General course information

The grade will be based on several problem sets.

Collaboration in small groups is allowed as long as it is fully disclosed.

We will not be using Canvas.

Feel free to interrupt me during the lecture to ask a question.

I am available via email and for in-person drop-ins.

No textbook. Main reference:

S. Fomin, L. Williams, and A. Zelevinsky, *Introduction to cluster algebras*, Chapters 1–7 (draft).

For additional references, see the course webpage.
This course introduces the basic notions, examples, and applications of the theory of cluster algebras from the combinatorial perspective.

Since their discovery in 2000, cluster algebras found numerous applications in diverse research areas, including:

- representation theory of Lie groups;
- Teichmüller theory & hyperbolic geometry;
- discrete and continuous integrable systems & Poisson geometry;
- classical invariant theory & incidence geometry;
- representations of quivers and finite-dimensional algebras; and
- mathematical physics & (quantum) dilogarithm identities.

We will focus on the general structural theory of cluster algebras, emphasizing elementary examples and combinatorial constructions.

We will begin by discussing two motivating examples.
Example 1: Grassmannian $\text{Gr}_{2,n+3}(\mathbb{C})$

Let $z$ be a $2 \times (n+3)$ matrix with generic complex entries:

$$z = \begin{bmatrix}
  z_{11} & z_{12} & \cdots & z_{1,n+3} \\
  z_{21} & z_{22} & \cdots & z_{2,n+3}
\end{bmatrix}.$$

**Plücker coordinates**

The Plücker coordinates $P_{ij} = P_{ij}(z)$, for $1 \leq i < j \leq n+3$, are given by

$$P_{ij} = \det \begin{bmatrix}
  z_{1i} & z_{1j} \\
  z_{2i} & z_{2j}
\end{bmatrix}.$$

**The Grassmannian $\text{Gr}_{2,n+3}(\mathbb{C})$**

The row span of a $2 \times (n+3)$ matrix $z$ is a 2-dim subspace in $\mathbb{C}^{n+3}$, i.e., a point in the complex Grassmannian $\text{Gr}_{2,n+3}(\mathbb{C})$.

The Plücker coordinates are homogeneous coordinates on $\text{Gr}_{2,n+3}(\mathbb{C})$. They can also be viewed as coordinates on the affine cone $\hat{\text{Gr}}_{2,n+3}(\mathbb{C})$. 
Grassmann-Plücker relations

Grassmann-Plücker relations for $\text{Gr}_{2,n+3}(\mathbb{C})$

\[ P_{ik} P_{j\ell} = P_{ij} P_{k\ell} + P_{i\ell} P_{jk} \quad (i < j < k < l) \]

Proof

We need to show that $P_{13} P_{24} = P_{12} P_{34} + P_{14} P_{23}$. We have:

\[
\begin{align*}
P_{12} P_{34} & - P_{13} P_{24} + P_{14} P_{23} \\
&= \begin{vmatrix}
    z_{11} & z_{12} \\
    z_{21} & z_{22}
\end{vmatrix}
\begin{vmatrix}
    z_{13} & z_{14} \\
    z_{23} & z_{24}
\end{vmatrix}
- \begin{vmatrix}
    z_{11} & z_{13} \\
    z_{21} & z_{23}
\end{vmatrix}
\begin{vmatrix}
    z_{12} & z_{14} \\
    z_{22} & z_{24}
\end{vmatrix}
+ \begin{vmatrix}
    z_{11} & z_{14} \\
    z_{21} & z_{24}
\end{vmatrix}
\begin{vmatrix}
    z_{12} & z_{13} \\
    z_{22} & z_{23}
\end{vmatrix}
\end{align*}
\]

\[
= \frac{1}{2}
\begin{vmatrix}
    z_{11} & z_{12} & z_{13} & z_{14} \\
    z_{21} & z_{22} & z_{23} & z_{24}
\end{vmatrix}
\begin{vmatrix}
    z_{11} & z_{12} & z_{13} & z_{14} \\
    z_{21} & z_{22} & z_{23} & z_{24}
\end{vmatrix}
= 0.
\]
The Plücker ring $A_n$

Definition

Let $A_n$ denote the commutative ring generated inside the polynomial ring $\mathbb{C}[z_{11}, \ldots, z_{2, n+3}]$ by the $\binom{n+3}{2}$ Plücker coordinates $P_{ij}$.

$A_n$ can also be defined using the Grassmann-Plücker relations.

Intrinsically, $A_n$ is the homogeneous coordinate ring of the Grassmannian $\text{Gr}_{2, n+3}(\mathbb{C})$ with respect to the Plücker embedding.

Alternatively, it is the ring of $\text{SL}_2$-invariants of a collection of $n+3$ vectors in $\mathbb{C}^2$.

The Plücker ring $A_n$ will be our first example of a cluster algebra.
Cluster monomials

Compatible Plücker coordinates

We say that two Plücker coordinates $P_{ik}$ and $P_{jl}$ are compatible unless $i < j < k < l$.

Cluster monomials

Any product of pairwise compatible (not necessarily distinct) Plücker coordinates is called a cluster monomial.

The following result can be traced back to classical 19th-century literature on invariant theory.

Theorem

Cluster monomials form a linear basis of $A_n$.

To work with this basis, one needs to understand the combinatorics of the compatibility relation.
Clusters

Coefficient variables
Each of the Plücker coordinates $P_{12}, P_{23}, P_{34}, \ldots, P_{n+2,n+3}, P_{1,n+3}$ is compatible with any $P_{ij}$. These $n + 3$ Plücker coordinates are the coefficient variables. (They are also called frozen variables.)

Cluster variables
The remaining generators $P_{ij}$ are the cluster variables.

Clusters
A maximal (by inclusion) collection of pairwise compatible cluster variables is called a cluster.

Fact
Each cluster has cardinality $n$. 
Clusters in the Plücker ring $\mathbb{C}[\hat{\text{Gr}}_{2,6}]$
Extended clusters

Definition
A maximal collection of pairwise compatible Plücker coordinates is called an extended cluster.

With this terminology, a cluster monomial is a monomial in the elements of some extended cluster.

An extended cluster is a cluster plus the $n + 3$ coefficient variables.

The size of each extended cluster is $2n + 3 = \dim \hat{\text{Gr}}_{2,n+3}$. 
Ptolemy relations

The \( \binom{n+3}{2} \) generators of the algebra \( \mathcal{A}_n \) can be labeled by the sides and diagonals of a convex \( (n + 3) \)-gon:

The defining relations of \( \mathcal{A}_n \) can be viewed as *Ptolemy relations*:

\[
x_a = P_{ij}
\]

\[
x_e x_{e'} = x_a x_c + x_b x_d
\]
Now, the picture becomes clear:

- coefficient variables correspond to the sides of the polygon;
- cluster variables correspond to the diagonals of the polygon;
- compatible means \textit{do not intersect};
- clusters (or extended clusters) correspond to triangulations;
- exchanges correspond to \textit{flips}. 

Flips

The 3-dimensional associahedron

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Clusters and exchange relations in $\mathbb{C}[\hat{\text{Gr}}_{2,6}]$

- Faces $\leftrightarrow$ cluster variables
- Vertices $\leftrightarrow$ clusters

Edges $\leftrightarrow$ Plücker relations

\[ P_{13} P_{25} = P_{12} P_{35} + P_{23} P_{15} \]
\[ P_{35} P_{14} = P_{34} P_{15} + P_{45} P_{13} \]
\[ P_{15} P_{36} = P_{56} P_{13} + P_{16} P_{35} \]
Mutations

Mutations in $\mathcal{A}_n$ are algebraic counterparts of flips.

**Mutation**

A *mutation* in $\mathcal{A}_n$ is a transformation that changes a cluster (resp., an extended cluster) by replacing one of its cluster variables by a new one using a flip in the underlying triangulation.

**Extended clusters as coordinate systems**

Each generator $x_a = P_{ij}$ of the ring $\mathcal{A}_n$ is uniquely expressed as a rational function in the elements of a given extended cluster.

**Proof**

Any two triangulations of a convex polygon are related to each other via flips. Hence any two extended clusters are related to each other via iterated mutations. Uniqueness follows by a dimension argument.