1. For some \( k > 0 \), the functions \( f(x) = 2e^x \) and \( g(x) = kx \) are tangent for a value of \( x > 0 \). Find the values of \( x \) and \( k \) that result in this condition being true. (4 points)

Solution: If \( f(x) \) and \( g(x) \) are tangent, we know that they intersect and that their slopes are the same at the point of intersection. Call the intersection point \( x = a \) (which is the point of tangency). Then, at \( x = a \), \( f(a) = g(a) \) and (to match slopes) \( f'(a) = g'(a) \). This requires that \( 2e^a = ka \) and \( 2e^a = k \). Dividing the first equation by the second, we get \( a = 1 \), so that, plugging back in to either equation, \( k = 2e^1 = 2e \).

2. Find the linear approximation to the function \( h(x) \) defined implicitly by \( x^2y + 3xy^4 = 10 \) if we are interested in values of \( x \) and \( y \) near the point \((2,1)\). (3 points)

Solution: The linear approximation is just the tangent line at \( x = 2 \). To find this, we need a slope and the point \((2,1)\). The slope we can find by implicit differentiation. Differentiating both sides of the equation, we get \( 2xy + x^2 \frac{dy}{dx} + 3y^4 + 12xy^3 \frac{dy}{dx} = 0 \), so that \( \frac{dy}{dx} = -\frac{2xy + 3y^4}{x^2 + 12xy^3} \). At \((x,y) = (2,1)\), this is \( \frac{dy}{dx} = -\frac{4 + 3}{4 + 24} = -\frac{7}{28} = -\frac{1}{4} \). Therefore, using point-slope form, the linear approximation (tangent line) is \( y = 1 - \frac{1}{4}(x - 2) = \frac{3}{2} - \frac{1}{4}x \).

3. Suppose that the figure to the right shows \( f'(x) \) for some function \( f(x) \). Identify all critical points, local maxima and minima, and inflection points of the function \( f(x) \). (3 points)

Solution: We know that critical points (where there may be local maxima or minima) occur when \( f'(x) = 0 \), so critical points are when \( x = 1 \) and \( x = 3 \). Then \( f'(x) \) changes sign (from negative to positive) at \( x = 3 \), so \( x = 3 \) is a local minimum, while \( x = 1 \) (where \( f'(x) \) does not change sign) is neither a local maximum or minimum. Inflection points of \( f(x) \) occur when \( f''(x) \) has a local maximum or minimum, so, from the figure, we know that there is an inflection point of \( f(x) \) at \( x = 1 \) and at the local minimum of \( f'(x) \), approximately \( x = 2.25 \).