Stochastic Differential Games in a Non-Markovian Setting

Erhan Bayraktar * H. Vincent Poor †

Princeton University

Abstract

Stochastic differential games are considered in a non-Markovian setting. Typically, in stochastic differential games the modulating process of the diffusion equation describing the state flow is taken to be Markovian. Then Nash equilibria or other types of solution such as Pareto equilibria are constructed using Hamilton-Jacobi-Bellman (HJB) equations. But in a non-Markovian setting the HJB method is not applicable. To examine the non-Markovian case, this paper considers the situation in which the modulating process is a fractional Brownian motion. Fractional noise calculus is used for such models to find the Nash equilibria explicitly. Although fractional Brownian motion is taken as the modulating process because of its versatility in modeling in the fields of finance and networks, the approach in this paper has the merit of being applicable to more general Gaussian stochastic differential games with only slight conceptual modifications. This work has applications in finance to stock price modeling which incorporates the effect of institutional investors, and to stochastic differential portfolio games in markets in which the stock prices follow diffusions modulated with fractional Brownian motion.

*Department of Electrical Engineering, Princeton University, Princeton, NJ 08544, ebayrak@ee.princeton.edu

†Department of Electrical Engineering, Princeton University, Princeton, NJ 08544, poor@princeton.edu

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1 Introduction

The study of stochastic differential games with controls is a part of game theory that is relatively unknown, even though it has significant potential for application as noted by Øksendal and Reikvam [34]. Prior work in this area has focused on the examination of such games in a Markovian setting (see below). In this paper we will study a type of non-Markovian stochastic differential game. In particular, we will consider a game in which the n-dimensional state X_t follows the following stochastic differential equation:

$$dX_t = \mu(t, X_t, \upsilon_1, ..., \upsilon_N)dt + \sigma(t, X_t, \upsilon_1, ..., \upsilon_N)dB_t^{(H)}, \tag{1}$$

$$\mu : [0,T] \times \mathbb{R}^{\mathrm{n}} \times \Upsilon_1 \times ... \times \Upsilon_{\mathrm{N}} \to \mathbb{R}^{\mathrm{n}}, \tag{2}$$

$$\sigma : [0,T] \times \mathbb{R}^{\mathrm{n}} \times \Upsilon_1 \times ... \times \Upsilon_{\mathrm{N}} \to \mathbb{R}^{\mathrm{n}} \times \mathbb{R}^{\mathrm{m}}, \tag{3}$$

where $\upsilon_i \in \Upsilon_i \subset \mathbb{R}^{\nu_i}$ is the control of ith player over the state and is adapted to the natural filtration of $B^{(H)}$. Here T is the expiration date of the game, and $B^{(H)}$ is an m-dimensional fractional Brownian motion (fBm) having independent components, with $H = (H_1, ..., H_m)$ denoting the vector of Hurst parameters of the components with $H_i \in (\frac{1}{2}, 1)$. That is, $B^{(H)}$ is an almost surely (a.s.) continuous zero mean Gaussian process having the autocorrelation structure given by

$$E\left\{B_t^{H_i} B_s^{H_j}\right\} = \frac{1}{2}\left\{|s|^{2H_i} + |t|^{2H_i} - |t-s|^{2H_i}\right\}\delta_{ij}, \qquad 1 \le i, j \le m, \tag{4}$$

where

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases} \tag{5}$$

Each agent wants to maximize its own pay-off (with this feature the problem differs from the usual optimal control problem):

$$J_i(x) = E^x\left\{\int_0^T f_i(t, X_t, \upsilon_t)dt + K_i(T, X_T)\right\}, \tag{6}$$

where $E^x\{\cdot\}$ denotes conditional expectation given $X_0 = x$.

Typically in this type of setting the modulating process in (1) is taken to be Brownian motion, *i.e.*, $H = (\frac{1}{2}, ..., \frac{1}{2})$ and the controls of the players are Markovian. Then Nash equilibria or other types of solution such as Pareto equilibria are constructed using the Hamilton-Jacobi-Bellman (HJB) equations. (See e.g. Friedman [11], Gaidov [12], [13], [14] Nilakantan [30], Øksendal and Reikvam [34], and Pravin [39].) However, fBm is not a Markov process for any H other than $\frac{1}{2}$, and therefore this approach does not work for the general case of (1). Here we will develop a quasi-martingale approach to the solution of this problem using the fractional noise calculus developed by Duncan et al. [9], and Øksendal and Hu [35] which generalizes White noise calculus (see [1]) to develop an integration theory with respect to fBm. The key to our solution will be the fractional Clark-Ocone formula developed by Øksendal and Hu [35]. The integrals in (1) are Wick type integrals (see Definition 3.4) rather than Stieltjes integrals (defined pathwise see e.g. [40]). The motivation for using Wick type integrals is as follows: The pathwise integral $\int_0^t f_s \delta B_s^H$ with respect to fBm does not in general have a zero mean, i.e.

$$E\left\{\int_0^t f_s \delta B_s^H\right\} \neq 0. \tag{7}$$

However the Wick type integral $\int_0^t f_s dB_s^H$ has zero mean, i.e.

$$E\left\{\int_0^t f_s dB_s^H\right\} = 0. \tag{8}$$

Therefore in a stochastic differential equation (SDE) of the form

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t^H, \tag{9}$$

the volatility term $\sigma(X_t)dB_t^H$ does not contribute to the mean rate of change, as it does with SDE's with pathwise integrals. Since seperating the random fluctuations from the mean rate of change is desirable for our purposes, we prefer to use Wick type integrals for defining the integrals with respect to fBm. (See [9].) (Note also that only in the Wick type calculus the standard tools of Itô calculus such as Clark-Ocone formula are available.) See [8] for applications of Wick calculus to pricing weather derivatives, [10] and [35] for further applications of Wick type calculus particularly in finance.

Fractional noise calculus reduces to White noise calculus when H is replaced by 1/2. Moreover the integrals of adapted processes in this framework are equal to the Itô integrals of these procesess with respect to Brownian motion. Hence our results hold in particular for the standard framework, i.e. when the modulator is a Brownian motion, and the integrals in (1) are taken to be Itô integrals.

This work has an immediate application in finance, to stock price modeling when the long-range dependence is accounted for in the model. The stock prices are considered to be states in this setting while the agents are *not price takers*, *i.e.* their trading change the price level. This models a market with institutional investors who make large transactions and therefore influence the prices of the stocks. These investors find themselves in a random environment due to the existence of small investors. The small investors are typically inert, i.e. that is they do not trade for long time intervals. A micro-structure model taking the inertness of the agents into account is constructed in [5]. It is shown that the prices arising fron the interaction of the small agents can be approximated by geometric fractional Brownian motion. The game theoretic setting in this paper is an extension to the results of [5] in the sense that, we start by assuming that the noise in the environment can be modeled by an fBm differential in the controlled stochastic differential equation (1), which is the noise due to the trades of the small investors.

Another possible application is stochastic differential portfolio games, which are studied by Browne in a Brownian motion setting in [6]. This formulation is applicable to the analysis of traders who are competing for a bonus, or to fund managers whose funds are invested in different markets, and who achieve rewards based on the relative performance of their funds. Yet another possible application is the stochastic goodwill problems (finding the optimal advertising policy for the maximization of the product image) in advertising when there are more than one good of the same kind in competition. (See [29] for stochastic goodwill problems in a stochastic optimal control setting.)

The rest of the paper is organized as follows. In Section 2 we give two Nash-equilibrium theorems, the first theorem is in a one-dimensional setting while the second one represents a more general type of setting where the states observed by the players and the source of randomness are multi-dimensional. In Section 3 we give a brief overview of fractional noise calculus. In Section 4 we give a proof of our first theorem, and in Section 5 we give a proof of our second theorem. Finally in Section 6 we give a sketch of how to extend the fractional noise machinery to more general Gaussian modulation processes.

2 Main Results

In this section we introduce two theorems which give explicitly the Nash-equilibrium strategies in two N player games. In the first game there is only one source of randomness and there is a common state observed by all players. The second game is a more general game where there are m sources of randomness and there are n states which are observed by all the players.

2.1 Nash Equilibrium in a Linear Game of N players

For the ease of exposition we consider first a one dimensional state equation, with the drift and diffusion coefficients controlled linearly by the players:

$$dX_t = rX_t dt + \sum_{i=1}^{N} \alpha_i(t)u_i(t)dt + C\sum_{i=1}^{N} \beta_i(t)v_i(t)dt + \sum_{i=1}^{N} \beta_i(t)v_i(t)dB^H(t), \tag{10}$$

where B^H denotes the one-dimensional version of $B^{(H)}$ with $H_1 = H$. The initial state will be denoted by $X_0 = x$. The pay-off function of player i will be of the form:

$$J_i(x) = E_\mu^x \left\{ \int_0^T \frac{c_i u_i^{\gamma_i}(t)}{\gamma_i} dt + \frac{b_i X_T^{\gamma_i'}}{\gamma_i'} \right\}; \tag{11}$$

that is, players are constant relative risk averse (CRRA). Here μ is the measure on the sample space under which the canonical process is an fBm. Player i controls the state by its choice of actions (u_i, v_i). We assume that $\alpha_i : [0,T] \to \mathbb{R}$ is bounded for each $i \in \{1, ..., N\}$. The coefficients functions $\beta_i : [0,T] \to \mathbb{R}$ will appear in the definition of admissible strategies.

Since this game is in a non-Markovian setting, it cannot be solved via the HJB method. Instead we will employ the recently developed fractional Wick calculus, which we describe briefly in Section 3 and the fractional Clark-Ocone formula (which is given along with the proof of the equilibrium theorem) to find Nash equilibria for this game. Observe that u_i affects the drift of the state and also appears in the pay-off function. It can be interpreted as a cost for the player, *i.e.* for gaining a certain amount of riskless increase the player pays an associated cost. Whereas by choice of v_i the player does not have to pay a cost for an associated gain (since this action does not appear in the pay-off function), but it is taking some risk (since v_i affects the diffusion coefficient in addition to the drift).

Let us introduce the following notation that is necessary to define the admissible strategies and for the statement of the theorem.

Define K as

$$K(t) = \frac{C(Tt - t^2)^{\frac{1}{2}-H}}{2H(2H-1)\Gamma(2H-1)\Gamma(2-2H)\cos(\pi(H-\frac{1}{2}))}, \quad \text{for } t \in [0,T]. \tag{12}$$

and define ζ by

$$\begin{gathered} \big((-\Delta)^{-(H-1/2)}\zeta_t\big)(s) = \big((-\Delta)^{-(H-1/2)}K\big)(s), \quad 0 \le s \le t, \\ \zeta_t(s) = 0 \quad s < 0 \quad \text{or} \quad s > t, \end{gathered} \tag{13}$$

where the operator $(-\Delta)^{-(H-1/2)}$ operating of a test function f is defined by

$$\big((-\Delta)^{-(H-1/2)}f\big)(x) = \frac{1}{2\Gamma(2H-1)\cos(\pi(H-1/2))}\int_{-\infty}^{\infty} |x-t|^{2H-2} f(t)dt. \tag{14}$$

where Γ is the gamma function and is given by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \quad x > 0. \tag{15}$$

The existence of such ζ is guaranteed by [18].

Define $\hat{\mu}$ and η by

$$\eta(T) := \frac{d\hat{\mu}}{d\mu} := \exp\left(-\int_0^T K(s)dB_s^H - \frac{1}{2}|K|_\phi^2\right), \tag{16}$$

$|K|_\phi^2$ is given by

$$|K|_\phi^2 = \int_{\mathbb{R}_+^2} K(s)K(t)\phi(s,t)dsdt, \tag{17}$$

where

$$\phi(s,t) = H(2H-1)|s-t|^{2H-2}; \quad s,t \in \mathbb{R}_+. \tag{18}$$

Note that integrals of deterministic functions w.r.t fBm are well defined, as will become clear in Section 3.

And finally define ρ by

$$\rho(t,w) = E_\mu\{\eta(T)\big|\mathcal{F}_t\}. \tag{19}$$

where $\mathcal{F}_t$ is the σ-algebra generated by $\{B_s^H, s \leq t\}$.

Now we will introduce the solution concept of Nash equilibrium in our context. We consider a set $\mathcal{A} = \mathcal{A}_1 \times ... \times \mathcal{A}_N$ of admissible strategies for which the admissibility conditions are adaptedness with respect to (w.r.t.) the filtration generated by fBm and the following integrability condition

$$\beta_i v_i \in \mathcal{L}_\phi^{1,2}(\hat{\mu}), \tag{20}$$

where $\mathcal{L}_\phi^{1,2}(\hat{\mu})$ denotes the completion of the set of all $\mathcal{F}_t$ adapted processes f such that

$$\| f \|_{\mathcal{L}_\phi^{1,2}(\hat{\mu})} := E_{\hat{\mu}} \left\{ \int_{\mathbf{R}} \int_{\mathbf{R}} f(s) f(t) \phi(s,t) ds dt \right\} + E_{\hat{\mu}} \left\{ \left(\int_{\mathbf{R}} D_s^\phi f(s) ds \right)^2 \right\} < \infty. \tag{21}$$

Here $D_s^\phi F = \int_{\mathbf{R}} \phi(s,t) D_t F dt$, where $D_s F$ denotes the Hida-Malliavin derivative of F, which will be introduced in Section 3.

Definition 2.1 *The strategy $z^e = (u^e, v^e) \in \mathcal{A}$ is called a Nash-equilibrium strategy if, for each i, player i's action $z_i^e = (u_i, v_i) \in \mathcal{A}_i$ is a best response to its opponents,* i.e.

$$J_i^x(z_1^e, ..., z_{i-1}^e, z_i, z_{i+1}^e, ..., z_N^e) \leq J_i^x(z^e), \tag{22}$$

for all x, for each player i and for all $z_i \in \mathcal{A}_i$.

Then we have the following Nash equilibrium theorem:

Theorem 2.1 *Consider the game given by (10) and (11). Then the following conditions ((23), (24)) are necessary and sufficient for the existence of a Nash-equilibrium:*

$$\gamma_i' = \gamma' \quad \textit{for } i = 1, ..., N \tag{23}$$

$$\begin{aligned} \sum_{i=1}^N \int_0^T m^{\frac{1}{\gamma_i - 1}} b_i^{\frac{1}{\gamma_i - 1}} \alpha_i(t)^{\frac{\gamma_i}{\gamma_i - 1}} e^{-rt\frac{\gamma_i}{\gamma_i - 1}} \exp\left(\frac{\gamma_i}{2(1-\gamma_i)^2} |\zeta_t|_\phi \right) dt \\ + m^{\frac{1}{\gamma' - 1}} e^{-rT\frac{\gamma'}{\gamma' - 1}} \exp\left(\frac{2\gamma' |K|_\phi}{2(1-\gamma')^2} \right) = x, \end{aligned} \tag{24}$$

has a solution $m^* \in \mathbb{R}$.

Let $\left((u_1^e, v_1^e), ..., (u_N^e, v_N^e)\right)$ denote the agents' Nash-equilibrium strategies. The first components of the equilibrium strategies uniquely determined by

$$u_i^e(t) = \left(\frac{m^* b_i \alpha_i(t)}{c_i} e^{-rt} \rho(t,w) \right)^{\frac{1}{\gamma_i - 1}}, \quad \textit{for } i = 1, ..., N, \tag{25}$$

while the second component of the players' strategies will be any of adapted (to the filtration of fBm) processes satisfying the following constraint:

$$
\begin{aligned}
e^{-rt}\sum_{i=1}^{N}\beta_i v_i^e(t) &= (m^*)^{\frac{1}{\gamma'-1}}\frac{K(t)}{1-\gamma'}\exp\Bigg(\frac{1}{1-\gamma'}\int_0^t K(s)dB_s^H - \frac{C}{1-\gamma'}\int_t^T K(s)ds \\
&+ \frac{2-\gamma}{2(1-\gamma^2)}|K|_\phi^2 - \frac{1}{1-\gamma}|K1_{[0,t]}|_\phi^2 - rT\frac{\gamma'}{\gamma'-1}\Bigg) - \int_0^T\sum_{i=1}^{N}\alpha_i(u)^{\frac{\gamma_i}{\gamma_i-1}}(\frac{m^*b_i}{c_i})^{\frac{1}{\gamma_i-1}} \\
&\times \frac{\zeta_u(t)}{1-\gamma_i}e^{-ru\frac{\gamma_i}{\gamma_i-1}}\exp\Bigg(\frac{1}{1-\gamma_i}\int_0^t\zeta_u(s)dB_s^H - \frac{C}{1-\gamma_i}\int_t^u\zeta_u(s)ds \\
&+ \frac{2-\gamma_i}{2(1-\gamma_i)^2}|\zeta_u|_\phi^2 - \frac{1}{1-\gamma_i}|\zeta_u 1_{[0,t]}|_\phi^2\Bigg)du,
\end{aligned}
\tag{26}
$$

where 1 stands for the indicator function. Finally, the state at time T at Nash equilibrium is given by

$$F^e = (m^*)^{\frac{1}{\gamma'-1}}\eta(T)^{\frac{1}{\gamma'-1}}e^{\frac{-rT}{\gamma'-1}}. \tag{27}$$

2.2 Multi-dimensional Games

In this section we will give the Nash equilibrium theorem for games with an arbitrary number of states and multiple sources of randomness affecting the evolution of the states. Here the sources will be independent fractional Brownian motions with different Hurst parameters as in (1)-(5).

We again consider linear controls, so that the dynamics (1) of the states are of the form

$$
\begin{aligned}
dX_t^\iota &= \Bigg(r_\iota X_t^\iota + \sum_{i=1}^{N}\alpha_i^\iota(t)u_i(t) + v_{\iota 1}^i(t)f_1(t) + ... + v_{\iota m}^i(t)f_m(t)\Bigg)dt \\
&+\Bigg(\sum_{j=1}^{N}v_{\iota 1}^j(t)dB_t^{H_1} + ... + v_{\iota m}^j(t)dB_t^{H_m}\Bigg), \qquad \text{for} \quad \iota = 1,...,n, \quad \text{and } X_0 = x.
\end{aligned}
\tag{28}
$$

Player i controls the state through the control inputs u_i and (v_{kj}^i) The pay-off function for player i is given by

$$J_i(x) = E_\mu^x\left\{\int_0^T g_i(u_i(t))dt + h_i(X_T)\right\}, \tag{29}$$

for some $g_i : \mathbb{R} \to \mathbb{R}$ and $h_i : \mathbb{R}^n \to \mathbb{R}$. We assume that

$$f_i : [0,T] \to \mathbb{R} \quad \text{i} = 1,...,\text{m are continuous,} \tag{30}$$

and

$$\alpha_i^\iota(t) : [0,T] \to \mathbb{R} \qquad \iota = 1,...,\text{n}; \quad \text{i} = 1,...,\text{N} \quad \text{are bounded.} \tag{31}$$

Each player is interested in maximizing its own pay-off function, and we will find the Nash equilibrium of this game. To be able to give closed form solutions we will assume that

$$g_i(u_i) = \frac{u_i^{\gamma_i}}{\gamma_i}, \quad \text{for} \quad i = 1, ..., N, \tag{32}$$

and

$$h_i(F^i) = h(F^i) = \frac{\| F^i \|^{\gamma'}}{\gamma'}, \quad \text{for} \quad i = 1, ..., N, \tag{33}$$

where $\| F^i \| = \sqrt{(F_1^i)^2 + ... + (F_n^i)^2}$, $0 < \gamma_i < 1$ and $0 < \gamma' < 1$ (CRRA utility functions).

We will now introduce some variables that are necessary to state the equilibrium theorem in this multi-dimensional case.

Suppose that $\hat{f}_i$ for $i = 1, ..., m$ is a solution to the integral equation

$$\int_0^T \hat{f}_i(s)\phi_i(s,t)ds = f_i(t), \quad 0 \leq t \leq T, \tag{34}$$

when ϕ_i is obtained from (18) by setting $H = H_i$ and set $\hat{f} = (\hat{f}_1, ..., \hat{f}_m)$. Note that we can compute $\hat{f}_i$ from f_i using (201) in the Appendix. Also define $\zeta_t = (\zeta_t^1, ..., \zeta_t^m)$ by,

$$\begin{aligned} \left((-\Delta)^{-(H-1/2)}\zeta_t^i\right)(s) &= \left((-\Delta)^{-(H-1/2)}\hat{f}_i\right)(s), \quad 0 \leq s \leq t, \\ \zeta_t^i(s) &= 0 \quad s < 0 \quad \text{or} \quad s > t, \quad \text{for } i = 1, ..., m, \end{aligned} \tag{35}$$

where Δ is given by (14).

We will denote by η the following random variable

$$\eta(T, w) = \exp\left(-\sum_{i=1}^m \int_{\mathbf{R}} \hat{f}_i(t)dB_t^{H_i} - \frac{1}{2}|\hat{f}|_\phi^2\right), \tag{36}$$

where

$$|\hat{f}|_\phi^2 = \sum_{i=1}^m \int_{\mathbf{R}^2} \hat{f}_i(s)\hat{f}_i(t)\phi_i(s,t)dsdt = \sum_{i=1}^m |\hat{f}_i|_{\phi_i}^2. \tag{37}$$

Finally let $\rho(t, w)$ denote the conditional expectation of $\eta(T, w)$ given the past up to time t, *i.e.*

$$\rho(t, w) = E\left\{\eta(T)|\mathcal{F}_t\right\}. \tag{38}$$

We call a strategy profile $\left((u_j), (v_{\iota i}^j)\right)$, admissible if each strategy is adapted to the natural filtration generated by $B^{(H)}$, and the following integrability condition is satisfied:

$$v_{\iota i}^j \in \mathcal{L}_{\phi_i}^{1,2}(\hat{\mu}) \quad \text{for } j = 1, ..., N; \ \iota = 1, ..., n; \ i = 1, ..., m; \tag{39}$$

where $\mathcal{L}_{\phi_i}^{1,2}(\hat{\mu})$ is the completion of all $\mathcal{F}_t$ adapted processes satisfying (21) with ϕ replaced by ϕ_i.

We can now state our multi-dimensional equilibrium theorem.

Theorem 2.2 *Consider the game given by (28) and (29). Then the following conditions given by (40) is necessary and sufficient for the existence of a Nash-equilibrium:*

$$
\begin{aligned}
&-\int_0^T \sum_{i=1}^{N} \alpha_i(t) \left(\lambda^T \alpha_i(t)\right)^{\frac{1}{\gamma_i-1}} \exp\left(\frac{rt\gamma_i}{1-\gamma_i} + \frac{\gamma_i}{2(1-\gamma_i)^2}|\zeta_t|_\phi^2\right) dt \\
&+ \lambda\|\lambda\|^{\frac{2-\gamma'}{\gamma'-1}} \exp\left(\frac{rT\gamma'}{1-\gamma'} + \frac{\gamma'}{2(1-\gamma')^2}|\hat{f}|_\phi^2\right) = x,
\end{aligned}
\tag{40}
$$

has a solution $\lambda^* \in \mathbb{R}^n$, where $\alpha_i = (\alpha_i^1, ..., \alpha_i^n)^T$.

Let u_i^e and $\left((v_{lk}^i)^e\right), l = 1, ..., n$ and $k = 1, ..., m$ denote the equilibrium strategy of player i. Then u_i^e is uniquely determined by

$$
u_i^e(t) = \left((\lambda^*)^T \alpha_i(t) e^{-rs} \rho(t, w)\right)^{\frac{1}{\gamma_i-1}}, \tag{41}
$$

and (v_{lk}^i) are any adapted (to the filtration generated by fBm) processes satisfying the following constraints:

$$
\begin{aligned}
e^{-r_l t} \sum_{i=1}^{N} v_{l,k}^i(t) &= \lambda_l^* \|\lambda^*\|^{\frac{2-\gamma'}{\gamma'-1}} \frac{1}{1-\gamma'} \hat{f}_k(t) \exp\left(\frac{rT\gamma'}{1-\gamma'} + \frac{1}{1-\gamma'}\int_0^t \hat{f}(s) f(s) ds\right. \\
&\quad \left. - \frac{1}{2(1-\gamma')^2}|\hat{f} 1_{[0,t]}|_\phi^2 + \frac{\gamma'}{2(1-\gamma')^2}|\hat{f}|_\phi^2\right) - \int_0^T \sum_{i=1}^{N} \alpha_i(s) \left((\lambda^*)^T \alpha_i(s)\right)^{\frac{1}{\gamma_i-1}} \\
&\quad \times \frac{1}{1-\gamma_i} \zeta_s^k(t) \exp\left(\frac{rs\gamma_i}{1-\gamma_i} + \frac{1}{1-\gamma_i}\sum_{j=1}^{m}\int_0^t \zeta_s^j(u) dB_u^{H_j}\right. \\
&\quad + \frac{1}{1-\gamma_i}\sum_{j=1}^{m}\int_0^t \zeta_s^j(u) f_j(u) du - \frac{1}{2(1-\gamma_i)^2}|\zeta_s 1_{[0,t]}|_\phi^2 + \frac{2-\gamma_i}{2(1-\gamma_i)^2}|\zeta_s|_\phi^2 \\
&\quad \left. + \frac{1}{\gamma_i - 1}\sum_{j}\int_0^T \zeta_s^j(u) f_j(u) du\right) ds
\end{aligned}
\tag{42}
$$

for $l = 1, ..., n$, and $k = 1, ..., m$ and $t \in [0, T]$. The final state at time T at the equilibrium is given by

$$
F^e = \eta(T)^{\frac{1}{\gamma'-1}} e^{\frac{rT}{1-\gamma'}} \|\lambda^*\|^{\frac{2-\gamma'}{\gamma'-1}} \lambda^* \tag{43}
$$

Before proving Theorems 2.1 2.2 we will, in the next section, give a brief introduction to fractional noise calculus, and introduce other essential background, mostly following the treatment by Hu and Øksendal [35]. (Although the original name given by Hu and Øksendal in [35] to this kind of calculus was fractional white noise calculus, we prefer to omit 'white' from the name since white suggests independence at each point in time, and here the noise considered is far from it.)

3 Fractional Noise Calculus

Following the traditional white noise calculus formulation [16], [17], and [23], we will take the probability space in the first game (10)-(11) (where there is only one source of randomness) be $\Omega = \mathcal{S}'(R)$, *i.e.* the space of tempered distributions (the dual space of $\mathcal{S}(\mathbb{R})$, the Schwartz space of rapidly decreasing functions) equipped with the weak-star topology. Then we will take the events to be Borel subsets of $\mathcal{S}'(R)$. In what follows $L^2_\phi(\mathbb{R})$ will denote the completion of the set of measurable functions satisfying

$$|f|^2_\phi = \int_{\mathbb{R}^2} f(s)f(t)\phi(s,t)dsdt < \infty. \tag{44}$$

Remark: It is shown by Taqqu and Pipiras [36] that the set of functions satisfying (44) is not a complete space.

For any $f, g \in L^2_\phi(\mathbb{R})$ their inner product is given by

$$(f,g)_\phi = \int_{\mathbb{R}^2} f(s)f(t)\phi(s,t)dsdt. \tag{45}$$

We can state the following result (see:[16]):

Theorem 3.1 *(The Bochner-Minlos Theorem)([16]) There exists a unique probability measure μ on Ω such that*

$$\int_\Omega e^{i<w,f>}d\mu(w) = e^{-\frac{1}{2}|f|^2_\phi}, \quad \textit{for all } f \in \mathcal{S}(\mathbb{R}), \tag{46}$$

where $< w, f >$ denotes the action of w on f.

Remark: The proof of this theorem uses the fact that the map $f \to \exp(-\frac{1}{2}|f|^2_\phi)$ is positive definite on $\mathcal{S}(R)$.

In our setting ϕ is taken as (see (18))

$$\phi(s,t) = H(2H-1)|s-t|^{2H-2}; \quad s,t \in \mathbb{R}. \tag{47}$$

Remark: By adapting the white noise machinery presented in this chapter, Theorems 2.1 and 2.2 can be shown to hold for more general Gaussian modulators in the state flow dynamics. However we state the results in terms of fBm to emphasize the fact that the game under consideration becomes non-Markovian, and also because this case admits an explicit equilibrium. Another motivation for this model is the fact that fBm is frequently used in modeling in finance ([3], [4], [28], and [35]), networks ([25] and [31]), and other applications([2], [32] and [33]).

Let us define a random process on Ω by

$$\tilde{B}_H(t) = < w, \chi_{[0,t]} > \quad t \geq 0, \tag{48}$$

where χ is given by

$$\chi_{[0,t]}(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq t \\ -1 & \text{if } t \leq s \leq 0, \text{ except for } t = s = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{49}$$

From (46) we see that $\tilde{B}_H$ is a zero mean Gaussian process with autocorrelation given by

$$E\left\{\tilde{B}_t^H \tilde{B}_s^H\right\} = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right). \tag{50}$$

Due to the Kolmogorov-Čentsov Theorem ([22]) there is a modification B^H of $\tilde{B}^H$ whose sample paths are Hölder continuous of order less than H, and therefore B^H is an fBm.

The stochastic integral of deterministic functions w.r.t. fBm are well defined by the following theorem.

Theorem 3.2 *([15]) If $f, g \in L^2_\phi(\mathbb{R})$, then $\int_{\mathbf{R}} f dB^H$ and $\int_{\mathbf{R}} g dB^H$ are well defined zero-mean Gaussian random variables, with variances $|f|^2_\phi$ and $|g|^2_\phi$ respectively, covariance $< f, g >_\phi$,* i.e. *the integral operator is a Hilbert space isomorphism.*

Note that

$$< w, f >= \int_{\mathbf{R}} f(t) dB_t^H \quad \text{for } f \in L^2_\phi(\mathbb{R}). \tag{51}$$

For Wiener chaos expansion of random variables in $L^2(\mu)$ one first has to find the orthonormal basis for $L^2_\phi(\mathbb{R})$. Let

$$h_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}\left(e^{\frac{-x^2}{2}}\right), \quad \text{for } n = 0,1,... \tag{52}$$

be the Hermite polynomials. Then the Hermite functions defined as

$$\tilde{h}_n(x) = \pi^{-\frac{1}{4}}\left((n-1)!\right)^{-\frac{1}{2}} h_{n-1}(\sqrt{2}x) e^{\frac{-x^2}{2}}, \quad n = 1, 2, ... \tag{53}$$

form an orthonormal basis for $L^2(\mathbb{R})$ ([23]).

Define the map from the space of functions satisfying (44) into $L^2(\mathbb{R})$ by

$$(I_\phi f)(u) = c_H \int_u^\infty (t-u)^{H-\frac{3}{2}} f(t) dt, \tag{54}$$

where

$$c_H = \sqrt{\frac{H(2H-1)\Gamma(\frac{3}{2}-H)}{\Gamma(H-\frac{1}{2})\Gamma(2-2H)}}, \tag{55}$$

(here Γ denotes the Gamma function). This map preserves the inner product, and the Hermite functions are in the range of this map. Let I_ϕ^{-1} denote the inverse map of I_ϕ. (For summable functions this inverse exists and is proportional to the Liouville differential of order $H - \frac{1}{2}$ [38], since $I_\phi(f)$ is proportional to the fractional integral of f of order $H - \frac{1}{2}$.)

The set $(e_n = I_\phi^{-1}(\tilde{h}_n))_{n=1,2,...}$ constitutes an orthonormal basis for $L^2_\phi(\mathbb{R})$. Let $\mathcal{J}$ denote the set of all (finite) multi-indices of non-negative integers. Then for

$$\alpha = (\alpha_1, \alpha_2, ..., \alpha_m) \in \mathcal{J}, \tag{56}$$

define

$$H_\alpha(w) := h_{\alpha_1}(< w, e_1 >)...h_{\alpha_m}(< w, e_m >). \tag{57}$$

Note that

$$\begin{aligned} E_\mu\{H_\alpha H_\beta\} &= 0 \quad \text{if } \alpha \neq \beta, \\ E_\mu\{H_\alpha^2\} &= \alpha!. \end{aligned} \tag{58}$$

Now we are ready to state the fractional Wiener-chaos expansion theorem.

Theorem 3.3 *([35]) Every $F \in L^2(\mu)$ has a unique representation*

$$F(w) = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha(w), \tag{59}$$

where $c_\alpha \in \mathbb{R}$ for all $\alpha \in \mathcal{J}$. The norm of F is given by

$$|F|^2_{L^2(\mu)} = \sum_{\alpha \in \mathcal{J}} \alpha! c_\alpha^2, \tag{60}$$

where $\alpha! = \alpha_1!...\alpha_m!$ for $\alpha = (\alpha_1, ..., \alpha_m)$.

The more familiar Wiener-chaos expansion theorem follows from Thm. 3.3 by using the following identity due to [9]

$$H_\alpha(w) = \int_{\mathbb{R}^n} e_1^{\otimes \alpha_1} \hat{\otimes} ... \hat{\otimes} e_m^{\otimes \alpha_m} (dB^H)^{\otimes n}, \tag{61}$$

where $n = \alpha_1 + ... + \alpha_m$, and $\hat{\otimes}$ denotes the symmetrized tensor product.

Define $\hat{L}^2_\phi(\mathbb{R}^n)$ as the space of functions $f : \mathbb{R}^n \to \mathbb{R}$ such that

$$f(x_{\sigma_1}, ..., x_{\sigma_n}) = f(x_1, ..., x_n) \tag{62}$$

for all permutations σ of $(1, 2, ..., n)$, and that

$$|f|_{L^2_\phi(\mathbb{R}^n)} := \int_{\mathbb{R}^{2n}} \phi(u_1, v_1)...\phi(u_n, v_n) f(u_1, ..., u_n) f(v_1, ..., v_n) du_1...du_n dv_1...dv_n < \infty. \tag{63}$$

Then we have the following form of chaos expansion:

Theorem 3.4 *[9] For every $F \in L^2(\mu)$ there exists $f_n \in \hat{L}^2_\phi(\mathrm{I\!R^n})$ for $n = 0, 1, ...$ such that*

$$
\begin{aligned}
F(w) &= \sum_{n=0}^{\infty} I_n(f_n), \\
|F|^2_{L^2_\mu} &= \sum_{n=0}^{\infty} n! |f_n|^2_{L^2_\phi(\mathbf{R}^n)},
\end{aligned}
\tag{64}
$$

where

$$
I_n(f) := \int_{\mathbf{R}^n} f(dB^H)^{\otimes n} := n! \int_{s_1 < ... < s_n} f(s_1, ..., s_n) dB^H_{s_1} ... dB^H_{s_n}. \tag{65}
$$

For defining the integration w.r.t fBm of random functions we will make use of the following spaces of measurable functions:

Definition 3.1 ***a)*** *The Hida test function space $(\mathcal{S})_H$ is defined to be the set of all*

$$
\begin{aligned}
\psi(w) &= \sum_{\alpha \in \mathcal{J}} a_\alpha H_\alpha(w) \in L^2(\mu) \quad \text{such that} \\
|\psi|^2_{(\mathcal{S})_{H,k}} &:= \sum_{\alpha \in \mathcal{J}} \alpha! a^2_\alpha (2\mathbb{N})^{k\alpha} < \infty \quad \text{for all } k \in \mathbb{N},
\end{aligned}
\tag{66}
$$

where

$$
(2\mathbb{N})^\beta = \prod_j (2j)^{\beta_j} \quad \text{if } \beta = (\beta_1, ..., \beta_m) \in \mathcal{J}. \tag{67}
$$

b) *The Hida distribution space (the dual of $(\mathcal{S})_H$, see [41]) $(\mathcal{S})^*_H$ is defined to be the set of all random variables*

$$
\begin{aligned}
G(w) &= \sum_{\beta \in \mathcal{J}} b_\beta H_\beta(w) \quad \text{such that} \\
|G|_{(\mathcal{S})^*_{H,q}} &:= \sum_{\beta \in \mathcal{J}} \beta! b^2_\beta (2\mathbb{N})^{-q\beta} < \infty \quad \text{for some } q \in \mathbb{N}.
\end{aligned}
\tag{68}
$$

The action of $G \in (\mathcal{S})^*_H$ on $\psi \in (\mathcal{S})_H$ is given by

$$
<< G, \psi >> := \sum_{\alpha \in \mathcal{J}} \alpha! a_\alpha b_\alpha. \tag{69}
$$

For defining the integral w.r.t fBm it is necessary to define $(\mathcal{S})^*_H$-valued Pettis integrals as follows.

Definition 3.2 *A function $Z : \mathrm{I\!R} \to (\mathcal{S})^*_{\mathrm{H}}$ is $(\mathcal{S})^*_H$ integrable if*

$$
<< Z(t), \psi >> \in L^1(\mathrm{I\!R}) \quad \text{for all } \psi \in (\mathcal{S})_H. \tag{70}
$$

Then the $(\mathcal{S})^*_H$ integral of Z denoted by $\int_{\mathbf{R}} Z(t)dt$, is the unique element in $(\mathcal{S})^*_H$ such that

$$
<< \int_{\mathbf{R}} Z(t)dt, \psi >> = \int_{\mathbf{R}} << Z(t), \psi >> dt, \quad \text{for all } \psi \in (\mathcal{S})_H. \tag{71}
$$

Remark: $t \to B_t^H$ is differentiable in $(\mathcal{S})_H^*$, *i.e.* fractional noise is a well-defined object and we denote it by (W_t^H).

Below we describe the Wick product which is the last ingredient necessary for describing the integration w.r.t fBm.

Definition 3.3 *Suppose $F, G \in (\mathcal{S})_H^*$ are given by*

$$F(w) = \sum_{\alpha \in \mathcal{J}} a_\alpha H_\alpha(w) \quad \text{and} \quad G(w) = \sum_{\beta \in \mathcal{J}} b_\beta H_\beta(w). \tag{72}$$

Then the Wick product $F \diamond G$ of F and G is defined as

$$(F \diamond G)(w) = \sum_{\alpha, \beta \in \mathcal{J}} a_\alpha b_\beta H_{\alpha+\beta}(w). \tag{73}$$

Remark: $(\mathcal{S})_H$, and $(\mathcal{S})_H^*$ are closed under Wick product.

The Wick exponential $\exp^\diamond$ is defined as

$$\exp^\diamond(X) = \sum_{n=0}^{\infty} \frac{X^{\diamond n}}{n!}, \tag{74}$$

provided the series converges in $(\mathcal{S})_H^*$, where $X^{\diamond n} = X \diamond ... \diamond X$ (n factors). Using the identity

$$\exp\left(tx - \frac{1}{2}t^2\right) = \sum_{n=0}^{\infty} \frac{t^n}{n!} h_n(x), \tag{75}$$

one can show that

$$\exp^\diamond(< w, f >) = \exp\left(< w, f > - \frac{1}{2}|f|_\phi^2\right), \quad f \in L_\phi^2(\mathbb{R}). \tag{76}$$

Definition 3.4 *Suppose $Y : \mathbb{R} \to (\mathcal{S})_{\mathrm{H}}^*$ is such that $Y(t) \diamond W_t^H$ is integrable in $(\mathcal{S})_H^*$. Then $\int_{\mathbf{R}} Y(t) dB_t^H$ is defined by*

$$\int_{\mathbf{R}} Y(t) dB_t^H := \int_{\mathbf{R}} Y(t) \diamond W_t^H dt. \tag{77}$$

Lemma 3.1 *Let $\mathcal{L}_\phi^{1,2}(\mu)$ denote the completion of the set of all $\mathcal{F}_t$ adapted processes f such that*

$$\| f \|_{\mathcal{L}_\phi^{1,2}(\mu)} := E_\mu \left\{ \int_{\mathbf{R}} \int_{\mathbf{R}} f(s) f(t) \phi(s,t) ds dt \right\} + E_\mu \left\{ \left(\int_{\mathbf{R}} D_s^\phi f(s) ds \right)^2 \right\} < \infty. \tag{78}$$

If $Y \in \mathcal{L}_\phi^{1,2}(\mu)$ then $\int_{\mathbf{R}} Y_t dB_t^H$ exists as an element of $L^2(\mu)$ and its norm is given by $\| Y \|_{\mathcal{L}_\phi^{1,2}(\mu)}$.

For finding the equilibrium strategies we also make use of the Hida derivative (which is called the Malliavin derivative in the context of Wiener space) which we will define below. We first define the directional derivative:

Definition 3.5 *Suppose that* $F : \mathcal{S}' \to \mathbb{R}$ *and* $\gamma \in \mathcal{S}'$*. Then the directional (Gateaux) derivative of* F *in the direction of* γ *is given by*

$$D_\gamma F(w) := \lim_{\epsilon \to 0} \frac{F(w + \epsilon\gamma) - F(w)}{\epsilon}, \tag{79}$$

if it exists in $(\mathcal{S})_H^*$.

Definition 3.6 $F : \mathcal{S}' \to \mathbb{R}$ *is said to be differentiable if there is a map* $K : \mathbb{R} \to (\mathcal{S})_H^*$ *such that*

$$\begin{aligned} &K(t,w)\gamma(t) \textit{ is } (\mathcal{S})_H^* \textit{ integrable} \\ &\textit{and} \quad D_\gamma F(w) = \int_{\mathbb{R}} K(t,w)\gamma(t)dt \textit{ for all } \gamma \in L^2(\mathbb{R}). \end{aligned} \tag{80}$$

Then $D_t F(w) := K(t,w)$ *is said to be the Hida derivative of* F.

Below we introduce the Pothoff-Timpel test functions and distributions [37], since we will make use of them to define quasi-conditional expectation in the following sections. Also we will see that the Hida derivative of the random variables in these spaces exist.

Definition 3.7 ***a)*** $f(w) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} f_n (dB^H)^{\otimes n}$ *belongs to the space* $\mathcal{G}$ *(test space) if*

$$|f|^2_{\mathcal{G},k} := \sum_{n=0}^{\infty} n! |f_n|^2_{L^2(\mathbb{R}^n)} e^{2kn} < \infty \quad \textit{for all } k \in \mathbb{N}. \tag{81}$$

b) $G = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} g_n (dB^H)^{\otimes n}$ *belongs to the distribution space* $\mathcal{G}^*$ *if*

$$|g|^2_{\mathcal{G}^*,-q} := \sum_{n=0}^{\infty} n! |g_n|^2_{L^2(\mathbb{R}^n)} e^{-2qn} < \infty \quad \textit{for some } q \in \mathbb{N}. \tag{82}$$

Remark: We have the embeddings:

$$(\mathcal{S})^* \supset \mathcal{G}^* \supset (L^2)^* = L^2 \supset \mathcal{G} \supset \mathcal{S}. \tag{83}$$

Remark: Let $F = \sum_\alpha c_\alpha H_\alpha(w) \in \mathcal{G}^*$. Then the Hida derivative exists and is given by

$$D_t F(w) = \sum_\alpha c_\alpha \sum_i \alpha_i H_{\alpha - \varepsilon^i}(w) e_i(t), \tag{84}$$

where $\varepsilon^i = (0, ..., 0, 1, 0, ..., 0)$ with the 1 in the ith component.

We proceed by defining the quasi-conditional expectation and then introducing the fractional Clark-Ocone theorem which will be crucial in reducing the dynamic optimization problems of the next section into static optimization problems.

Definition 3.8 *[35] If $F \in \mathcal{G}^*(\mu)$ has the following expansion*

$$F(w) = \sum_{n=0}^{\infty} \int_{[0,T]^n} f_n (dB^H)^{\otimes n}, \tag{85}$$

then its quasi-conditional expectation is given by

$$\tilde{E}_\mu \{F \big| \mathcal{F}_t\} = \sum_{n=0}^{\infty} \int_{[0,t]^n} f_n (dB^H)^{\otimes n}. \tag{86}$$

Note that $\tilde{E}_\mu \{F \big| \mathcal{F}_t\} \neq E_\mu \{F \big| \mathcal{F}_t\}$ in general. (Only for $H = \frac{1}{2}$ is the quasi conditional expectation operator the same as the conditional expectation operator on $L^2(\mu)$.) However the following property holds,

$$\tilde{E}\{F \big| \mathcal{F}_t\} = F \quad \text{a.s.} \Leftrightarrow F \quad \text{is} \quad \mathcal{F}_t \quad \text{measurable}. \tag{87}$$

The following property of the quasi conditional expectation will be helpful in the computations:

$$\tilde{E}\{F \diamond G | \mathcal{F}_t\} = \tilde{E}\{F | \mathcal{F}_t\} \diamond \tilde{E}\{G | \mathcal{F}_t\} \quad \text{for } F, G \in \mathcal{G}^*, \tag{88}$$

We will also need the notion of a quasi martingale which is defined as follows:

Definition 3.9 *Suppose M_t is an $(\mathcal{F}_t)$ adapted process in $\mathcal{G}^*$. It is called a quasi-martingale if*

$$\tilde{E}\{M_t \big| \mathcal{F}_s\} = M_s, \quad \textit{for all } t \geq s. \tag{89}$$

Lemma 3.2 *[19] Let $F \in \mathcal{L}_\phi^{1,2}(\mu)$. Then*

$$M_t = \int_0^t F_s dB_s^H, \tag{90}$$

is a quasi-martingale.

Now we will state the fractional Clark-Ocone theorem.

Theorem 3.5 *[35] Suppose $G(w) \in L^2(\mu)$ is $\mathcal{F}_T$ measurable. Define*

$$\psi(t, w) = \tilde{E}_\mu \{D_t G \big| \mathcal{F}_t\}, \tag{91}$$

where $D_t G$ is the Hida derivative of G at t, which exists as an element of $\mathcal{G}^*(\mu)$. Then $\psi \in \mathcal{L}_\phi^{1,2}(\mu)$ and

$$G(w) = E_\mu\{G\} + \int_0^T \psi(t, w) dB_t^H. \tag{92}$$

For multi-dimensional games where there are multiple sources of randomness we will need multi-dimensional fractional white noise theory, a treatment of which is given in [7]. Here we will give a short introduction.

Let μ be the probability measure on $\mathcal{S}'$ as defined before. For a fixed $m \in \mathbb{N}$ define a new probability space by

$$\begin{aligned} &\Omega : = \mathcal{S}' \times .... \times \mathcal{S}' \ (m \text{ factors}), \\ &\mu_m = \mu \times ... \times \mu \ (m \text{ factors}). \end{aligned} \tag{93}$$

The action of $w = (w_1, ..., w_m) \in \Omega$ on $f = (f_1, ..., f_m) \in \mathcal{S} := \mathcal{S}(\mathbb{R}) \times ... \times \mathcal{S}(\mathbb{R})$ is defined to be $< w, f >= \sum_{i=1}^{m} < w_i, f_i >$. Then we have

$$\int_{\Omega} e^{i<w,f>} d\mu_m = e^{-\frac{1}{2}|f|^2}, \tag{94}$$

where $|f|^2 = \sum_{i=1}^{m} |f_i|^2_{L^2(\mathbb{R})}$. m-dimensional fBm is given by the continuous modification of

$$\tilde{B}_t^H = (< w_1, 1_{[0,t]} >, ..., < w_m, 1_{[0,t]} >). \tag{95}$$

The multi-dimensional fractional noise calculus goes through, and the main ingredient of the generalization from the one dimensional case are the following random variables which are the multi-dimensional versions of (57). Let $\mathcal{J}^m = \mathcal{J} \times ... \times \mathcal{J}$ (m factors) be the set of m-tuples $\Gamma = (\gamma^1, ..., \gamma^m)$ with $\gamma^i \in \mathcal{J}$. Then we define

$$H_\Gamma = \prod_{i=1}^{m} H_{\gamma^i}, \tag{96}$$

and using it instead of (57) the multi-dimensional case goes through with only slight modifications.

4 Proof of Theorem 2.1

Recall that in Theorem 2.1, we consider the one dimensional state equation (10). where the pay-off function of player i is of the form (11).

As noted previously, we will employ the fractional Clark-Ocone formula and the Wick calculus introduced in Section 3 to find Nash equilibria for this type of game. We begin this development by stating a fractional version of Girsanov's theorem, which is given by [35].

Theorem 4.1 *([35]) Suppose $T > 0$ and $u : [0, T] \to \mathbb{R}$ is continuous. Suppose further that $K : [0, T] \to \mathbb{R}$ satisfies the equation*

$$\int_0^T K(s)\phi(s, t)ds = u(t), \quad 0 \le t \le T, \tag{97}$$

where ϕ is given by (47). Extend K to $\mathbb{R}$ by putting $K(s) = 0$ outside $[0,T]$. Define the probability measure $\hat{\mu}$ on F_T by

$$\frac{d\hat{\mu}(w)}{d\mu(w)} = \exp\Big(-\int_0^T K(s)dB_s^H - \frac{1}{2}|K|_\phi^2\Big). \tag{98}$$

Then

$$\hat{B}_t^H = \int_0^t u(s)ds + B_t^H, \tag{99}$$

is an fBm with respect to $\hat{\mu}$.

The dynamics of the state (10) can be written as

$$d(e^{-rt}X_t) - e^{-rt}\sum_{i=1}^N \alpha_i(t)u_i(t)dt = e^{-rt}\sum_{i=1}^N \beta_i(t)v_i(t)(Cdt + dB_t^H). \tag{100}$$

Let η and $\hat{\mu}$ be defined as in (16); *i.e.*,

$$\begin{aligned}\eta(T) = \frac{d\hat{\mu}}{d\mu} &= \exp\left(-\int_0^T K(s)dB_s^H - \frac{1}{2}|K|_\phi^2\right) \\ &= \exp^\diamond\left(-\int_0^T K(s)dB_s^H\right),\end{aligned} \tag{101}$$

where K is from (12). Then since K solves (97) for $u(t) = C$ (see Appendix Lemma 8.1) and by the fractional Girsanov formula, the process

$$\hat{B}_t^H = Ct + B_t^H, \tag{102}$$

is an fBm with respect to $\hat{\mu}$ having the same Hurst parameter as the modulating process in (10). Thus, the differential equation describing the flow of the state is given in terms of $\hat{B}^H$ as,

$$e^{-rt}X_t - \int_0^t e^{-rs}\sum_{i=1}^N \alpha_i(s)u_i(s)dt = x + \int_0^t e^{-rs}\sum_{i=1}^N \beta_i(s)v_i(s)d\hat{B}_s^H. \tag{103}$$

To be able to find a Nash equilibrium, we will use the quasi- martingale approach to stochastic control in the proof. (For another application of this approach see [19].) We first find the best response of a player to the given strategies of other players and for that we will use the fractional Clark-Ocone Theorem (Thm 3.5).

First note that by Lemma 3.2 and (20),

$$\int_0^t e^{-rs}\sum_{i=1}^N \beta_i(s)v_i(s)d\hat{B}_s^H, \tag{104}$$

is a quasi-martingale. Therefore we have

$$E_{\hat{\mu}}\left\{e^{-rt}X_t - \int_0^t e^{-rs}\sum_{i=1}^{N}\alpha_i(s)u_i(s)dt\right\} = x. \tag{105}$$

Now let G be given by

$$G = e^{-rT}F_i - \int_0^T e^{-rs}\sum_{j=1}^{N}\alpha_j(s)u_j(s)ds. \tag{106}$$

Assume $G \in L^2(\hat{\mu})$. Then if

$$E_{\hat{\mu}}\{G\} = x, \tag{107}$$

then by the fractional Clark-Ocone formula (92) we have

$$G = x + \int_0^T \tilde{E}_{\hat{\mu}}\left\{D_sG\middle|\mathcal{F}_s\right\}d\hat{B}_s^H. \tag{108}$$

If we choose v_i such that

$$v_i(s) = \frac{-\sum_{j\neq i}\beta_j(s)v_j(s) + e^{rs}\tilde{E}_{\hat{\mu}}\left\{D_sG\middle|\mathcal{F}_s\right\}}{\beta_i(s)}, \tag{109}$$

in (103), from (108) we see that $X_T = F_i$.

By the above argument we can change the dynamic optimization problem of maximizing (11) under the dynamics (10) into a static optimization problem. In particular, given the other players' strategies, player i wishes to solve the following maximization problem:

$$\begin{aligned} K_i(x) = \sup_{u_i,F_i}\Bigg\{ & E_\mu\left\{\int_0^T \frac{c_iu_i(t)^{\gamma_i}}{\gamma_i}dt + \frac{b_iF_i^{\gamma_i'}}{\gamma_i'}\right\};\ \text{given that} \\ & E_{\hat{\mu}}\left\{-\int_0^T e^{-rs}\sum_{j=1}^{N}\alpha_j(s)u_j(s)ds + e^{-rT}F_i\right\} = x\Bigg\}, \end{aligned} \tag{110}$$

where the supremum is taken over F_i and (u_i) such that

$$e^{-rT}F_i - \int_0^T e^{-rs}\sum_{j=1}^{N}\alpha_j(s)u_j(s)ds \in L^2(\hat{\mu}). \tag{111}$$

This optimization problem can be solved by first considering for each $\lambda_i > 0$ the following unconstrained problem,

$$C_i(x,\lambda) = \sup_{u_i,F_i}\left\{E_\mu\left\{\int_0^T \frac{c_iu_i(t)^{\gamma_i}}{\gamma_i}dt + \frac{b_iF_i^{\gamma_i'}}{\gamma_i'}\right\} + \lambda_iE_{\hat{\mu}}\left\{-\int_0^T e^{-rs}\sum_{j=1}^{N}\alpha_j(s)u_j(s)ds + e^{-rT}F_i\right\}\right\}, \tag{112}$$

and then solving for λ_i from the slackness condition:

$$E_{\hat{\mu}}\left\{-\int_0^T e^{-rs}\sum_{j=1}^{N}\alpha_j(s)u_j(s)ds + e^{-rT}F_i\right\} = x. \tag{113}$$

Let us define, as before, the following random variable

$$\rho(t,w) = E_\mu\{\eta(T)\big|\mathcal{F}_t\}, \tag{114}$$

where η is from (101). Using the fact that

$$E_\mu\{\eta(T)u_i(t)\} = E_\mu\{\rho(t)u_i(t)\}, \tag{115}$$

we can solve (112) by maximizing pointwise, *i.e.* for each t and w, the functions,

$$g_i(u_i) = \frac{c_i u_i^{\gamma_i}}{\gamma_i} - \lambda_i\rho(t,w)e^{-rt}\sum_{j=1}^{N}\alpha_j(t)u_j, \tag{116}$$

and

$$h_i(F_i) = \frac{b_i F_i^{\gamma_i'}}{\gamma_i'} - \lambda_i\eta(T,w)e^{-rT}F_i. \tag{117}$$

Since $0 < \gamma_i < 1$, these functions are concave, and therefore we can solve $g_i'(u_i) = 0$ and $h_i'(F_i) = 0$ to find the maximum points, which are given by,

$$u_i(t) = \left(\frac{\lambda_i\rho(t,w)e^{-rt}\alpha_i(t)}{c_i}\right)^{\frac{1}{\gamma_i-1}}, \tag{118}$$

and

$$F_i = \left(\frac{\lambda_i\eta(T,w)e^{-rT}}{b_i}\right)^{\frac{1}{\gamma_i'-1}}. \tag{119}$$

Since $\alpha_i(t)$ is bounded by assumption, (111) is satisfied. Note that at the Nash equilibrium F_i is independent from the player index i, *i.e.* $F_i = F^e$ for all i. We will use this condition to show that the Lagrange multipliers at the equilibrium are necessarily linear in b_i and then use the slackness condition to actually find their value. First we will find $E_\mu\left\{\eta(T)^{\frac{1}{\gamma_i'-1}}\right\}$. Note that

$$\begin{aligned}\eta(T)^{\frac{1}{\gamma_i'-1}} &= \exp\left(\frac{1}{1-\gamma_i'}\int_0^T K(s)dB_s^H + \frac{1}{2(1-\gamma_i')}|K|_\phi^2\right)\\ &= \exp\left(\frac{1}{1-\gamma_i'}\int_0^T K(s)dB_s^H - \frac{1}{2(1-\gamma_i')^2}|K|_\phi^2\right)\exp\left(\frac{2-\gamma_i'}{2(1-\gamma_i')^2}|K|_\phi^2\right).\end{aligned} \tag{120}$$

Since

$$E\left\{\exp^\diamond\left(\int_0^T f(s)dB_s^H\right)\right\} = 1, \tag{121}$$

for all $f \in L^2(\mu)$ we thus have

$$E\left\{\eta(T)^{\frac{1}{\gamma_i'-1}}\right\} = \exp\left(\frac{2-\gamma_i'}{2(1-\gamma_i')^2}|K|_\phi^2\right). \tag{122}$$

Hence

$$EF_i = \left(\frac{\lambda_i}{b_i}\right)^{\frac{1}{\gamma_i'-1}} \exp\left(\frac{2-\gamma_i'}{2(1-\gamma_i')^2}|K|_\phi^2 - \frac{rT}{\gamma_i'-1}\right). \tag{123}$$

Then

$$F_i = E\{F_i\} \exp^\diamond\left(\frac{1}{1-\gamma_i'}\int_0^T K(s)dB_s^H\right). \tag{124}$$

From (124) we see that for a Nash-equilibrium to exist necessarily we have $\gamma_i' = \gamma'$, and $\lambda_i^e = mb_i$. From (4) we see that m is to be found from the slackness condition:

$$E_\mu\Big\{\int_0^T e^{-rt}\rho(t)\Big(\sum_{i=1}^N \alpha_i(t)\left(\frac{\lambda_i^e\rho(t)e^{-rt}\alpha_i(t)}{c_i}\right)^{\frac{1}{\gamma_i-1}}\Big)dt + e^{-rT}\eta(T)\left(\frac{\lambda_i^e\eta(T)e^{-rT}}{b_i}\right)^{\frac{1}{\gamma_i'-1}}\Big\} = x. \tag{125}$$

Note that by (101) we have the following

$$\begin{aligned}\eta(T)^{\frac{\gamma'}{\gamma'-1}} &= \exp\left(\frac{\gamma'}{1-\gamma'}\int_0^T K(s)dB_s^H + \frac{\gamma'}{2(1-\gamma')}|K|_\phi^2\right)\\ &= \exp^\diamond\left(\frac{\gamma'}{1-\gamma'}\int_0^T K(s)dB_s^H\right)\exp\left(\frac{\gamma'}{2(1-\gamma')^2}|K|_\phi^2\right).\end{aligned} \tag{126}$$

$$E\left\{\eta(T)^{\frac{\gamma'}{\gamma'-1}}\right\} = \exp\left(\frac{\gamma'}{2(\gamma'-1)^2}|K|_\phi^2\right). \tag{127}$$

Using Thm. 3.2 of [18] $\rho(t,w)$ can be written as

$$\rho(t,w) = \exp\Big(-\int_0^t \zeta_t(s)dB_s^H - \frac{1}{2}|\zeta_t|_\phi^2\Big), \tag{128}$$

where ζ_t is given by the following:

$$\begin{aligned}\big((-\Delta)^{-(H-1/2)}\zeta_t\big)(s) &= \big((-\Delta)^{-(H-1/2)}K\big)(s), \quad 0\le s\le t,\\ \zeta_t(s) &= 0 \quad s<0 \quad \text{or} \quad s>t,\end{aligned} \tag{129}$$

with the operator $(-\Delta)^{-(H-1/2)}$ on $L^2(\mu)$ defined by (14).

Thus

$$E\left\{\rho_t^{\frac{\gamma_i}{\gamma_i-1}}\right\} = E\Big\{\exp\Big(\frac{\gamma_i}{1-\gamma_i}\int_0^T \zeta_t(s)dB_s^H - \frac{\gamma_i^2}{2(1-\gamma_i)^2}|\zeta_t|_\phi^2 + \frac{\gamma_i}{2(1-\gamma_i)}|\zeta_t|_\phi^2 + \frac{\gamma_i^2}{2(1-\gamma_i)^2}|\zeta_t|_\phi^2\Big)\Big\}, \tag{130}$$

from which we conclude that

$$E\left\{\rho(t)^{\frac{\gamma_i}{\gamma_i-1}}\right\} = \exp\Big(\frac{\gamma_i}{2(1-\gamma_i)^2}|\zeta_t|_\phi\Big), \tag{131}$$

so that m can be solved from (125), which leads to the following equation:

$$\sum_{i=1}^{N}\int_0^T m^{\frac{1}{\gamma_i-1}} b_i^{\frac{1}{\gamma_i-1}} \alpha_i(t)^{\frac{\gamma_i}{\gamma_i-1}} e^{-rt\frac{\gamma_i}{\gamma_i-1}} \exp\big(\frac{\gamma_i}{2(1-\gamma_i)^2}|\zeta_t|_\phi\big)dt \\ +m^{\frac{1}{\gamma'-1}} e^{-rT\frac{\gamma'}{\gamma'-1}} \exp\left(\frac{2\gamma'|K|_\phi}{2(1-\gamma')^2}\right) = x. \tag{132}$$

After solving for m using (132), then by (118) and (119) we have the final state at the equilibrium and strategy u_i for player i leading to that state, given respectively by,

$$F^e = m^{\frac{1}{\gamma'-1}} \eta(T)^{\frac{1}{\gamma'-1}} e^{\frac{-rT}{\gamma'-1}}, \tag{133}$$

and

$$u_i^e(t) = \left(\frac{mb_i\alpha_i(t)}{c_i} e^{-rt}\rho(t,w)\right)^{\frac{1}{\gamma_i-1}}. \tag{134}$$

Observe that these controls are not Markovian. (In a Brownian motion setting the controls were assumed to be Markovian at the outset so that the HJB equations for an equilibrium solution can be developed [6], [12] and [34].) Note that

Now we will proceed to find (v_i) at the equilibrium, which is the second component of the players' strategies. For this we will again make use of the fractional Clark-Ocone formula.

Suppose G^e is given by

$$G^e = e^{-rT}F^e - \int_0^T e^{-rs}\sum_{i=1}^{N}\alpha_i u_i^e(s)ds. \tag{135}$$

Since there is a unique adapted process $\psi(t,w)$ such that

$$G^e = E_\mu\{G^e\} + \int_0^T \psi(t,w)d\hat{B}_t^H, \tag{136}$$

which, from the Clark-Ocone formula, is given by

$$\psi(t,w) = \tilde{E}_{\hat{\mu}}\big\{D_tG^e\big|\mathcal{F}_t\big\}, \tag{137}$$

it can now be seen immediately that any adapted (v_i^e) that satisfies

$$\tilde{E}_\mu\big\{D_tG^e\big|\mathcal{F}_t\big\} = e^{-rt}\sum_{i=1}^{N}\beta_i v_i^e(t), \tag{138}$$

is an equilibrium strategy.

To obtain a more explicit expression, we will compute $\tilde{E}_{\hat{\mu}}\{D_tG^e|\mathcal{F}_t\}$. Using (133) and (134), G^e is given by

$$G^e(T,w) = m^{\frac{1}{\gamma'-1}} e^{-rT\frac{\gamma'}{\gamma'-1}} \eta(T,w)^{\frac{1}{\gamma'-1}} - \int_0^T\sum_{i=1}^{N}\alpha_i(t)^{\frac{\gamma_i}{\gamma_i-1}}\left(\frac{mb_i}{c_i}\right)^{\frac{1}{\gamma_i-1}} e^{-rt\frac{\gamma_i}{\gamma_i-1}}\rho(t,w)^{\frac{1}{\gamma_i-1}}dt. \tag{139}$$

To calculate the quasi conditional expectation of the Hida derivative of G^e we will first find it for the stochastic part of the first term on the right-hand side of (139). Define R as

$$R = \exp\left(\frac{2-\gamma'}{(1-\gamma')^2}|K|_\phi^2 - \frac{C}{1-\gamma'}\int_0^T K(s)ds\right). \tag{140}$$

Using the chain rule, (102) and (86), we have

$$\begin{aligned}
\tilde{E}_{\hat{\mu}}\left\{D_t\eta(T)^{\frac{1}{\gamma'-1}}\big|\mathcal{F}_t\right\} &= \tilde{E}_{\hat{\mu}}\left\{\frac{K(t)}{1-\gamma'}\eta(T)^{\frac{1}{\gamma'-1}}\big|\mathcal{F}_t\right\} \\
&= \frac{K(t)}{1-\gamma'}R\tilde{E}_{\hat{\mu}}\left\{\exp^\diamond\left(\frac{1}{1-\gamma'}\int_0^T K(s)d\hat{B}_s^H\right)|\mathcal{F}_t\right\} \\
&= \frac{K(t)}{1-\gamma'}R\exp^\diamond\left(\frac{1}{1-\gamma'}\int_0^t K(s)d\hat{B}_s^H\right) \\
&= \frac{K(t)}{1-\gamma'}\exp\left(\frac{1}{1-\gamma'}\int_0^T K(s)dB_s^H - \frac{C}{1-\gamma'}\int_t^T K(s)ds\right. \\
&+ \left.\frac{2-\gamma}{2(1-\gamma^2)}|K|_\phi^2 - \frac{1}{1-\gamma}|K1_{[0,t]}|_\phi^2\right).
\end{aligned} \tag{141}$$

Now we will find the quasi conditional expectation of the Hida derivative of the stochastic part of second term on the right-hand side of (139) using (128). *I.e.*,

$$\begin{aligned}
&\tilde{E}_{\hat{\mu}}\Big\{D_t\big(e^{-ru\frac{\gamma_i}{\gamma_i-1}}\rho(u)^{\frac{1}{\gamma_i-1}}\big)\big|\mathcal{F}_t\Big\} = \\
&\frac{\zeta_u(t)}{1-\gamma_i}e^{-ru\frac{\gamma_i}{\gamma_i-1}}\exp\left(\frac{1}{1-\gamma_i}\int_0^t \zeta_u(s)dB_s^H - \frac{C}{1-\gamma_i}\int_t^u \zeta_u(s)ds\right. \\
&+ \left.\frac{2-\gamma_i}{2(1-\gamma_i)^2}|\zeta_u|_\phi^2 - \frac{1}{1-\gamma_i}|\zeta_u 1_{[0,t]}|_\phi^2\right).
\end{aligned} \tag{142}$$

Using (138) and (139) we have the result for the second component for the players' equilibrium strategies, and this concludes the proof of Theorem 2.1. □

5 Proof of Theorem 2.2

We now turn to the multi-dimensional model (28)-(33), and prove Theorem 2.2. The key tools in this pursuit are the multidimensional fractional Girsanov formula, and the multidimensional fractional Clark-Ocone formula, both of which are given by Biagini and Øksendal [7].

First we give the multidimensional fractional Girsanov formula.

Theorem 5.1 *([7]) Suppose* $\gamma = (\gamma_1, ..., \gamma_m) \in (L^2(\mathbb{R}))^{\mathrm{m}}$ *and* $\hat{\gamma} = (\hat{\gamma}_1, ..., \hat{\gamma}_m) \in L_\phi^2$ *are related by*

$$\gamma_k(t) = \int_{\mathbb{R}} \hat{\gamma}_k(s)\phi_k(s,t)ds; \quad t \in \mathbb{R}, \quad \mathrm{k} = 1, ..., \mathrm{m}, \tag{143}$$

where ϕ_k is given by

$$\phi_k(s,t) = H_k(2H_k - 1)|s-t|^{2H_k-2}, \quad k = 1,2,...,m. \tag{144}$$

Suppose further that $G \in L^2(\mu)$. Then

$$E\{G(\omega+\gamma)\} = E\left\{G(\omega)\exp^{\diamond}(< w, \hat{\gamma} >)\right\} := E\Big\{G(w)\varepsilon\left(\int_{\mathbf{R}} \hat{\gamma}(t)dB_t^{(H)}\right)\Big\}. \tag{145}$$

Corollary 5.1 *Suppose $g : \mathbb{R}^{\mathrm{m}} \to \mathbb{R}$ is bounded and γ is given as in Thm. 5.1. Then with ε defined as in (145) we have*

$$E_\mu\left\{g\left(B^{(H)} + \int_0^t \gamma(s)ds\right)\right\} = E_\mu\left\{g(B^{(H)})\varepsilon\left(\int_{\mathbf{R}} \hat{\gamma}(t)dB_t^{(H)}\right)\right\}. \tag{146}$$

PROOF:
If

$$G(w) = g(< w_1, 1_{[0,t]} >, ..., < w_m, 1_{[0,t]} >) = g(B^{(H)}(t)), \tag{147}$$

then

$$\begin{aligned} G(w+\gamma) &= g(< w_1 + \gamma_1, 1_{[0,t]} >, ..., < w_m + \gamma_m, 1_{[0,t]} >) \\ &= g(< w_1, 1_{[0,t]} > + < \gamma_1, 1_{[0,t]} >, ..., < w_m, 1_{[0,t]} > + < \gamma_m, 1_{[0,t]} >). \end{aligned} \tag{148}$$

Then by (95) we have

$$G(w+\gamma) = g\left(B^{(H)} + \int_0^t \gamma(s)ds\right). \tag{149}$$

Using Thm. 5.1 the result follows. □

By Cor. 5.1 we conclude that

$$\hat{B}_t^{(H)} = \Big(B_t^{H_1} + \int_0^t f_1(s)ds, ..., B_t^{H_m} + \int_0^t f_m(s)ds\Big), \tag{150}$$

is an fBm under the measure $\hat{\mu}$, whose Radon-Nikodym derivative with respect to μ is given by

$$\eta(T,w) = \frac{d\hat{\mu}}{d\mu} = \varepsilon\Big(-\int_{\mathbf{R}} \hat{f}(t)dB_t^{(H)}\Big), \tag{151}$$

where $\hat{f} = (\hat{f}_1, ..., \hat{f}_m)$, and $\hat{f}_i$ for $i = 1, ..., m$ is a solution to the integral equation (34).

We can write

$$\begin{aligned} \eta(T,w) &= \exp^{\diamond}\left(-\int_{\mathbf{R}} \hat{f}(t)dB_t^{(H)}\right) \\ &= \exp^{\diamond}\left(-\sum_{i=1}^m \int_{\mathbf{R}} \hat{f}_i(t)dB_t^{H_i}\right) \\ &= \exp\left(-\sum_{i=1}^m \int_{\mathbf{R}} \hat{f}_i(t)dB_t^{H_i} - \frac{1}{2}|\hat{f}|_\phi^2\right), \end{aligned} \tag{152}$$

where

$$|\hat{f}|_{\phi}^{2} = \sum_{i=1}^{m} \int_{\mathbf{R}^2} \hat{f}_i(s)\hat{f}_i(t)\phi_i(s,t)dsdt = \sum_{i=1}^{m} |\hat{f}_i|_{\phi_i}^{2}. \tag{153}$$

The dynamics of the state equation in terms of $\hat{B}^{(H)}$ can be written as

$$dX_t^i = \big(r_i X_t^i + \sum_{j=1}^{N} \alpha_j^i u_j(t)\big)dt + \big(\sum_{j=1}^{N} v_{i1}^j(t)d\hat{B}_t^{H_1} + ... + v_{im}^j(t)d\hat{B}_t^{H_m}\big), \qquad \text{for} \quad i = 1, ..., n. \tag{154}$$

On writing $X_t = (X_t^1, ..., X_t^n)^T$, $r = diag(r_1, ..., r_n)$, $\alpha_i = (\alpha_i^1, ..., \alpha_i^n)^T$, and

$$v_i = \begin{pmatrix} v_{11}^i & ... & v_{1m}^i \\ ... & ... & ... \\ v_{1n}^i & ... & v_{nm}^i \end{pmatrix}, \tag{155}$$

the state equation can be written as

$$e^{-rt}X_t - \int_0^t e^{-rs} \sum_{i=1}^{N} \alpha_i(s)u_i(s)ds = x + \int_0^t e^{-rs} \sum_{i=1}^{N} v_i(s)d\hat{B}_s^{(H)} \tag{156}$$

where x is the state at time 0. This form looks exactly like the one-dimensional case, and to be able to solve the problem the same way we need a multi-dimensional version of the Clark-Ocone formula. The following is a trivial extension of the Clark-Ocone formula for multiple source of randomness given by [7] to incorporate the case of multiple states.

Theorem 5.2 *Suppose $G = (G_1, ..., G_n)$ with $G_j \in L^2(\mu)$ for $j = 1, ..., n$. We can write*

$$G(w) = E_\mu\{G\} + \sum_{i=1}^{m} \int_0^T \tilde{E}_\mu\{D_{i,t}G \big| \mathcal{F}_t\} dB_t^{H_i}, \tag{157}$$

where $D_{k,t}F = \frac{\partial F}{\partial w_k}(t, w)$ is the vector of Hida derivative of the components of F with respect to w_k at (t, w).

Suppose G^i is given by

$$G^i = e^{-rT}F^i - \int_0^T e^{-rs} \sum_{j=1}^{N} \alpha_j(s)u_j(s)ds, \quad i = 1, ..., N. \tag{158}$$

If $E_{\hat{\mu}}\{G\} = x$, then, given the other players' strategies, player i can choose v^i (by the Clark-Ocone formula) as

$$v_{j,k}^i = e^{r_j t}\tilde{E}_{\hat{\mu}}\{\hat{D}_{k,t}G_j^i | F_t\} - \sum_{l \neq i} v_{j,k}^l, \quad j = 1, ..., n, \quad k = 1, ..., m, \tag{159}$$

so that the final state F^i is attained. ($\hat{D}_{k,t}$ denotes the Hida derivative under the measure $\hat{\mu}$.) Here G_j^i denotes the jth component of the n dimensional vector G^i. Therefore, as in the one-dimensional case, instead of having a dynamic optimization problem of maximizing the pay-off of the form (29) we have a static optimization problem:

$$\begin{aligned}\kappa_i(x) &= \sup_{u_i, F^i} E_{\mu}\Big\{ \int_0^T g_i(u_i)dt + h(F^i)\Big\} \\ &+ \lambda_i^T E_{\hat{\mu}}\Big\{ -\int_0^T e^{-rs}\sum_{j=1}^N \alpha_j(s)u_j(s)ds + e^{-rT}F^i\Big\}, \quad i = 1,..,N.\end{aligned} \tag{160}$$

λ_i is an $n \times 1$ matrix which is to be found from the constraint condition

$$E_{\hat{\mu}}\Big\{ -\int_0^T e^{-rs}\sum_{j=1}^N \alpha_j(s)u_j(s)ds + e^{-rT}F^i\Big\} = x. \tag{161}$$

The optimization (160) can be carried out pointwise and can be solved by maximizing the functions:

$$\kappa(u_i) = g_i(u_i) - \lambda_i^T\rho(t,w)e^{-rs}\sum_{j=1}^N \alpha_j(t)u_j, \tag{162}$$

and

$$\beta(F^i) = h(F^i) - \lambda_i^T\eta(T)e^{-rT}F^i. \tag{163}$$

Note that g_i and h_i are given by (32) and (33), respectively. Solving for $\nabla\beta(F^i) = 0$ gives

$$F^i = \eta(T)e^{-rT}\|F^i\|^{2-\gamma'}\lambda_i. \tag{164}$$

At the Nash equilibrium the final state has to be independent of the player index i, *i.e.* $F^i = F^e$. This immediately implies that the Lagrange multiplier vector at the Nash-equilibrium has to be the same for each player which will be denoted by λ. Then

$$F^e = \lambda\eta(T)e^{-rT}\left(\|\lambda\|\eta(t,w)e^{-rT}\right)^{\frac{2-\gamma'}{\gamma'-1}}. \tag{165}$$

Using (162) we have

$$u_i(t) = \left(\lambda^T\alpha_i(t)e^{-rt}\rho(t,w)\right)^{\frac{1}{\gamma_i-1}}. \tag{166}$$

The slackness condition (161) at the equilibrium is then given by

$$-\int_0^T \sum_{i=1}^N e^{\frac{rt\gamma_i}{1-\gamma_i}}\alpha_i(t)\left(\lambda^T\alpha_i(t)\right)^{\frac{1}{\gamma_i-1}} E\left\{\rho(t)^{\frac{\gamma_i}{\gamma_i-1}}\right\}dt + e^{\frac{rT\gamma'}{1-\gamma'}}\lambda\|\lambda\|^{\frac{2-\gamma'}{\gamma'-1}}E\left\{\eta(T)^{\frac{\gamma'}{\gamma'-1}}\right\} = x. \tag{167}$$

We need to calculate $E\left\{\eta(T)^{\frac{\gamma'}{\gamma'-1}}\right\}$ and $E\left\{\rho(t)^{\frac{\gamma_i}{\gamma_i-1}}\right\}$, which is immediate from the following expressions,

$$\begin{aligned}\eta(T,w)^{\frac{\gamma'}{\gamma'-1}} &= \exp\left(\frac{\gamma'}{1-\gamma'}\int_0^T \hat{f}(s)dB_s^{(H)} + \frac{\gamma'}{2(1-\gamma')}|\hat{f}|_\phi^2\right) \\ &= \exp\left(\frac{\gamma'}{1-\gamma'}\int_0^T \hat{f}(s)dB_s^{(H)} - \frac{\gamma'^2}{2(1-\gamma')^2}|\hat{f}|_\phi^2\right)\exp\left(\frac{\gamma'}{2(1-\gamma')^2}|\hat{f}|_\phi^2\right),\end{aligned} \tag{168}$$

and

$$\rho(t,w)^{\frac{\gamma_i}{\gamma_i-1}} = \exp\left(- \frac{\gamma_i}{\gamma_i-1}\int_0^t \zeta_t(s)dB_s^{(H)} - \frac{1}{2}\frac{\gamma_i}{\gamma_i-1}|\zeta_t|_\phi^2 \right)$$
$$= \exp\left(- \frac{\gamma_i}{\gamma_i-1}\int_0^t \zeta_t(s)dB_s^{(H)} - \frac{1}{2}\frac{\gamma_i^2}{(\gamma_i-1)^2}|\zeta_t|_\phi^2 \right) \exp\left(\frac{\gamma_i}{2(\gamma_i-1)^2}|\zeta_t|_\phi^2 \right), \tag{169}$$

since the stochastic terms in (168) and (169) have unit mean. Therefore we conclude that λ has to satisfy (40).

Finding λ we can proceed by finding the second component of the players' strategies, *i.e.* $(v_{j,k}^i)$ by using (159). Using (158) we can see that at the Nash equilibrium G^i is independent of i. Denote it by G^e which is then given by

$$G^e = \lambda\|\lambda\|^{\frac{2-\gamma'}{\gamma'-1}} \exp\left(\frac{rT\gamma'}{1-\gamma'}\right)\eta(T,w)^{\frac{1}{\gamma'-1}} - \int_0^T \sum_{i=1}^N \alpha_i(s)e^{\frac{rs\gamma_i}{1-\gamma_i}}\left(\lambda^T\alpha_i(s)\right)^{\frac{1}{\gamma_i-1}}\rho(s,w)^{\frac{1}{\gamma_i-1}}ds \tag{170}$$

At the Nash equilibrium the second component of the players' strategies will be any adapted strategy satisfying the constraint:

$$\sum_{i=1}^N v_{j,k}^i(t) = e^{r_jt}\tilde{E}_{\hat{\mu}}\{\hat{D}_{k,t}G_j^e|F_t\}, \quad j=1,...,n, \quad k=1,...,m. \tag{171}$$

Let us first calculate the quasi-conditional expectation of the Hida derivative of the stochastic component of the first term of the equation (170). We first have to write the relevant variable in terms of $\hat{\mu}$-fBm.

$$\begin{aligned}\eta(T,w)^{\frac{1}{\gamma'-1}} &= \exp\left(\frac{1}{\gamma'-1}\int_0^T \hat{f}(s)dB_s^{(H)} + \frac{1}{2(1-\gamma')}|\hat{f}|_\phi^2\right)\\ &= \exp\left(\frac{1}{\gamma'-1}\int_0^T \hat{f}(s)d\hat{B}_s^{(H)} - \frac{1}{2(1-\gamma'))}|\hat{f}|_\phi^2\right),\end{aligned} \tag{172}$$

where we have used (150) for the second equality.

Taking the Hida derivative $\hat{D}_{k,t}$ of $\eta(T,w)^{\frac{1}{\gamma'-1}}$ is equivalent to multiplying it with $\frac{1}{1-\gamma'}\hat{f}(t)$ which is due to the chain rule, and

$$\hat{D}_{k,t}\left(\sum_{i=1}^m \int_0^T f_i(t)d\hat{B}_t^{H_i}\right) = f_k(t), \quad k=1,...,m, \text{ for } f\in L_\phi^2, \tag{173}$$

We proceed by taking the quasi conditional expectation. Using (86), (88) and the definition of the Wick exponential we get

$$\begin{aligned}\tilde{E}_{\hat{\mu}}\left\{\hat{D}_{k,t}\left(\eta(T)^{\frac{1}{\gamma'-1}}\right)|\mathcal{F}_t\right\} &= \frac{1}{1-\gamma'}\hat{f}_k(t)\exp\left(\frac{1}{1-\gamma'}\int_0^t \hat{f}(s)d\hat{B}_s^{(H)} - \frac{1}{2(1-\gamma')^2}|\hat{f}1_{[0,t]}|_\phi^2\right.\\ &\quad \left.+\frac{\gamma'}{2(1-\gamma')^2}|\hat{f}|_\phi^2\right)\\ &= \frac{1}{1-\gamma'}\hat{f}_k(t)\exp\left(\frac{1}{1-\gamma'}\int_0^t \hat{f}(s)dB_s^{(H)} + \int_0^t \hat{f}(s)f(s)ds\right.\\ &\quad \left.-\frac{1}{2(1-\gamma')^2}|\hat{f}1_{[0,t]}|_\phi^2 + \frac{\gamma'}{2(1-\gamma')^2}|\hat{f}|_\phi^2\right)\end{aligned} \tag{174}$$

Now let us proceed by finding the quasi-conditional expectation of the Hida derivative of stochastic component of the second term of (170), viz.,

$$\tilde{E}_{\hat{\mu}}\left\{\hat{D}_{k,t}\left(\rho(s,w)^{\frac{1}{\gamma_i-1}}\right)|\mathcal{F}_t\right\}. \tag{175}$$

It will be convenient to write the relevant variable in terms of $\hat{\mu}$ fBm,

$$\begin{aligned}\rho(s,w)^{\frac{1}{\gamma_i-1}} &= \exp\left(\frac{1}{1-\gamma_i}\sum_{j=1}^m\int_0^T \zeta_s^j(u)d\hat{B}_u^{H_j} - \frac{1}{1-\gamma_i}\sum_{j=1}^m\int_0^T \zeta_s^j(u)f_j(u)du\right.\\ &\quad\left.+\frac{1}{2(1-\gamma_i)}|\zeta_s|_\phi^2\right).\end{aligned} \tag{176}$$

For the purpose of taking the conditional expectation it will be convenient to write (176) as

$$\begin{aligned}\rho(s,w)^{\frac{1}{\gamma_i-1}} &= \exp\left(\frac{1}{1-\gamma_i}\sum_{j=1}^m\int_0^T \zeta_s^j(u)d\hat{B}_u^{H_j} - \frac{1}{2(1-\gamma_i)^2}|\zeta_s|_\phi^2\right)\exp\left(\frac{2-\gamma_i}{2(\gamma_i-1)^2}|\zeta_s|_\phi^2\right)\\ &\quad\times\exp\left(\frac{1}{\gamma_i-1}\sum_{j=1}^m\int_0^T \zeta_s^j(u)f_j(u)du\right).\end{aligned} \tag{177}$$

Taking the Hida derivative $\hat{D}_{k,t}$ of $\rho(s,w)^{\frac{1}{\gamma'-1}}$ is equivalent to multiplying it with $\frac{1}{1-\gamma'}\zeta_s^k(t)$

which is due to the chain rule, and (173).

$$\tilde{E}_{\hat{\mu}}\left\{\hat{D}_{k,t}\left(\rho(s,w)^{\frac{1}{\gamma_i-1}}\right)|\mathcal{F}_t\right\} = \frac{1}{1-\gamma_i}\zeta_s^k(t)\exp\left(\frac{1}{1-\gamma_i}\sum_{j=1}^m\int_0^t \zeta_s^j(u)d\hat{B}_u^{H_j} - \frac{1}{2(\gamma_i-1)^2}|\zeta_s 1_{[0,t]}|_\phi^2\right)$$
$$\times\exp\left(\frac{2-\gamma_i}{2(1-\gamma_i)^2}|\zeta_s|_\phi^2 + \frac{1}{\gamma_i-1}\sum_{j=1}^m\int_0^T \zeta_s^j(u)f_j(u)du\right)$$
$$= \exp\left(\frac{1}{1-\gamma_i}\sum_{j=1}^m\int_0^t \zeta_s^j(u)dB_u^{H_j} + \frac{1}{1-\gamma_i}\sum_{j=1}^m\int_0^t \zeta_s^j(u)f_j(u)du\right)$$
$$\times\exp\left(-\frac{1}{2(1-\gamma_i)^2}|\zeta_s 1_{[0,t]}|_\phi^2 + \frac{2-\gamma_i}{2(1-\gamma_i)^2}|\zeta_s|_\phi^2\right.$$
$$\left. + \frac{1}{\gamma_i-1}\sum_{j=1}^m\int_0^T \zeta_s^j(u)f_j(u)du\right). \tag{178}$$

Using (170), (171), (174) and (178) we have (26) which concludes the proof of Theorem 2.2. □

6 Extension of the Wick Calculus to Arbitrary Gaussian Processes

Although the results of the preceding sections have considered the explicit case in which the modulator in (1) is fBm, these results can be extended to the situation in which the modulator is a more general Gaussian process within sufficient regularity. This requires the extension of the Wick calculus to more general Gaussian processes. In this section, we sketch how this extension can be accomplished. The first step in extending the fractional noise machinery introduced in Section 3 to more general Gaussian processes is the following theorem due to Loève [27] for integrating deterministic functions with respect to second order processes :

Theorem 6.1 *([27]) Suppose that X is a zero-mean process such that $E\{X_t^2\} < \infty$ for all t, and denote its covariance function by R. Then, for $-\infty < a < b < \infty$,*

$$\int_a^b f(t)dX_t \tag{179}$$

exists as the L^2-limit of Riemann sums if and only if

$$\int_a^b\int_a^b f(t)f(s)d^2R(s,t) < \infty. \tag{180}$$

Henceforth X will denote a Gaussian process. By the Bochner-Minlos Theorem 3.1, there exists a unique probability measure on the space of tempered distributions such that (46) holds with $|f|_\phi^2$ replaced by (180).

We will denote $L^2(\mu)$ by $L^2(X)$ and $H(X)$ will denote the linear space of X, *i.e.* the closed subspace of $L^2(X)$ spanned by X_t for all $t \in [a, b]$ (*i.e.* the first Wiener chaos.) As in [20] we construct $\Lambda(R)$, a Hilbert space of deterministic integrable 'functions' isomorphic to $H(X)$ by completing the pre-Hilbert space of step functions $\mathbb{S}$ with the following inner product,

$$< f, g >_{\mathbb{S}} = \int\int f(t)g(s)d^2R(t,s), \tag{181}$$

for any $f, g \in \mathbb{S}$. Then the integration operator defined on the set of step functions (the integration with respect to X) can be extended to an isomorphism between $H(X)$ and $\Lambda(R)$. (The elements of $\Lambda(R)$ are generalized functions, *i.e.* distributions [36].)

As a second step we will define the Wick-integrability of a random process with respect to a Gaussian process. This is done by utilizing the tensor product structure of the space $L^2(X)$. Let us define the tensor product of Hilbert spaces.

Definition 6.1 *The algebraic tensor product $H_1 \otimes H_2$ of Hilbert spaces H_1 and H_2 is a pre-Hilbert space with the following inner product*

$$< h_1 \otimes h_2, g_1 \otimes g_2 >_{H_1 \otimes H_2} := < h_1, g_1 >_{H_1} < h_2, g_2 >_{H_2}, \tag{182}$$

for $g_i, h_i \in H_i$ and $i = 1, 2$. The closure of this pre-Hilbert space is the tensor product of Hilbert spaces, which will still be denoted by $H_1 \otimes H_2$. $H_1 \tilde{\otimes} H_2$ will denote the symmetrized tensor product.

Then we have the following Wiener chaos isomorphism theorem:

Theorem 6.2 *([21]) $\oplus_{p\geq 0} H^{\tilde{\otimes} p}(X)$ is isomorphic to $L^2(X)$ with the unique isomorphism Φ defined by*

$$\Phi(\xi_1^{\tilde{\otimes}\alpha_1} \tilde{\otimes} ... \tilde{\otimes} \xi_k^{\tilde{\otimes}\alpha_k}) = \frac{1}{\sqrt{p!}} \prod_{j=1}^{k} h_{\alpha_j}(\xi_j), \tag{183}$$

where $\xi_i \in H(X)$ for all i are orthonormal; $p = |\alpha| = \alpha_1 + ... + \alpha_k$ (α is given by (56)). Here h_n is the Hermite polynomial of degree n which is defined by (52).

Note that for a random variable $\xi \in H(X)$ with unit variance we have

$$\Phi(e^{\tilde{\otimes}\xi}) = \exp\left(\xi - \frac{1}{2}\right), \tag{184}$$

where the exponential is defined by

$$e^{\tilde{\otimes}\xi} = \sum_{p\geq 0} \frac{\xi^{\tilde{\otimes} p}}{\sqrt{p!}}. \tag{185}$$

We proceed as in [20], and in order to define the integral of a stochastic process with respect to X, we first define a tensor product integral, denoted by $\int F_t \otimes dX_t$, and its domain, denoted by $\Lambda(R)_{L^2(X)}$.

Suppose $\mathbb{S}_{L^2(X)}$ is the pre-Hilbert space of the $L^2(X)$ valued step functions F_t,

$$F_t = \sum_{i=1}^{N} F_i 1_{(t_i, t_{i+1}]}, \tag{186}$$

for $(t_i, t_{i+1}] \in [a, b]$, and $F_i \in L^2(X)$, equipped with the inner product

$$< F, G >= \int \int < F_t, G_s >_{L^2(X)} d^2 R(t, s). \tag{187}$$

Let $\Lambda(R)_{L^2(X)}$ denote the completion of $\mathbb{S}_{L^2(X)}$. For the $F \in \mathbb{S}_{L^2(X)}$ given in (186) define the integral $I_\otimes$ as

$$\int F_t \otimes dX_t = \sum_{i=1}^{N} F_{t_i} \otimes (X_{t_{i+1}} - X_{t_i}). \tag{188}$$

Since this integral is a norm preserving linear map, it has a unique extension to an isomorphism from $\Lambda(R)_{L^2(X)}$ into $L^2(X) \otimes H(X)$. We will construct a map Ψ from $L^2(X) \otimes H(X)$ into $L^2(X)$ and call the composition of the two maps, $\Psi(I_\otimes)$ the stochastic integral. We start by defining the following linear map

$$\Psi_p : H^{\tilde{\otimes} p}(X) \otimes H(X) \to H^{\tilde{\otimes} p+1}(X) \tag{189}$$

by

$$\Psi_p \left(\left(\xi_1^{\tilde{\otimes}\alpha_1} \tilde{\otimes} ... \tilde{\otimes} \xi_k^{\tilde{\otimes}\alpha_k} \right) \otimes \xi_l \right) = (p+1)^{\frac{1}{2}} \xi_1^{\tilde{\otimes}\alpha_1} \tilde{\otimes} ... \tilde{\otimes} \xi_k^{\tilde{\otimes}\alpha_k} \tilde{\otimes} \xi_l, \tag{190}$$

where $(\xi_i) \in H(X)$ is an orthonormal set of random variables, $\alpha_1 + ... + \alpha_k = p$. Ψ_p can be extended uniquely to a bounded linear map with norm $(p+1)^{1/2}$ from $H^{\tilde{\otimes} p}(X) \otimes H(X)$ onto $H^{\tilde{\otimes} p+1}(X)$.

Now define Ψ^* as the map from $\oplus_{p \geq 0} H^{\tilde{\otimes} p}(X) \otimes H(x)$ onto $\oplus_{p \geq 1} H^{\tilde{\otimes} p}(X)$, by $\Psi^* = \oplus_{p \geq 0} \Psi_p$, *viz.* the restriction of Ψ^* to $H^{\tilde{\otimes} p}(X) \otimes H(X)$ is Ψ_p. The domain of the operator Ψ^* is given by

$$\mathcal{D}^* = \left\{ \eta \in \left(H^{\tilde{\otimes}\alpha_1}(X) \oplus ... \oplus H^{\tilde{\otimes}\alpha_m}(X) \right) \otimes H(X) : \alpha_1 + ... + \alpha_m < \infty \right\}, \tag{191}$$

so that $\sum_{p \geq 0} \|\Psi_p(\eta_p)\|^2 < \infty$, where η_p is the projection of η on $H^{\tilde{\otimes} p}(X) \otimes H(X)$.

By Thm. 6.2, $\oplus_{p \geq 0} H^{\tilde{\otimes} p}(X)$ is isomorphic to $L^2(X)$. Therefore $\left(\oplus_{p \geq 0} H^{\tilde{\otimes} p}(X) \right) \otimes H(X)$ is isomorphic to $L^2(X) \otimes H(X)$. Denote this isomorphism by Φ_0. Let $\mathcal{D} = \Phi_0(\mathcal{D}^*)$, which is a proper subset of $L^2(X) \otimes H(X)$. Then define Ψ by

$$\Psi = \Phi \circ \Psi^* \circ \Phi_0^{-1}. \tag{192}$$

We define the Wick product of $V \in L^2(X)$ and $W \in H(X)$ as

$$V \diamond W := \Psi(V \otimes W). \tag{193}$$

Note that $V \diamond W$ is in $L^2(X)$ iff $V \otimes W \in \mathcal{D} \otimes H(X)$.

The integral $\int F_t \diamond dX_t$ is then defined by

$$\int_a^b F_t \diamond dX_t = \Psi \circ I_{\otimes}(F) \tag{194}$$

for all F such that $I_{\otimes}(F) = \int F_t \otimes dX_t \in \mathcal{D}$. The set of all F's in the domain of integration is denoted by $\Lambda(R)^*_{L^2(X)}$. Then we have the Itô representation formula as a result of the Multiple Wiener Integral (MWI) representation of the random variables in $L^2(X)$ ([20]), and the fact that each MWI can be written as an iterated integral:

Theorem 6.3 *([20]) Every $\theta \in L^2(X)$ has the following representation*

$$\theta = E\{\theta\} + \int_a^b F_t \diamond dX_t, \tag{195}$$

for an $F \in \Lambda(R)^*_{L^2(X)}$ that is adapted to the filtration generated by X.

Now let us define the Wick product of two elements in $L^2(X)$. As a first step we define $\Upsilon_{p,q}$,

$$\Upsilon_{p,q} : H^{\tilde{\otimes} p}(X) \otimes H^{\tilde{\otimes} q}(X) \to H^{\tilde{\otimes} p+q}(X), \tag{196}$$

as

$$\Upsilon_{p,q}\left(\left(\xi_{\gamma_1}^{\tilde{\otimes}\alpha_1} \tilde{\otimes} \ldots \tilde{\otimes} \xi_{\gamma_k}^{\tilde{\otimes}\alpha_k}\right) \otimes \left(\xi_{\lambda_1}^{\tilde{\otimes}\beta_1} \tilde{\otimes} \ldots \tilde{\otimes} \xi_{\lambda_l}^{\tilde{\otimes}\beta_l}\right)\right) = \sqrt{\frac{(p+q)!}{p!q!}}\, \xi_{\gamma_1}^{\tilde{\otimes}\alpha_1} \tilde{\otimes} \ldots \tilde{\otimes} \xi_{\gamma_k}^{\tilde{\otimes}\alpha_k} \tilde{\otimes} \xi_{\lambda_1}^{\tilde{\otimes}\beta_1} \tilde{\otimes} \ldots \tilde{\otimes} \xi_{\lambda_l}^{\tilde{\otimes}\beta_l}, \tag{197}$$

for any (ξ_γ) that is an orthonormal set in $H(X)$.

Let $\Upsilon = \oplus_{p \geq 0} \oplus_{q \geq 0} \Upsilon_{p,q}$ then the we define the Wick-product of the $W, V \in L^2(X)$ as

$$W \diamond V := \Phi\left(\Upsilon\left(\Phi^{-1}(W) \otimes \Phi^{-1}(V)\right)\right). \tag{198}$$

Note that $L^2(X)$ is not closed under $\diamond$, since the tensor product of the random variables may not be in the domain of Υ. Then one can define the spaces of generalized random variables as in Defn. 3.1 and Defn. 3.7, use (198) as the definition of the Wick-product over these spaces, and see that the Wick product so defined is closed over these spaces.

The main machinery we use to develop strategies leading to a Nash equilibrium are the Girsanov formula (the absolute continuity of the translated measure w.r.t. the original measure), and the Clark-Ocone formula. These can be extended to more genreal Gaussian modulator with regularity. The Girsanov theorem, Thm. 4.1 can be stated for an sufficiently regular Gaussian processes.(The proof of the Girsanov thm. in [35] does not make use of the explicit expression for ϕ.) The derivation of the Clark-Ocone theorem (Thm. 3.5) is done by using only the tensor product structure of the space L^2 (Thm. 6.2) and the spaces of generalized random variables. Defining the Hida derivative and the quasi-conditional expectation operator (w.r.t. which X is a quasi-martingale) for the Gaussian process X, we can restate the Clark-Ocone theorem. Now replacing ζ_t in Thm. 2.1 by ϑ_t such that

$$E\left\{\int_0^T K(s)dX_s \middle| \mathcal{F}_t\right\} = \int_0^t \vartheta_t(s)dX_s, \tag{199}$$

we have a Nash equilibrium theorem for a general Gaussian process. Note that, unlike the case of fBm we cannot in general write ϑ explicitly in terms of K. Hence, we cannot give an explicit solution for the Nash equilibrium. A general multi-dimensional theorem (Thm. 2.2) can also be restated for a multi-dimensional Gaussian process with independent components (the components do not have to be identical) by making the conceptual modifications as in the one-dimensional case.

7 Conclusion

In this paper we have explicitly found Nash equilibria for two stochastic differential games in a non-Markovian setting. All the agents observe the states, and they control the state through modifying the drift and the volatility. The agents are heterogeneous in their controls and the utility functions. We took the modulating processes to be fractional Brownian motions, because of fBm is versatile in modeling long-range dependence phenomenon in finance and networks. In the first game there is only one source of randomness and agents observe a single state, while in the second game there are multiple sources of randomness (fBm's with different Hurst parameters) and multiple states.

Since the diffusion is modulated by a non-Markovian process, the usual way of finding the Nash equilibria via Hamilton-Jacobi-Bellman equations is not available. Therefore we have made use of the fractional noise calculus to calculate the agents' strategies in the Nash-equilibrium. Although we took the modulating process of the diffusions to be fBm, our results hold for a more general Gaussian modulating process with only slight modifications to white noise machinery.

Our results are applicable to financial markets in which the stock price dynamics are modulated with fractional Brownian motion. One of the candidate applications is stock price modeling when each agent's activities in the market affect the price flow (institutional investors are such examples), or if there are transaction costs. Also it is applicable to stochastic portfolio games, in which agents compete for a bonus.

8 Appendix

Lemma 8.1 *([24]) Let $f : [0,T] \to \mathbb{R}$ be a continuous function and introduce the following integral equation:*

$$\int_0^T \hat{f}(s)\phi(s,t)ds = f(t) \quad \textit{for } t \in [0,T], \tag{200}$$

where ϕ is given by (18). The solution to this equation is given by

$$\hat{f}(t) = -\frac{1}{d_H} t^{\frac{1}{2}-H} \frac{d}{ds} \int_t^T dw w^{2H-1}(w-t)^{\frac{1}{2}-H} \frac{d}{dw} \int_0^w dz z^{\frac{1}{2}-H}(w-z)^{\frac{1}{2}-H} f(z), \tag{201}$$

where

$$d_H = 2H(2H-1)\left(\Gamma(\frac{3}{2}-H)\right)^2 \Gamma(2H-1)\cos(\pi(H-\frac{1}{2})) \tag{202}$$

Corollary 8.1 *If we take f(t)=C on $[0,T]$ in the integral equation given by (200) then the solution $\hat{f}(t)$ is given by*

$$\hat{f}(t) = \frac{C}{k_H} t^{\frac{1}{2}-H}(T-t)^{\frac{1}{2}-H}, \tag{203}$$

where

$$k_H = 2H(2H-1)\Gamma(2-2H)\Gamma(2H-1)\cos(\pi(H-\frac{1}{2})). \tag{204}$$

Proof: The proof can be found in [35], but we present it here for the sake of completeness.

$$\hat{f}(t) = -\frac{1}{d_H} C t^{\frac{1}{2}-H} \frac{d}{ds} \int_t^T dw w^{2H-1}(w-t)^{\frac{1}{2}-H} \frac{d}{dw} \int_0^w dz z^{\frac{1}{2}-H}(w-z)^{\frac{1}{2}-H}. \tag{205}$$

Note that

$$\frac{\int_0^w z^{\frac{1}{2}-H}(w-z)^{\frac{1}{2}-H}dz}{w^{2-2H}} = B(\frac{3}{2},\frac{3}{2}) = \frac{\Gamma\left(\frac{3}{2}-H\right)^2}{\Gamma(3-H)}, \tag{206}$$

where $B(\cdot,\cdot)$ is the beta function given by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt. \tag{207}$$

Hence

$$\frac{d}{dw}\int_0^w z^{\frac{1}{2}-H}(w-z)^{\frac{1}{2}-H}dz = \frac{\Gamma\left(\frac{3}{2}-H\right)^2}{\Gamma(2-H)} w^{1-2H}. \tag{208}$$

Using (208) it is not hard to evaluate (205) to get (203). □

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