

THE SEVERAL COMPLEX VARIABLES PROBLEM LIST IT'S BACK!!!!!!!

September 1995

During the past two years, we got many requests from people interested in the list, but we did NOT get many new problems. Since we believe that this list is worth pursuing, we are making a new attempt. The SCV problem list is an evolving, growing part of our subject. Please contribute your ideas for helping to keep it lively. In this come-back edition we request from you:

- 1.- **NEW PROBLEMS!!**
- 2.- Tell us if you know that any of the problems have been recently solved. We will announce it in the next edition.

To simplify the reading, we are stating the new problems for this edition at the beginning of the list, right after the introduction.

1 Introduction

The purpose of this bulletin board is to collect problems in higher dimensional complex analysis. We are interested both in basic research questions as well as interactive questions with other fields and sciences. We encourage everybody to submit problems to the list. This includes not only those coming up in your own work, but also others- maybe well known and classical- that you see missing, but that you think workers in the field should be aware of.

Not only are we searching for basic research type questions in several complex variables, we also solicit questions exploring relations to other mathematical fields, one complex variable, partial differential equations, differential geometry, dynamics, etc. and to other sciences such as physics, engineering, biology etc. While some questions fall rather naturally into one of the subject areas in this problem list and may lead to a publishable paper, other questions may be non-specific or of a transient or technical quality. Thus we have a section called “Scratchpad” for conversational questions, vaguely formulated questions, or questions to which you may hope to get a quick answer. In the “open prize problems” section, you are welcome to offer a nice little prize to whoever does it. The miscellaneous section is for announcements of conferences, jokes, remarks on the general state of the field, etc.

This file is in Latex. (In order to get a table of contents you should run the file twice.) If a problem appears in the problem list the first time or has been changed, it will be marked by the word **New**.

To submit problems and other suggestions, email to:

- Gregery Buzzard at gbuzzard@iu-math.math.indiana.edu,
- John Erik Fornaess at fornaess@math.lsa.umich.edu ,
- Estela A. Gavosto at gavosto@math.lsa.umich.edu or to
- Steven G. Krantz at sk@math.wustl.edu

ANNOUNCEMENTS:

- Academic Year 95/96, Special Year in Several Complex Variables, MSRI, Berkeley.
- January, 10-13, 1996 Orlando, FL (1996 Joint Meetings)
 - AMS Special Session on Analytic Methods in SCV.
 - AMS Special Session on Multidimensional Complex Dynamics.
- March 22-23, 1996 Iowa City, IA . AMS Special Session on Geometric and Analytic Methods in SCV

2 NEW PROBLEMS IN THIS EDITION:

Problem 2.0.1 (New) (See Section 7.2.) Let V be a germ at 0 of an irreducible complex variety in \mathbb{C}^n . For any small enough $\epsilon > 0$, let $E = E_\epsilon := V \cap \mathbb{R}^n \cap \{|z| < \epsilon\}$. Define the extremal function U_E on $\{|z| < \epsilon\} : U_E(z) := \sup\{u(z); u \leq 0 \text{ on } E, 0 \leq u \leq 1 \text{ on } \{|z| < \epsilon\}$. The problem is to classify those V for which we have a constant $A > 0$ for which $(\alpha) U_E(z) \leq A|\Im z|, |z| < \epsilon/2$.

Problem 2.0.2 (New) (See Section 8.2.) What is the lowest degree of a polynomial mapping $P : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ for which the (real) Jacobian conjecture fails, i.e. the Jacobian of P near vanishes while p fails to be invertible.

Problem 2.0.3 (New) (See Section 11.3) (Liouville type assertion on CR 3-manifolds) On a CR 3-manifold one can always define a Carnot-Caratheodory metric. A homeomorphism between two strongly pseudoconvex CR 3-manifolds is called conformal if it maps infinitesimal spheres with respect to a Carnot-Caratheodory metric to infinitesimal spheres. Is a conformal homeomorphism CR? -Puqi Tang, tang@math.purdue.edu

Problem 2.0.4 (New) (See Section 11.3) A diffeomorphism between two strongly pseudoconvex CR $(2n + 1)$ -manifolds is called quasiconformal if its differential preserves the underlying contact structures and distorts the CR structures boundedly. When Hermitian metrics on the contact bundles are fixed, this distortion can be measured by checking how spheres in the contact space is mapped to ellipsoids. However, this measurement depends on the choices of the Hermitian metrics if $n > 1$. Fixing a quasiconformal diffeomorphism, can we choose Hermitian metrics so that the distortion is minimal? - Puqi Tang, tang@math.purdue.edu

Contents

1	Introduction	1
2	NEW PROBLEMS IN THIS EDITION:	3
3	Envelopes of holomorphy of domains in \mathbb{C}^n	6
3.1	C-R functions	6
3.2	The Future Tube	6
3.3	Schlichtness of envelopes of holomorphy	6
4	The Levi Problem	7
4.1	Locally Stein open subsets of Stein Spaces	7
4.2	The Runge Problem	7
4.3	The Union Problem	7
5	Holomorphic function theory of domains in \mathbb{C}^n	9
5.1	Approximation Problems	9
5.2	The corona problem for the ball or the polydisc	9
5.3	Hyper-holomorphic functions	9
5.4	Peak Points and Peak Sets	10
5.5	Representing measures and polynomial hulls	10
5.6	Zero sets of holomorphic functions	10
6	Existence and regularity properties of the Cauchy Riemann operator	11
6.1	The $\bar{\partial}$ Neumann problem	11
6.2	Hölder estimates for $\bar{\partial}$ on pseudoconvex domains of finite type in \mathbb{C}^n	11
6.3	L^p Estimates for $\bar{\partial}$	12
6.4	Uniform estimates for $\bar{\partial}$ on pseudoconvex domains	12
7	Geometric and Topological Properties of Pseudoconvex Domains	13
7.1	Topological Properties of Pseudoconvex Domains	13
7.2	The Bergman Metric B	13
7.3	The Caratheodory Metric c	13
7.4	Finite Type	13

7.5	The Kobayashi Metric k	14
7.6	Regular Domains	14
8	Plurisubharmonic Functions	15
8.1	The Complex Monge Ampere Operator	15
8.2	Extremal Functions	15
9	Holomorphic maps	16
9.1	Automorphisms	16
9.2	The Jacobian Conjecture	16
9.3	Proper correspondences	16
9.4	Real analytic domains (not necessarily pseudoconvex) in \mathbb{C}^n , $f : U \rightarrow V$ biholomorphic	16
9.5	Stein manifolds	16
10	Dynamical properties of holomorphic maps	18
10.1	Dynamics of entire holomorphic maps	18
10.2	The invariant set K^+ of a polynomial complex Henon map . .	18
10.3	Dynamics on \mathbb{P}^k	19
11	Complex Analytic Varieties	20
11.1	Embedding Varieties	20
11.2	Varieties over the unit disc	20
11.3	Complex Differential Geometry	20
12	CR manifolds	21
12.1	Embedding	21
12.2	Extensions	21
12.3	Quasiconformal mappings	21
13	Scratchpad	22
14	Open prize problems	23

3 Envelopes of holomorphy of domains in \mathbb{C}^n

3.1 C-R functions

Problem 3.1.1 *Let M be a smooth real hypersurface in \mathbb{C}^n with defining function r . Let p_0 be a point in M , and let U be a neighborhood in \mathbb{C}^n of p_0 . When do C-R functions on $M \cap U$ extend to $\{r > 0\}$?*

-B. Stenones, berit@math.lsa.umich.edu

3.2 The Future Tube

3.3 Schlichtness of envelopes of holomorphy

Problem 3.3.1 *Let D be a bounded domain in \mathbb{C}^n with smooth boundary, with envelope of holomorphy D^\sim . A) Is D^\sim finitely sheeted over \mathbb{C}^n ? B) Does D^\sim have finite volume?*

-B. Stenones, berit@math.lsa.umich.edu.

Problem 3.3.2 *If D is a strictly pseudoconvex domain in \mathbb{C}^2 diffeomorphic to \mathbb{R}^4 and A is an embedded analytic disc outside D with $bA \subset bD$, then there is an embedded smooth disc M in D , with $bM = bA$, and such that M is totally real except at one elliptic point. With such a configuration (M, A) in \mathbb{C}^2 , does it follow that the envelope of holomorphy of M contains A ?*

-F. Forstneric, forstner@math.wisc.edu

4 The Levi Problem

4.1 Locally Stein open subsets of Stein Spaces

Problem 4.1.1 *Let U be an open subset of a Stein space. If U is locally Stein, is U Stein?*

Problem 4.1.2 *Let M be a compact complex manifold. Find some appropriate conditions on M such that if U is an open subset of M which is locally Stein, then U is Stein.*

-N. Sibony, sibony@anh.matups.fr

4.2 The Runge Problem

Problem 4.2.1 *Let X be a complex subvariety of the unit ball \mathbb{B} with an isolated singularity at the origin. Suppose Ω_t is a continuously increasing family of Stein open subsets of X . Also, suppose that $0 \in \partial\Omega_{1/2}, 0 \in \Omega_t, t > 1/2$. Is Ω_s Runge in $\Omega_t \forall s < t$?*

Problem 4.2.2 *Let $K^{\text{compact}} \subset X \setminus \{0\}$, X as above, $0 \in \hat{K}$, the polynomially convex hull of K . Suppose that $L \subset X$, L contains a neighborhood of K in X . Does \hat{L} contain a neighborhood of 0 in X ? (This is equivalent to the previous problem.)*

4.3 The Union Problem

Problem 4.3.1 *Let $\{U_n\}$ be an increasing sequence of Stein open subsets of a Stein Space X . Is $\bigcup_{n=1}^{\infty} U_n$ Stein? (If X is a complex manifold, not necessarily Stein, the answer is in general no.)*

Problem 4.3.2 *Suppose M is a complex manifold of dimension 2, and suppose $\forall K^{\text{compact}} \subset M$ there exists an open subset U , $K \subset U \subset M$ such that U is biholomorphic to the unit ball. Does this imply that M is equivalent to the unit ball \mathbb{B} , $\Delta \times \mathbb{C}$ or \mathbb{C}^2 ? (This is false in \mathbb{C}^3).*

Problem 4.3.3 *Is “long” \mathbb{C}^2 biholomorphic to \mathbb{C}^2 ? (A complex manifold is a long \mathbb{C}^2 if it is the union of proper subsets that are biholomorphic to \mathbb{C}^2 .)*

5 Holomorphic function theory of domains in \mathbb{C}^n

5.1 Approximation Problems

Problem 5.1.1 *Let $\Omega \subset\subset \mathbb{C}^n$ be a smooth pseudoconvex domain. Let $A^k(\bar{\Omega}) = H(\Omega) \cap C^k(\bar{\Omega})$ with C^k topology, $1 \leq k \leq \infty$. Is $A^\infty(\bar{\Omega})$ dense in $A^k(\bar{\Omega})$?*

-N. Sibony, sibony@anh.matups.fr

Problem 5.1.2 *Let $\Omega \subset\subset \mathbb{C}^n$ be a smooth pseudoconvex domain. Assume that $\bar{\Omega}$ has a Stein neighborhood basis. Is every function in $A^\infty(\bar{\Omega})$ uniformly approximable by holomorphic functions in a neighborhood of $\bar{\Omega}$?*

-N. Sibony, sibony@anh.matups.fr

5.2 The corona problem for the ball or the polydisc

Problem 5.2.1 *Let f_1, \dots, f_k be bounded holomorphic functions on Ω . Suppose $\sum |f_i| > c > 0$. The Corona problem is whether there exist bounded holomorphic functions g_1, \dots, g_k on Ω such that $\sum f_i g_i \equiv 1$. Can the Corona problem be solved if Ω is the unit ball or the unit polydisc?*

5.3 Hyper-holomorphic functions

Problem 5.3.1 *Let $\Phi = \{\Phi^0, \Phi^1, \Phi^2, \Phi^3\}$ be a subset of the body (= skew - field) \mathbb{H} of the real quaternions with the conditions*

$$\Phi^p \bar{\Phi}^q + \Phi^q \bar{\Phi}^p = 2\delta_{p,q},$$

Ω is a domain in $\mathbb{R}^4 = \mathbb{C}^2$. On the \mathbb{H} -bimodule $C^1(\Omega; \mathbb{H})$ the operator $D_\Phi = \sum_{k=0}^3 \Phi^k \frac{\partial}{\partial x_k}$ defines the set of the (left - Φ)-hyper-holomorphic functions $M_\Phi(\Omega; \mathbb{H}) = \text{Ker} D_\Phi$. Let $\hat{\Phi} = \{1, i, j, -k\}$ with i, j, k being the ordinary imaginary units in \mathbb{H} . It is known that

$$\text{Hol}(\Omega; \mathbb{C}^2) \subset M_{\hat{\Phi}}(\Omega; \mathbb{H}),$$

where $\text{Hol}(\Omega; \mathbb{C}^2)$ is the subset of all holomorphic (in the usual sense) mappings.

1) Describe the set of all hyper-holomorphic but non-holomorphic functions in Ω (it is not empty).

2) Describe "locations" of the sets Hol and $M_{\Phi}\text{Hol}$ inside M_{Φ} .

3) How do they depend on Ω ?

-H. Shapiro and N. Vasilevsky, *matemat@cinvesmx.bitnet*

5.4 Peak Points and Peak Sets

Problem 5.4.1 *Is every boundary point of a bounded pseudoconvex domain D in \mathbb{C}^n of finite type a peak point for $A(D)$?*

Problem 5.4.2 *Describe peak sets on weakly pseudoconvex domains of finite type.*

5.5 Representing measures and polynomial hulls

Problem 5.5.1 *Characterize the representing measures for the origin in the unit ball in \mathbb{C}^n .*

Problem 5.5.2 *Let $X \subset \mathbb{C}^n$ be compact and totally disconnected. Assume that X has finite one-dimensional Hausdorff measure. Then is X polynomially convex?*

-H. Alexander, *U22330@UICVM.bitnet*

5.6 Zero sets of holomorphic functions

Problem 5.6.1 *Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded, pseudoconvex domain. Let $V \subseteq \Omega$ be a divisor which is bounded away from the Silov boundary in $\partial\Omega$. Prove that there is a bounded holomorphic function on Ω that vanishes on V . (This is known for the polydisc—a result of Rudin.)*

-S. Krantz, *sk @math.wustl.edu*

Problem 5.6.2 *Give a geometric characterization of zero sets of H^p functions on strongly pseudoconvex domains.*

6 Existence and regularity properties of the Cauchy Riemann operator

6.1 The $\bar{\partial}$ Neumann problem

Problem 6.1.1 *Solve the $\bar{\partial}$ Neumann problem in H^s .*

Problem 6.1.2 *Solve $\bar{\partial}$ Neumann in $L^2(d\mu)$, $d\mu$ a measure, even on the unit disc.*

Problem 6.1.3 *Let Ω be a smooth bounded planar domain and let $T : L^2(\Omega) \mapsto L^2(\Omega)$ be the Kohn solution operator for $\partial/\partial\bar{z}$. For $h \in L^\infty(\Omega)$ we define the compact non-self-adjoint operator $S_h : L^2(\Omega) \mapsto L^2(\Omega)$, $f \mapsto \bar{h}Tf$. Under what circumstance will S_h have eigenvalues? Consider in particular the case where h is holomorphic. If h has a single-valued holomorphic primitive H then it is not hard to see that S_h has no eigenvalues. (Just note that $e^{-H/\lambda}Tf$ is holomorphic and orthogonal to holomorphic functions.) Does the converse hold? That is: do periods of H guarantee the existence of eigenvalues? (For annuli, the answer is “yes.”)*

-D. Barrett, barrett@math.lsa.umich.edu

Problem 6.1.4 *If D is a real analytically bounded pseudoconvex domain in \mathbb{C}^n , is the $\bar{\partial}$ Neumann problem globally real analytic hypoelliptic up to the boundary?*

- D. S. Tartakoff, U22393@UICVM.bitnet

Problem 6.1.5 *If M is a compact real analytic pseudoconvex CR manifold of real dimension at least 5 is \square_b globally real analytic hypoelliptic (say on $(0,1)$ forms) on M ?*

- D. S. Tartakoff, U22393@UICVM.bitnet

6.2 Hölder estimates for $\bar{\partial}$ on pseudoconvex domains of finite type in \mathbb{C}^n

Problem 6.2.1 *Can one solve $\bar{\partial}$ with Hölder estimates on any real analytic bounded pseudoconvex domain in \mathbb{C}^n ?*

6.3 L^p Estimates for $\bar{\partial}$

Problem 6.3.1 Solve $\bar{\partial}$ in L^p in bounded pseudoconvex domains, without necessarily smooth boundary, $1 < p < 2$.

Problem 6.3.2 Solve $\bar{\partial}$ with L^p estimates, $1 < p < \infty$ for smooth convex domains in \mathbb{C}^n .

6.4 Uniform estimates for $\bar{\partial}$ on pseudoconvex domains

Problem 6.4.1 Can one solve $\bar{\partial}u = f$ in sup norm on smooth bounded convex domains in \mathbb{C}^2 ?

Problem 6.4.2 Which geometric properties of the domain do uniform estimates depend on? [e.g. L^2 estimates depend on the diameter and the dimension].

Problem 6.4.3 If $\Omega \subset\subset \mathbb{C}^n$ is pseudoconvex with smooth boundary, does $\bar{\partial} : \Lambda^\alpha(\bar{\Omega}) \mapsto \Lambda_{(0,1)}^\beta(\bar{\Omega})$ have closed range if $\alpha < \beta$?

Problem 6.4.4 Let Ω in \mathbb{C}^n be a bounded biholomorphic image of the unit ball in \mathbb{C}^n . Does the $\bar{\partial}$ satisfy uniform estimates on Ω ? The problem is of interest mainly when Ω has very rough boundary.

-B. Berndtsson, bob@math.chalmers.se

7 Geometric and Topological Properties of Pseudoconvex Domains

7.1 Topological Properties of Pseudoconvex Domains

Problem 7.1.1 *Does there exist a pseudoconvex domain in \mathbb{C}^2 with the same homotopy type as S^2 ?*

-F. Forstneric, forstner@math.wisc.edu

7.2 The Bergman Metric B

Problem 7.2.1 *Is B complete on bounded pseudoconvex domains with less than C^1 boundary?*

-J. McNeal, mcneal@math.princeton.edu

Problem 7.2.2 *For which domains is the Bergman kernel function $K(z, w)$ nonvanishing everywhere? This is related to Lu Qi-Keng conjecture.*

7.3 The Caratheodory Metric c

Problem 7.3.1 *Is c complete on a bounded smooth pseudoconvex domain?*

Problem 7.3.2 *Is there a smoothly bounded pseudoconvex domain in \mathbb{C}^n on which the Kobayashi and Caratheodory metrics are not comparable?*

-S. Krantz, SK@math.wustl.edu

7.4 Finite Type

Problem 7.4.1 *Is the regular order of contact of a smooth pseudoconvex domain the same as the commutator type, one vector field at a time.*

-J. McNeal, mcneal@math.princeton.edu

Problem 7.4.2 *Are finite type pseudoconvex domains in \mathbb{C}^2 locally convexifiable with biholomorphic maps with continuous boundary values?*

7.5 The Kobayashi Metric k

Problem 7.5.1 *Is k complete on a bounded smooth pseudoconvex domain?*

Problem 7.5.2 *What's the asymptotic behavior of k on finite type domains in \mathbb{C}^n ?*

Problem 7.5.3 *Let $D = \{z \in \mathbb{C}^n; h(z) < 1\}$ be a bounded pseudoconvex complete-circular domain with Minkowski function h . Characterize the Caratheodory, the Bergman or the Kobayashi completeness of D via properties of h . [The following is known: a) If h is continuous, then D is Bergman complete. b) If $n \geq 3$, continuity of h does not imply the Kobayashi completeness of D .]*

-P. Pflug, pflugvec@dosuni1.bitnet

Problem 7.5.4 *Given a compact complex hypersurface X in \mathbb{P}^n with degree at least $n+2$ and only normal crossing singularities. Show that the infinitesimal Kobayashi pseudometric on $\mathbb{P}^n \setminus X$ is degenerate at most on a complex hypersurface.*

-N. Sibony, sibony@anh.matups.fr

Problem 7.5.5 *Find (or prove it does not exist) a complete hyperbolic manifold not biholomorphic to a convex domain where the Kobayashi and Carathéodory distances agree everywhere.*

-M. Abate, abate@vaxsns.infn.it

Problem 7.5.6 *Can you recover certain derivatives of the Levi form at a weakly pseudoconvex point in terms of the weighted boundary behavior of the Kobayashi metric?*

-S. Krantz, sk@math.wustl.edu

7.6 Regular Domains

Problem 7.6.1 *Let $\Omega \subset \subset \mathbb{C}^n$ be pseudoconvex with smooth boundary. For $K \subset \bar{\Omega}$ compact, let $\hat{K} := \{z \in \bar{\Omega}; \phi(z) \leq \sup_K \phi, \phi \text{ continuous on } \bar{\Omega}, \phi \text{ plurisubharmonic on } \Omega\}$. Assume $\hat{K} \cap \partial\Omega = K \cap \partial\Omega$. Then $\bar{\Omega}$ has a Stein neighborhood basis.*

-N. Sibony, sibony@anh.matups.fr

8 Plurisubharmonic Functions

8.1 The Complex Monge Ampere Operator

Problem 8.1.1 *Find conditions insuring that if a sequence of plurisubharmonic functions $u_m \rightarrow u$ in \mathbb{C}^n then $(dd^c u_m)^n \rightarrow (dd^c u)^n$.*

8.2 Extremal Functions

Problem 8.2.1 (New) Let V be a germ at 0 of an irreducible complex variety in \mathbb{C}^n . For any small enough $\epsilon > 0$, let $E = E_\epsilon := V \cap \mathbb{R}^n \cap \{|z| < \epsilon\}$. Define the extremal function U_E on $\{|z| < \epsilon\}$: $U_E(z) := \sup\{u(z); u \leq 0 \text{ on } E, 0 \leq u \leq 1 \text{ on } \{|z| < \epsilon\}\}$. The problem is to classify those V for which we have a constant $A > 0$ for which $(\alpha) U_E(z) \leq A|\Im z|, |z| < \epsilon/2$.

9 Holomorphic maps

9.1 Automorphisms

Problem 9.1.1 Which smooth domains have non compact automorphism groups?

9.2 The Jacobian Conjecture

Problem 9.2.1 If $P: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ is a holomorphic polynomial map with non-vanishing Jacobian, then P is biholomorphic.

Problem 9.2.2 (New) What is the lowest degree of a polynomial mapping $P: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ for which the (real) Jacobian conjecture fails, i.e. the Jacobian of P never vanishes while p fails to be invertible.

9.3 Proper correspondences

Problem 9.3.1 Classify Proper Holomorphic Correspondences from the unit disc to itself.

Problem 9.3.2 Is every proper holomorphic self-mapping of a smooth bounded domain in $\mathbb{C}^n (n > 1)$ biholomorphic?

9.4 Real analytic domains (not necessarily pseudoconvex) in \mathbb{C}^n , $f: U \rightarrow V$ biholomorphic

Problem 9.4.1 Does f have a continuous extension to \bar{U} ? (This is true if U, V are pseudoconvex.)

9.5 Stein manifolds

Problem 9.5.1 If M is an n -dimensional parallelizable Stein manifold, can M be holomorphically immersed in \mathbb{C}^n ? [True for $n=1$ (Gunning and Narasimhan); also such an M can be immersed in \mathbb{C}^{n+1} (Eliashberg-Gromov)].
-R. Narasimhan, Dept. of Math., Univ. of Chicago, Chicago, IL 60637

Problem 9.5.2 *Can $SL(n, \mathbb{C})$ be holomorphically immersed in \mathbb{C}^{n^2-1} . [Cannot be done algebraically, true if $n=2$]
-R. Narasimhan, Dept. of Math., Univ. of Chicago, Chicago, IL 60637*

Problem 9.5.3 *Can the algebraic hypersurface*

$$w^3 + wg(w_1, \dots, w_n) + f(w_1, \dots, w_n) = 1,$$

*where g is homogeneous of degree 2 and f homogeneous of degree 3, be holomorphically immersed in \mathbb{C}^n [Not true for algebraic immersion]
-R. Narasimhan, Dept. of Math., Univ. of Chicago, Chicago, IL 60637*

Problem 9.5.4 *Suppose that the polynomial equation*

$$F(w_0, w_1, \dots, w_n) = 1$$

*defines a smooth affine algebraic hypersurface (these are always parallelizable (Murthy, Swan, $n=2$; Suslin for general n). Can they be holomorphically immersed in \mathbb{C}^n ?
-R. Narasimhan, Dept. of Math., Univ. of Chicago, Chicago, IL 60637*

10 Dynamical properties of holomorphic maps

10.1 Dynamics of entire holomorphic maps

Problem 10.1.1 *Do there exist Fatou–Bieberbach domains which are not Runge? A Fatou–Bieberbach domain is a proper subdomain of \mathbb{C}^2 which is biholomorphic to \mathbb{C}^2 .*

-F. Forstneric, forstner@math.wisc.edu

10.2 The invariant set K^+ of a polynomial complex Henon map

Problem 10.2.1 *Do there exist wandering domains for complex Henon maps?*

Problem 10.2.2 *The Fatou Set of a Polynomial Automorphism of \mathbb{C}^2 . Let f be a polynomial automorphism of \mathbb{C}^2 , and suppose that f is not conjugate to an elementary map. Let Ω be a connected component of the set of points where the forward iterates $\{f^n : n = 1, 2, 3, \dots\}$ are locally bounded, which is the Fatou set of f . Let us suppose that Ω is periodic, i.e. $f^n\Omega = \Omega$ for some $n \neq 0$. Let us first consider the case in which Ω is recurrent, i.e. Ω contains a point whose orbit does not converge to $\partial\Omega$. In this case Ω is either (a) a basin of attraction of a sink orbit, (b) the basin of attractin of a Siegel disk, or (c) the basin of a Herman ring. It is not hard to construct examples of (a) and (b).*

(i) *Can case (c) occur? Nothing at all seems to be known about periodic domains which are not recurrent.*

(ii) *If Ω is not recurrent, does there exist $P \in \partial\Omega$ such that $\lim_{k \rightarrow \infty} f^{nk}(Q) = P$ for all $Q \in \Omega$?*

(iii) *If Ω is not recurrent, does there exist $P \in \partial\Omega$ such that $f^n P = P$, and one of the eigenvalues of $DF(P)$ is $e^{2\pi i p/q}$?*

-E. Bedford, bedford@iubacs.bitnet or bedford@ucs.indiana.edu.

Problem 10.2.3 *Let $F_c : (z, w) \rightarrow (z^2 + c - w, z)$, $c \in \mathbb{C}$ a symplectic automorphism of \mathbb{C}^2 , i.e. F_c is biholomorphic and $F_c^*(dz \wedge dw) = dz \wedge dw$. Can there exist a $c \in \mathbb{C}$ and a periodic orbit $\{z_i\}_{i=0}^k$ for F_c such that z_0 belongs to a Siegel domain ?*

-M. Herman

10.3 Dynamics on \mathbf{P}^k

Problem 10.3.1 Suppose f is a holomorphic map on \mathbf{P}^2 of degree $d \geq 2$. Must f have a repelling fixed point.

Problem 10.3.2 Does every holomorphic map $F : \mathbf{P}^k \rightarrow \mathbf{P}^k$ of degree at least two have a repelling periodic point? Do meromorphic maps have periodic orbits? -sibony@anh.matups.fr

Problem 10.3.3 Is the support of μ equal to the closure of the repelling periodic orbits?

Problem 10.3.4 Does there exist a holomorphic map $F : \mathbf{P}^k \rightarrow \mathbf{P}^k$ with a wandering Fatou component U , i.e. $F^n(U) \cap F^m(U) = \emptyset$ for all $n \neq m$?

Problem 10.3.5 Classify the dynamics around a fixed point of a holomorphic map on \mathbf{P}^k or even just defined in a neighborhood of p . -sibony@anh.matups.fr

Problem 10.3.6 Let $f : \mathbf{P}^2 \rightarrow \mathbf{P}^2$ be a holomorphic map of degree $d \geq 2$. Assume K is a totally invariant set. Let C denote the critical set of f . Assume $\overline{\bigcup_{n=1}^{\infty} f^n(C)} \cap K = \emptyset$. Is f hyperbolic on K ?

Problem 10.3.7 Let H_d denote the space of holomorphic maps $f : \mathbf{P}^2 \rightarrow \mathbf{P}^2$ of degree d . This is a finite dimensional space parametrized by the coefficients. Does the set of $f \in H_d$ with infinitely many attractive basins have measure zero?

Problem 10.3.8 Let $f : \mathbf{P}^2 \rightarrow \mathbf{P}^2$ be a holomorphic map of degree $d \geq 2$ and let $\lambda_1 \leq \lambda_2$ be the Lyapunov exponents for the ergodic measure $\mu = T \wedge T$. Is $\lambda_1 > 0$?

Problem 10.3.9 Classify critically finite maps on \mathbf{P}^3 .

11 Complex Analytic Varieties

11.1 Embedding Varieties

Problem 11.1.1 *Is every compact reduced analytic space biholomorphically equivalent to a subvariety of a complex manifold? If so, can the manifold be chosen to be compact?*

-E. L. Stout, stout@math.washington.edu

Problem 11.1.2 *Let C be a smooth cubic in \mathbb{P}^2 . Then $\mathbb{P}^2 \setminus C$ is algebraically parallelizable, but cannot be algebraically immersed in \mathbb{C}^2 . Can it be holomorphically immersed in \mathbb{C}^2 ? [There are smooth cubics for which it is true. For related problems see the section called "Stein manifolds".]*

-R. Narasimhan, Dept. of Math., Univ. of Chicago, Chicago, IL 60637

11.2 Varieties over the unit disc

Problem 11.2.1 *Consider the following statement: "Let Y be a compact subset of \mathbb{C}^2 which lies over the circle $|\lambda| = 1$, and let U be an open neighborhood of K , the polynomially convex hull of Y . Then there exists a finitely sheeted Riemann surface S lying over $|\lambda| < 1$ as a branched covering such that $S \subset U$." Is this statement true for every choice of Y and U ?*

-H. Alexander, u22330@wicvm.bitnet and J. Wermer, Brown University.

11.3 Complex Differential Geometry

Problem 11.3.1 *Let M be a Kähler manifold. Assume that all curvatures are between $-a^2$ and $-b^2$. Construct nontrivial bounded holomorphic functions on M .*

-N. Sibony, sibony@anh.matups.fr

12 CR manifolds

12.1 Embedding

Problem 12.1.1 *Are strongly pseudoconvex CR surfaces of real dimension 5 locally embeddable?*

12.2 Extensions

Problem 12.2.1 *Let $M \subset P^n$ be a compact odd dimensional CR manifold. Find conditions on M which ensure that M bounds a complex variety. (This is well understood in \mathbb{C}^n .)*

-N.Sibony, sibony@anh.matups.fr

12.3 Quasiconformal mappings

Problem 12.3.1 (New) (Liouville type assertion on CR 3-manifolds) On a CR 3-manifold one can always define a Carnot-Caratheodory metric. A homeomorphism between two strongly pseudoconvex CR 3-manifolds is called conformal if it maps infinitesimal spheres with respect to a Carnot-Caratheodory metric to infinitesimal spheres. Is a conformal homeomorphism CR?

-Puqi Tang, tang@math.purdue.edu

Problem 12.3.2 (New) A diffeomorphism between two strongly pseudoconvex CR $(2n + 1)$ -manifolds is called quasiconformal if its differential preserves the underlying contact structures and distorts the CR structures boundedly. When Hermitian metrics on the contact bundles are fixed, this distortion can be measured by checking how spheres in the contact space is mapped to ellipsoids. However, this measurement depends on the choices of the Hermitian metrics if $n > 1$. Fixing a quasiconformal diffeomorphism, can we choose Hermitian metrics so that the distortion is minimal?

- Puqi Tang, tang@math.purdue.edu

13 Scratchpad

Problem 13.0.3 *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain with complete Kobayashi metric. Is there a constant $C = C(p) > 0$ such that if f is holomorphic and $p > 0$ then for all $z \in \Omega$ and all $r > 0$ small it holds that*

$$|f(z)|^p \leq C \frac{1}{|B(z, r)|} \int_{B(z, r)} |f(\zeta)|^p dV(\zeta).$$

Here f is any holomorphic function and $B(z, R)$ is the Kobayashi metric ball. The measure being used is up for grabs. It can be Euclidean or one of the canonical measures associated with the Kobayashi metric construction (such as that due to Eisenman or Bun Wong). In case Ω is finite type in \mathbb{C}^2 or strongly pseudoconvex in any dimension then the results is true just because metric balls are comparable to polydiscs and then classical arguments of Stein (or see my SCV book) will suffice. In general I think that something like this should be true but I have no idea how to prove it.

-S. Krantz, sk@math.wustl.edu

14 Open prize problems

Prize Problem 1 *Prove that if Ω is strongly pseudoconvex then there is an absolute constant $K > 0$ such that the integrated Kobayashi distance between any two points can be realized by a Kobayashi chain of K discs. [For instance, on a convex domain, $K = 1$.]*

Prize \$50 -S. Krantz. sk @math.wustl.edu

Prize Problem 2 *Prove that if Ω is a smooth bounded pseudoconvex domain in \mathbb{C}^n which is finite type in the sense of D'Angelo, then it is of finite type in the sense of Kohn (ideal type).*

Prize \$50 -J. McNeal, mcneal@math.princeton.edu