1. (4 points) Let \( f(x) \) be a function that is everywhere differentiable. Suppose that you know the values for \( f'(x) \) given in the table below.

\[
x = \begin{array}{cccccc}
-3 & -2 & -1 & 0 & 1 \\
2 & 0.5 & -0.5 & -1 & -0.5 \\
\end{array}
\]

a. Identify the location of any critical points and local maxima or local minima, if any, that this data indicates \( f(x) \) has.

b. If possible, identify the location of any inflection points of \( f(x) \), and the concavity of the graph of \( f(x) \). If it is not possible, briefly explain why.

**Solution:**
a. We know that \( f(x) \) has critical points at points where \( f'(x) \) is zero or undefined. Because \( f(x) \) is everywhere differentiable there are no undefined points, and given that \( f'(-2) = 0.5 \) and \( f'(-1) = -0.5 \) we know that there is a point between \( x = -2 \) and \( x = -1 \) where \( f'(x) = 0 \). So there is a critical point between these two \( x \)-values. Because the sign of \( f'(x) \) changes from positive to negative there, we know that this is a local maximum.

b. Inflection points occur where \( f(x) \) changes concavity, which is where \( f'(x) \) has a local maximum or minimum. From the data it is clear that \( f'(x) \) has a local minimum between \( x = -1 \) and \( x = 1 \), so there is an inflection point in the graph of \( f(x) \) between those two \( x \)-values. Because \( f''(x) \) is negative where \( f'(x) \) is decreasing, we know that \( f(x) \) is concave down before the inflection point and concave up thereafter.

2. (4 points) Consider the family of functions given by \( y = a \ln(x) + bx^2 \), with \( a \) and \( b \) both positive. If the graph of a member of this family has an inflection point at \( x = 3 \), what can you say about \( a \) and \( b \)?

**Solution:** We know that an inflection point is where \( f''(x) \) changes sign. Here, \( \frac{dy}{dx} = \frac{a}{x} + 2bx \) and \( \frac{d^2y}{dx^2} = -\frac{a}{x^2} + 2b \). Inflection points will occur when \( \frac{d^2y}{dx^2} = 0 \), which is \( -\frac{a}{x^2} + 2b = 0 \), or \( x = \sqrt{\frac{a}{2b}} \). (“Clearly”\(^\dagger\) as \( x \) goes through \( \sqrt{\frac{a}{2b}} \), \( \frac{a}{x^2} \) changes from greater than to less than \( 2b \), or vice versa, so \( \frac{d^2y}{dx^2} \) must change sign at both of these points. Thus they must actually be inflection points.)

Then, if the family is to have an inflection point at \( x = 3 \), we must have \( 3 = \sqrt{\frac{a}{2b}} \) or, \( 9 = \frac{a}{2b} \). Thus the condition \( a = 18b \) will result in members of the family having an inflection point at \( x = 3 \).

\(^\dagger\) Really it’s \( x = \pm \sqrt{\frac{a}{2b}} \), but because we started with a term involving \( \ln(x) \) the negative value doesn’t make sense.

\(^\dagger\) Check to make sure that this makes sense to you—try plugging in some values for \( a \) and \( b \) if you’re not sure about it.