe ratio* of portfolio Π is defined as

$$\alpha(r_t, \lambda_t, t) \triangleq \frac{\lim_{h\to 0} \frac{1}{h}(\mathbb{E}[\Pi_{t+h}|\mathcal{F}_t] - \Pi_t) - r\Pi_t}{\sqrt{\lim_{h\to 0} \frac{1}{h}\mathbf{Var}(\Pi_{t+h}|\mathcal{F}_t)}}. \quad (2.13)$$

We assume that the insurance company selects a constant $\alpha \in [0, \sqrt{\lambda}]$ and sets the price of the basket of contracts so that the instantaneous Sharpe ratio $\alpha(r, \lambda, t) = \alpha$. The prices can then be obtained from the equation

$$\begin{aligned} &-\mathcal{D}^{a^{\mathbb{Q}}} P^{m,n} + r P_r^{m,n} \frac{F}{F_r} = r\Pi \\ &+ \alpha\sqrt{\sigma^2(\lambda - \underline{\lambda})^2 \left(P_\lambda^{m,n}\right)^2 + m\lambda(P^{m,n} - P^{m-1,n} - F)^2 + n\lambda(P^{m,n} - P^{m,n-1})^2}. \end{aligned} \quad (2.14)$$

Because $\Pi = -P^{m,n} + \pi F$, it follows that $P^{m,n}$, for $m, n \geq 1$, satisfies the non-linear PDE given by

$$\begin{aligned} &P_t^{m,n} + a^{\mathbb{Q}} P_r^{m,n} + \frac{1}{2} b^2 P_{rr}^{m,n} + \mu(\lambda - \underline{\lambda}) P_\lambda^{m,n} + \frac{1}{2}\sigma^2(\lambda - \underline{\lambda})^2 P_{\lambda\lambda}^{m,n} \\ &\quad - m\lambda(P^{m,n} - P^{m-1,n} - F) - n\lambda(P^{m,n} - P^{m,n-1}) \\ &= rP^{m,n} - \alpha\sqrt{\sigma^2(\lambda - \underline{\lambda})^2 (P_\lambda^{m,n})^2 + m\lambda(P^{m,n} - P^{m-1,n} - F)^2 + n\lambda(P^{m,n} - P^{m,n-1})^2}, \end{aligned} \quad (2.15)$$

with $P^{m,n}(r, \lambda, T) = n$.

Similarly, we can derive the price $P^{1,0}$ for a single life insurance policy and the price $P^{0,1}$ for a single pure endowment to be the solutions, respectively, of

$$\begin{aligned} &P_t^{1,0} + a^{\mathbb{Q}} P_r^{1,0} + \frac{1}{2} b^2 P_{rr}^{1,0} + \mu(\lambda - \underline{\lambda}) P_\lambda^{1,0} + \frac{1}{2}\sigma^2(\lambda - \underline{\lambda})^2 P_{\lambda\lambda}^{1,0} - \lambda(P^{1,0} - F) \\ &= rP^{1,0} - \alpha\sqrt{\sigma^2(\lambda - \underline{\lambda})^2 (P_\lambda^{1,0})^2 + \lambda(P^{1,0} - F)^2}, \quad P^{1,0}(r, \lambda, T) = 0, \end{aligned} \quad (2.16)$$

and

$$\begin{aligned} &P_t^{0,1} + a^{\mathbb{Q}} P_r^{0,1} + \frac{1}{2} b^2 P_{rr}^{0,1} + \mu(\lambda - \underline{\lambda}) P_\lambda^{0,1} + \frac{1}{2}\sigma^2(\lambda - \underline{\lambda})^2 P_{\lambda\lambda}^{0,1} - \lambda P^{0,1} \\ &= rP^{0,1} - \alpha\sqrt{\sigma^2(\lambda - \underline{\lambda})^2 (P_\lambda^{0,1})^2 + \lambda(P^{0,1})^2}, \quad P^{0,1}(r, \lambda, T) = 1. \end{aligned} \quad (2.17)$$

Note that the price $P^{m,n}$, for $m, n \geq 0$ with at least one of m and n strictly greater than 0, can be written as $P^{m,n}(r, \lambda, t) = F(r, t)\varphi^{m,n}(\lambda, t)$, in which $\varphi^{m,n}$ solves

$$\begin{aligned}\varphi_t^{m,n} + \mu(\lambda - \underline{\lambda})\varphi_\lambda^{m,n} + \frac{1}{2}\sigma^2(\lambda - \underline{\lambda})^2\varphi_{\lambda\lambda}^{m,n} - m\lambda(\varphi^{m,n} - \varphi^{m-1,n} - 1) - n\lambda(\varphi^{m,n} - \varphi^{m,n-1})\\ = -\alpha\sqrt{\sigma^2(\lambda - \underline{\lambda})^2(\varphi_\lambda^{m,n})^2 + m\lambda(\varphi^{m,n} - \varphi^{m-1,n} - 1)^2 + n\lambda(\varphi^{m,n} - \varphi^{m,n-1})^2},\end{aligned} \tag{2.18}$$

in which $\varphi^{m,n}(\lambda, T) = n$ and $\varphi^{0,0} \equiv 0$.

Assumption 2.2 *Throughout the paper, we assume that the Sharpe ratio satisfies*

$$\alpha \in [0, \sqrt{\underline{\lambda}}]. \tag{2.19}$$

Remark 2.1 *To see the necessity of making the assumption in* (2.19)*, consider the case when λ is deterministic, $\sigma \equiv 0$. Suppose $\lambda(t)$ is the solution of $d\lambda = \mu(\lambda, s)(\lambda - \underline{\lambda})ds$ with $\lambda(0) = \lambda$. Then the equation for $\varphi^{0,1}$ becomes a linear ordinary differential equation*

$$\varphi_t^{0,1} - (\lambda(t) - \alpha\sqrt{\lambda(t)})\varphi^{0,1}(t) = 0, \quad \varphi^{0,1}(T) = 1.$$

The solution of this equation is $\varphi^{0,1} = \exp\left(-\int_t^T \left(\lambda(s) - \alpha\sqrt{\lambda(s)}\right) ds\right)$. In this case $\varphi^{0,1}$ is less than or equal to one only if $\alpha \in [0, \sqrt{\underline{\lambda}}]$. If $\varphi^{0,1}$ is greater than one, than the price of a pure endowment becomes greater than the price of a default-free zero-coupon bond, which yields an arbitrage opportunity. Therefore, $\sqrt{\underline{\lambda}}$ might be seen as a good deal bound on the Sharpe ratio*, see e.g. Bjork and Slinko (2005).*

In the next section, we prove that holding a basket of m life insurance and n pure endowment contracts is cheaper than holding them separately; that is, $\varphi^{m,n}$ is *subadditive.* This confirms the intuition that pure endowments provide a partial hedge for life insurance, and vice versa.

3 Properties of $\varphi^{m,n}$: Monotonicity and Subadditivity

To demonstrate the subadditivity property of $\varphi^{m,n}$ we will rely on a comparison theorem for non-linear PDEs. We begin by stating a relevant one-sided Lipschitz condition.

Definition 3.1 *We say that a function $g : (\underline{\lambda}, \infty) \times [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ belongs to $\mathcal{G}$ if there exist functions $c : (\underline{\lambda}, \infty) \times [0, T] \to \mathbb{R}_+$ and $d : (\underline{\lambda}, \infty) \times [0, T] \to \mathbb{R}_+$ satisfying*

$$c(\lambda, t) \leq K(1 + (\log(\lambda - \underline{\lambda}))^2), \quad \textit{and} \quad d(\lambda, t) \leq K(\lambda - \underline{\lambda})(1 + (\log(\lambda - \underline{\lambda}))), \tag{3.1}$$

for some $K \geq 0$ such that

$$g(\lambda, t, v, p) - g(\lambda, t, w, q) \leq c(\lambda, t)(v - w) + d(\lambda, t)|p - q|, \tag{3.2}$$

for $v, w, p, q \in \mathbb{R}$ such that $v > w$.

We will rely on the following comparison principle, which we obtain from Walter (1970), Section 28.

Theorem 3.1 *Let $D = (\underline{\lambda}, \infty) \times [0, T]$, and denote by $\mathbb{D}$ the collection of functions on D that are C^2 in their first variable and C^1 in their second. Define a differential operator $\mathcal{L}$ on $\mathbb{D}$ by*

$$\mathcal{L}v = v_t + \frac{1}{2}\sigma^2(t)(\lambda - \underline{\lambda})^2 v_{\lambda\lambda} + g(\lambda, t, v, v_\lambda), \tag{3.3}$$

in which $g \in \mathcal{G}$. Suppose that $v, w \in \mathbb{D}$ are such that there exists a constant $K \geq 0$ with $v(\lambda, t) \leq \exp(K(\log(\lambda - \underline{\lambda}))^2)$ and $w(\lambda, t) \geq -\exp(K(\log(\lambda - \underline{\lambda}))^2)$ for large λ and for λ close to $\underline{\lambda}$ and for all $t > 0$. Then, if (a) $\mathcal{L}v \geq \mathcal{L}w$ on D, and if (b) $v(\lambda, T) \leq w(\lambda, T)$ for all $\lambda > \underline{\lambda}$, then $v \leq w$ on D.

Proof: The proof of this theorem follows from the comparison theorem in Walter (1970), Section 28, after we transform the variables λ and t in (3.3) to $y = \log(\lambda - \underline{\lambda})$ and $\tau = T - t$ and write $\tilde{v}(y, \tau) = v(\lambda, t)$. The logarithmic transformation eliminates the unbounded coefficient $(\lambda - \underline{\lambda})^2$ of the second-order derivative $v_{\lambda\lambda}$ in (3.3), and $\tau = T - t$ reverses time so that we can apply the initial-valued comparison theorem for (non-linear) PDEs in Walter (1970). □

Lemma 3.1 *Define $g_{m,n}$ for $m, n \geq 1$, by*

$$\begin{aligned} g_{m,n}(\lambda, t, v, p) \triangleq\ & \mu(\lambda, t)(\lambda - \underline{\lambda})p - m\lambda(v - \varphi^{m-1,n} - 1) - n\lambda(v - \varphi^{m,n-1}) \\ & + \alpha\sqrt{\sigma^2(\lambda - \underline{\lambda})^2 p^2 + m\lambda(v - \varphi^{m-1,n} - 1)^2 + n\lambda(v - \varphi^{m,n-1})^2}, \end{aligned} \tag{3.4}$$

in which $\varphi^{m-1,n}$ and $\varphi^{m,n-1}$ solve equation (2.18) *when m is replaced by $m-1$ and n is replaced by $n-1$, respectively. Then, $g_{m,n} \in \mathcal{G}$, if there exists $K > 0$ such that*

$$|\mu(\lambda, t)| \leq K(1 + |\log(\lambda - \underline{\lambda})|). \tag{3.5}$$

Proof: Suppose $v > w$, then

$$\begin{aligned} g_{m,n}(\lambda, t, v, p) - g_{m,n}(\lambda, t, w, q) &= \mu(\lambda, t)(\lambda - \underline{\lambda})(p - q) - (n + m)\lambda(v - w) \\ & + \alpha\sqrt{\sigma^2(\lambda - \underline{\lambda})^2 p^2 + m\lambda(v - \varphi^{m-1,n} - 1)^2 + n\lambda(v - \varphi^{m,n-1})^2} \\ & - \alpha\sqrt{\sigma^2(\lambda - \underline{\lambda})^2 q^2 + m\lambda(w - \varphi^{m-1,n} - 1)^2 + n\lambda(w - \varphi^{m,n-1})^2} \\ & \leq (|\mu(\lambda, t)| + \alpha\sigma(t))(\lambda - \underline{\lambda})|p - q| - ((n + m)\lambda - \alpha(\sqrt{n\lambda} + \sqrt{m\lambda}))(v - w) \\ & \leq (|\mu(\lambda, t)| + \alpha\sigma(t))(\lambda - \underline{\lambda})|p - q| \\ & \leq K(\lambda - \underline{\lambda})(1 + |\log(\lambda - \underline{\lambda})|)|p - q|. \end{aligned} \tag{3.6}$$

The first inequality follows from the fact that $\left|\sqrt{A^2 + B^2 + C^2} - \sqrt{X^2 + Y^2 + Z^2}\right| \leq |A - X| + |B - Y| + |C - Z|$, for any $A, B, C, X, Y, Z \in \mathbb{R}$. The second inequality follows from $\alpha \in (0, \sqrt{\underline{\lambda}})$ and $\lambda > \underline{\lambda}$. The last inequality follows from the assumptions on the functions μ. □

Assumption 3.1 *In the rest of the paper, we assume that* (3.5) *holds.*

By using the above comparison principle, it is not hard to prove that $\varphi^{m,n} \geq 0$, for $m, n \in \mathbb{N}$, or equivalently that $P^{m,n} \geq 0$. Also, it follows from the comparison principle that the price $P^{m,n}$ increases if the Sharpe ratio α increases. We now prove the monotonicity of $\varphi^{m,n}$ with respect to m and n.

Theorem 3.2 *For a fixed $m \in \mathbb{N}$ and $(\lambda, t) \in (\underline{\lambda}, \infty) \times [0, T]$, $\varphi^{m,n}(\lambda, t)$ is an increasing sequence in n. Similarly, for a fixed $n \in \mathbb{N}$, $\varphi^{m,n}(\lambda, t)$ is an increasing sequence in m.*

Proof: We will only prove the first assertion of the theorem; the second assertion can be proved in a similar fashion. To prove the first assertion, we apply an induction argument in three steps. First, note that Lemma 4.3 of Milevsky et al. (2005) implies that $(\varphi^{0,n}(\lambda, t))_{n \in \mathbb{N}}$ is an increasing sequence.

Second, for $m \geq 0$, we require that $\varphi^{m,1} \geq \varphi^{m,0}$ implies that $\varphi^{m+1,1} \geq \varphi^{m+1,0}$. Since the proof of this step is similar to the derivation below, we omit it.

In the third, and final, step for $m \geq 1$ and $n \geq 2$, we assume that $\varphi^{m,n-1} \geq \varphi^{m,n-2}$ and that $\varphi^{m-1,n} \geq \varphi^{m-1,n-1}$ and show that $\varphi^{m,n} \geq \varphi^{m,n-1}$. Define the differential operator $\mathcal{L}$ on $\mathbb{D}$ by (3.3) with $g = g_{m,n}$ in (3.4). Since $\varphi^{m,n}$ solves (2.18), we have that $\mathcal{L}\varphi^{m,n} = 0$. On the other hand,

$$\begin{aligned}
&\mathcal{L}\varphi^{m,n-1} = (n-1)\lambda(\varphi^{m,n-1} - \varphi^{m,n-2}) + m\lambda(\varphi^{m-1,n} - \varphi^{m-1,n-1}) \\
&+ \alpha\sqrt{\sigma^2(\lambda - \underline{\lambda})(\varphi_\lambda^{m,n-1})^2 + m\lambda(\varphi^{m,n-1} - \varphi^{m-1,n} - 1)^2} \\
&- \alpha\sqrt{\sigma^2(\lambda - \underline{\lambda})(\varphi_\lambda^{m,n-1})^2 + m\lambda(\varphi^{m,n-1} - \varphi^{m-1,n-1} - 1)^2 + (n-1)\lambda(\varphi^{m,n-1} - \varphi^{m,n-2})^2} \\
&\geq (m\lambda - \alpha\sqrt{m\lambda})(\varphi^{m-1,n} - \varphi^{m-1,n-1}) + ((n-1)\lambda - \alpha\sqrt{(n-1)\lambda})(\varphi^{m,n-1} - \varphi^{m,n-2}) \\
&\geq 0.
\end{aligned} \tag{3.7}$$

Here the first inequality follows since $\left|\sqrt{A^2 + B^2 + C^2} - \sqrt{X^2 + Y^2 + Z^2}\right| \leq |A - X| + |B - Y| + |C - Z|$, for any $A, B, C, X, Y, Z \in \mathbb{R}$. The second inequality follows from the induction hypothesis and from $\alpha \in (0, \sqrt{\underline{\lambda}})$ and $\lambda > \underline{\lambda}$. We also have that $\varphi^{m,n}(\lambda, T) = \varphi^{m,n-1}(\lambda, T)$. From Theorem 3.1, we conclude that $\varphi^{m,n} \geq \varphi^{m,n-1}$. $\square$

Theorem 3.2 agrees with our intuition that as we add more policyholders to our portfolio, then the price increases.

Next, we show that the price $P^{m,n}$ is subadditive, or equivalently, that $\varphi^{m,n}$ is subadditive. To this end, we require the following technical lemma.

Lemma 3.2 *Suppose* $A, B, A_1, A_2, B_1, B_2, B_\lambda, C_\lambda \in \mathbb{R}$ *are such that* $A \geq A_1$, $A \geq A_2$, $B \geq B_1$, *and* $B \geq B_2$. *Then for any nonnegative integers* m_1, m_2, n_1, n_2,

$$\begin{aligned}
\sqrt{(B_\lambda + C_\lambda)^2 + (n_1 + n_2)A^2 + (m_1 + m_2)B^2} &\leq \sqrt{B_\lambda^2 + n_1 A_1^2 + m_1 B_1^2} \\
&+ \sqrt{C_\lambda^2 + n_2 A_2^2 + m_2 B_2^2} + \sqrt{n_1}(A - A_1) + \sqrt{n_2}(A - A_2) \\
&+ \sqrt{m_1}(B - B_1) + \sqrt{m_2}(B - B_2).
\end{aligned} \tag{3.8}$$

PROOF: Square both sides of (3.8) to get

$$\begin{aligned}
(B_\lambda + C_\lambda)^2 + (n_1 + n_2)A^2 + (m_1 + m_2)B^2 &\leq B_\lambda^2 + n_1 A_1^2 + m_1 B_1^2 + C_\lambda^2 + n_2 A_2^2 + m_2 B_2^2 \\
&+ n_1(A - A_1)^2 + n_2(A - A_2)^2 + m_1(B - B_1)^2 + m_2(B - B_2)^2 + 2K,
\end{aligned}$$

which simplifies to

$$B_\lambda C_\lambda + n_1 A_1(A - A_1) + n_2 A_2(A - A_2) + m_1 B_1(B - B_1) + m_2 B_2(B - B_2) \leq K, \tag{3.9}$$

in which $K \geq 0$ is defined by

$$\begin{aligned}
K &= \sqrt{B_\lambda^2 + n_1 A_1^2 + m_1 B_1^2}\Bigg[\sqrt{C_\lambda^2 + n_2 A_2^2 + m_2 B_2^2} + \sqrt{m_1}(B - B_1) + \sqrt{m_2}(B - B_2) \\
&\qquad + \sqrt{n_1}(A - A_1) + \sqrt{n_2}(A - A_2)\Bigg] \\
&+ \sqrt{C_\lambda^2 + n_2 A_2^2 + m_2 B_2^2}\left[\sqrt{m_1}(B - B_1) + \sqrt{m_2}(B - B_2) + \sqrt{n_1}(A - A_1) + \sqrt{n_2}(A - A_2)\right] \\
&+ \sqrt{m_1}(B - B_1)\left[\sqrt{m_2}(B - B_2) + \sqrt{n_1}(A - A_1) + \sqrt{n_2}(A - A_2)\right] \\
&+ \sqrt{m_2}(B - B_2)\left[\sqrt{n_1}(A - A_1) + \sqrt{n_2}(A - A_2)\right] + \sqrt{n_1}(A - A_1)\sqrt{n_2}(A - A_2)
\end{aligned}$$

If the left-hand side of (3.9) is negative, then we are done. Otherwise, by squaring both sides of (3.9), we obtain the equivalent expression

$$\begin{aligned}
&B_\lambda^2 C_\lambda^2 + n_1^2 A_1^2(A - A_1)^2 + n_2^2 A_2^2(A - A_2)^2 + m_1^2 B_1^2(B - B_1)^2 + m_2^2 B_2^2(B - B_2)^2 \\
&\quad + 2B_\lambda C_\lambda\left[n_1 A_1(A - A_1) + n_2 A_2(A - A_2) + m_1 B_1(B - B_1) + m_2 B_2(B - B_2)\right] \\
&\quad + 2n_1 A_1(A - A_1)\left[n_2 A_2(A - A_2) + m_1 B_1(B - B_1) + m_2 B_2(B - B_2)\right] \\
&\quad + 2n_2 A_2(A - A_2)\left[m_1 B_1(B - B_1) + m_2 B_2(B - B_2)\right] + 2m_1 B_1(B - B_1)m_2 B_2(B - B_2) \\
&\leq K^2.
\end{aligned}$$

This inequality simplifies to

$$
\begin{aligned}
& n_1 n_2 \left[A_1^2 A_2^2 + (A_1(A - A_2) - A_2(A - A_1))^2 + (A - A_1)^2 (A - A_2)^2\right] \\
& \quad + n_1 m_1 \left[A_1^2 B_1^2 + (A_1(B - B_1) - B_1(A - A_1))^2 + (A - A_1)^2 (B - B_1)^2\right] \\
& \quad + n_1 m_2 \left[A_1^2 B_2^2 + (B_2(A - A_1) - A_1(B - B_2))^2 + (A - A_1)^2 (B - B_2)^2\right] \\
& \quad + n_2 m_1 \left[A_2^2 B_1^2 + (A_2(B - B_1) - B_1(A - A_2))^2 + (A - A_2)^2 (B - B_1)^2\right] \\
& \quad + n_2 m_2 \left[A_2^2 B_2^2 + (A_2(B - B_2) - B_2(A - A_2))^2 + (A - A_2)^2 (B - B_2)^2\right] \\
& \quad + m_1 m_2 \left[B_1^2 B_2^2 + ((B - B_1)(B - B_2) - B_1 B_2)^2 + (B - B_1)^2 (B - B_2)^2\right] \\
& \quad + n_1 \left[(C_\lambda A_1 - B_\lambda (A - A_1))^2 + C_\lambda^2 (A - A_1)^2\right] \\
& \quad + n_2 \left[(B_\lambda A_2 - C_\lambda (A - A_2))^2 + B_\lambda^2 (A - A_2)^2\right] \\
& \quad + m_1 \left[(C_\lambda B_1 - B_\lambda (B - B_1))^2 + C_\lambda^2 (B - B_1)^2\right] \\
& \quad + m_2 \left[(B_\lambda B_2 - C_\lambda (B - B_2))^2 + B_\lambda^2 (B - B_2)^2\right] + \kappa \\
& \geq 0,
\end{aligned}
$$

in which $\kappa \geq 0$ is a sum of nonnegative square-root terms. The last inequality always holds because each term in the sum is nonnegative. □

Theorem 3.3 *For all $m_1, m_2, n_1, n_2 \in \mathbb{N}$, we have that*

$$\varphi^{m,n} \leq \varphi^{m_1,n_1} + \varphi^{m_2,n_2} \tag{3.10}$$

on D, in which $m_1 + m_2 = m$ and $n_1 + n_2 = n$.

Proof: We prove the assertion by induction. First, $\varphi^{0,n} \leq \varphi^{0,n_1} + \varphi^{0,n_2}$, for all $n_1, n_2 \in \mathbb{N}$ such that $n = n_1 + n_2$, follows from Theorem 4.11 of Milevsky et al. (2005). Next, $\varphi^{m,0} \leq \varphi^{m_1,0} + \varphi^{m_2,0}$, for all $m_1, m_2 \in \mathbb{N}$ such that $m = m_1 + m_2$, can be proved similarly.

Next, we assume that for $m_1, m_2, n_1, n_2 \in \mathbb{N}$ with $m_1 + m_2 = m$ and $n_1 + n_2 = n$, we have

$$
\begin{aligned}
\varphi^{m,n-1} &\leq \varphi^{m_1,n_1} + \varphi^{m_2,n_2-1}, \quad & \varphi^{m,n-1} &\leq \varphi^{m_1,n_1-1} + \varphi^{m_2,n_2}, \\
\varphi^{m-1,n} &\leq \varphi^{m_1-1,n_1} + \varphi^{m_2,n_2}, \quad & \varphi^{m-1,n} &\leq \varphi^{m_1,n_1} + \varphi^{m_2-1,n_2},
\end{aligned}
\tag{3.11}
$$

with the convention that if say $m_1 = 0$, then we do not consider the inequality $\varphi^{m-1,n} \leq \varphi^{-1,n_1} + \varphi^{m_2,n_2}$ because φ^{-1,n_1} is not defined. Under the assumption in (3.11), we now show that $\varphi^{m,n} \leq \varphi^{m_1,n_1} + \varphi^{m_2,n_2}$. To this end, define the differential operator $\mathcal{L}$ on $\mathbb{D}$ by (3.3) with $g = g_{m,n}$ from (3.4). Because $\varphi^{m,n}$ solves (2.18), we have that $\mathcal{L}\varphi^{m,n} = 0$. Define $\eta \triangleq \varphi^{m_1,n_1} + \varphi^{m_2,n_2}$ on D. Then, if $m_1, m_2, n_1, n_2 \geq 1$, we have

$$
\begin{aligned}
\mathcal{L}\eta &= n_1\lambda(\varphi^{m,n-1} - \varphi^{m_1,n_1-1} - \varphi^{m_2,n_2}) + n_2\lambda(\varphi^{m,n-1} - \varphi^{m_1,n_1} - \varphi^{m_2,n_2-1}) \\
&\quad + m_1\lambda(\varphi^{m-1,n} - \varphi^{m_1-1,n_1} - \varphi^{m_2,n_2}) + m_2\lambda(\varphi^{m-1,n} - \varphi^{m_1,n_1} - \varphi^{m_2-1,n_2}) \\
&\quad + \alpha\sqrt{\sigma^2(\lambda - \underline{\lambda})^2 \eta_\lambda^2 + n\lambda\left(\eta - \varphi^{m,n-1}\right)^2 + m\lambda\left(\eta - \varphi^{m-1,n} - 1\right)^2} \\
&\quad - \alpha\sqrt{\sigma^2(\lambda - \underline{\lambda})^2 \left(\varphi_\lambda^{m_1,n_1}\right)^2 + n_1\lambda\left(\varphi^{m_1,n_1} - \varphi^{m_1,n_1-1}\right)^2 + m_1\lambda\left(\varphi^{m_1,n_1} - \varphi^{m_1-1,n_1} - 1\right)^2} \\
&\quad - \alpha\sqrt{\sigma^2(\lambda - \underline{\lambda})^2 \left(\varphi_\lambda^{m_2,n_2}\right)^2 + n_2\lambda\left(\varphi^{m_2,n_2} - \varphi^{m_2,n_2-1}\right)^2 + m_2\lambda\left(\varphi^{m_2,n_2} - \varphi^{m_2-1,n_2} - 1\right)^2} \\
&\leq (n_1\lambda - \alpha\sqrt{n_1\lambda})(\varphi^{m,n-1} - \varphi^{m_1,n_1-1} - \varphi^{m_2,n_2}) \\
&\quad + (n_2\lambda - \alpha\sqrt{n_2\lambda})(\varphi^{m,n-1} - \varphi^{m_1,n_1} - \varphi^{m_2,n_2-1}) \\
&\quad + (m_1\lambda - \alpha\sqrt{m_1\lambda})(\varphi^{m-1,n} - \varphi^{m_1-1,n_1} - \varphi^{m_2,n_2}) \\
&\quad + (m_2\lambda - \alpha\sqrt{m_2\lambda})(\varphi^{m-1,n} - \varphi^{m_1,n_1} - \varphi^{m_2-1,n_2}) \\
&\leq 0.
\end{aligned}
\tag{3.12}
$$

Note that the first inequality follows from an application of Lemma 3.2 by assigning $B_\lambda = \sigma(\lambda - \underline{\lambda})\varphi^{m_1,n_1}$, $C_\lambda = \sigma(\lambda - \underline{\lambda})\varphi^{m_2,n_2}$, $A = \sqrt{\lambda}(\eta - \varphi^{m,n-1})$, $A_1 = \sqrt{\lambda}(\varphi^{m_1,n_1} - \varphi^{m_1,n_1-1})$, $A_2 = \sqrt{\lambda}(\varphi^{m_2,n_2} - \varphi^{m_2,n_2-1})$, $B = \sqrt{\lambda}(\eta - \varphi^{m-1,n} - 1)$, $B_1 = \sqrt{\lambda}(\varphi^{m_1,n_1} - \varphi^{m_1-1,n_1} - 1)$, and $B_2 = \sqrt{\lambda}(\varphi^{m_2,n_2} - \varphi^{m_2-1,n_2} - 1)$. It follows from the induction hypothesis in (3.11) that $A \geq A_1$, $A \geq A_2$, $B \geq B_1$ and $B \geq B_2$. The last inequality in (3.12) follows from the induction hypothesis and from $\alpha \leq \sqrt{\underline{\lambda}}$ and $\lambda > \underline{\lambda}$. If any of m_1, m_2, n_1, or n_2 is 0, then the corresponding term in (3.12) is 0. We also have that $\varphi^{m,n}(\lambda, T) = n = n_1 + n_2 = \varphi^{m_1,n_1}(\lambda, T) + \varphi^{m_2,n_2}$. From Theorem 3.1, we conclude that $\varphi^{m,n} \leq \varphi^{m_1,n_1} + \varphi^{m_2,n_2}$. □

Because $P^{m,n} = F\varphi^{m,n}$, it follows from (3.10) that $P^{m,n} \leq P^{m_1,n_1} + P^{m_2,n_2}$; that is, the price $P^{m,n}$ is subadditive. Subadditivity is a reasonable property because if it did not hold, then buyers of insurance could insure risks separately and thereby save money.

In our framework we define the relative price of a pure endowment at time t as the difference $P^{m,n+1}(r, \lambda, t) - P^{m,n}(r, \lambda, t) = F(r, t)(\varphi^{m,n+1}(\lambda, t) - \varphi^{m,n}(\lambda, t))$, for $m, n \in \mathbb{N}$, given that the insurer already holds m life insurances and n pure endowments. This relative, or marginal, price is nonnegative as a result of Theorem 3.2. The corollary below gives a tight upper bound on the relative price of a pure endowment.

Corollary 3.1 *Let us introduce*

$$h(t) \triangleq \exp(-(\underline{\lambda} - \alpha\sqrt{\underline{\lambda}})(T - t)), \tag{3.13}$$

which solves the pde for $\varphi^{0,1}$ *when we set* $\lambda_t = \underline{\lambda}$ *for all* $t \geq 0$. *Then for all* $m, n \in \mathbb{N}$

$$\varphi^{m,n+1}(\lambda, t) - \varphi^{m,n}(\lambda, t) \leq h(t), \quad \lambda \geq \underline{\lambda}, t \geq 0, \tag{3.14}$$

where h *is as in (3.13).*

PROOF: The proof of is a direct consequence of subadditivity (Theorem 3.3). Indeed $\varphi^{m,n+1} \leq \varphi^{m,n} + \varphi^{0,1}$ and $\varphi^{0,1} \leq h$ as a result of Theorem 3.5 of Milevsky et al. (2005).
□

Note that the upper bound h in (3.14) is attained when $\lambda_t = \underline{\lambda}$ (and thereby $\lambda_s = \underline{\lambda}$ for all $s \geq t$).

The relative price of a life insurance contract when the insurer holds m life insurance contracts and n pure endowment contracts is given by $F\left(\varphi^{m+1,n} - \varphi^{m,n}\right)$. As a corollary to Theorem 3.3, we have an upper bound for this relative price.

Corollary 3.2 *For all* $m, n \in \mathbb{N}$, $\varphi^{m+1,n} - \varphi^{m,n} \leq 1$.

Thus, it follows from Corollary 3.2 that the relative price for a life insurance contract is bounded above by the price of a default-free zero-coupon bond. As λ becomes arbitrarily large, this upper bound is attained because the individual is very likely to die immediately.

4 Conclusion

In this paper, we developed a relative pricing methodology for pure endowments and life insurances, assuming that the insurer has already sold a portfolio of these instruments. In our model, we take the hazard rate and the interest rate to be stochastic. The market is incomplete since the insurer can not fully hedge the against the stochastic mortality risk. The insurer values a portfolio of pure endowment and insurance contracts by specifying the instantaneous Sharpe ratio of the optimal local hedging portfolio.

We first show that the value of the portfolio with m life insurances and n pure endowments, $P^{m,n}$, satisfies subaddivity; that is, $P^{m,n} \leq P^{m_1,n_1} + P^{m_2,n_2}$ for any $m_1, m_2, n_1, n_2 \in \mathbb{N}$ such that $m_1 + m_2 = m$ and $n_1 + n_2 = n$. This property shows that for given market prices for insurance and pure endowments, the insurer is better off selling both the pure endowment and the life insurance instead of selling only one type of contract; that is, life insurance and pure endowment hedge each other partially.

We also found that the relative price, $P^{m,n+1} - P^{m,n}$, for a pure endowment satisfies

$$0 \leq P^{m,n+1} - P^{m,n} \leq hF.$$

Here F is the price of a zero coupon bond and $h \leq 1$ is a risk-adjusted probability of survival; see equation (3.13). Similarly, the relative price of a life insurance, $P^{m+1,n} - P^{m,n}$, satisfies

$$0 \leq P^{m+1,n} - P^{m,n} \leq F.$$

In future work, we plan to investigate the limits,

$$\lim_{n \to \infty} \lim_{m \to \infty} (P^{m,n+1} - P^{m,n}) \quad \text{and} \quad \lim_{m \to \infty} \lim_{n \to \infty} (P^{m+1,n} - P^{m,n}),$$

which are the relative prices of pure endowment and life insurance contracts, respectively, given that the insurer has sold arbitrarily many of these contracts (the saturation prices). We are also interested in determining the corresponding rates of convergence to see how well these limits might approximate the prices of contracts in real life.

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