Math 566
Problem Set 1

(1) Prove that the number of walks of length $\ell$ between two distinct vertices of the complete graph $K_p$ differs by 1 from the number of closed walks of length $\ell$ originating at a given vertex.

(2) Prove that the diameter of a connected graph is less than the number of its distinct eigenvalues.

(3) Prove that the largest eigenvalue $\lambda_{\text{max}}$ of a graph $G$ is at least the average degree of the vertices in $G$.

[Hint: Use that, for a symmetric real matrix $M$, one has $\max_{|x|=1} x^T M x = \lambda_{\text{max}}$.]

(4) Let $G$ be a graph obtained by removing $n$ disjoint edges from the complete bipartite graph $K_{n,n}$. Compute the eigenvalues of $G$.

(5) Find the eigenvalues of the graph obtained by removing $n$ disjoint edges from the complete graph $K_{2n}$.

(6) Find the number of marked closed walks of length $\ell$ in the graph below.

(7) Find the eigenvalues of the 1-skeleton of an octagonal prism.

(8) Find a combinatorial proof (using generating functions) of the formula for the number of marked closed walks of length $\ell$ in the $n$-cube:

$$\sum_{i=0}^{n} \binom{n}{i} (n-2i)^\ell.$$ 

(9) Find the probability that a simple random walk on the discrete torus with $n^2$ vertices (the direct product of two $n$-cycles) returns to its origin after $\ell$ steps.