1. (3 points) A stained glass window is to be made in the shape shown to the right, with a rectangular section surmounted by a semi-circular top. If \( P \) ft of border material are available, what should the dimensions of the window be to maximize its area? (You may assume that \( P > 2 \). The circumference of a circle is \( C = 2\pi r \).)

**Solution:** The area of the window is the sum of the area of the rectangular section and the area of the semi-circular top, so 
\[
A = 2rh + \frac{1}{2}\pi r^2.
\]
To get this to a single variable, use the constraint that the perimeter is \( P \) ft long:
\[
P = 2h + 2r + \frac{1}{2}\pi r = 2h + (2 + \pi)r,
\]
so, solving for \( h \),
\[
h = \frac{P}{2} - (1 + \frac{\pi}{2})r.
\]
Plugging this in to \( A \),
\[
A = \frac{P}{2}r - (1 + \frac{\pi}{2})\frac{1}{2}\pi r^2
\]
This will be maximized at the end points (\( r = 0 \), where \( h = \frac{P}{2} \), or \( r = \frac{P}{\pi} \), where \( h = 0 \)), or at critical points. Critical point(s) are where \( A'(r) = P - (4 + \pi)r = 0 \), or \( r = \frac{P}{4 + \pi} \). Here \( A''(r) = -(4 + \pi) \), so we know that \( A(r) \) is concave down, and therefore, because this is the only critical point, we know this will give the maximum value for \( A \). For this \( r \), \( h = \frac{P}{2} - (1 + \frac{\pi}{2})(\frac{P}{4 + \pi}) \).

*What’s with the \( P > 2 \)? It’s a red herring.*

2. (2 points) Suppose that the velocity of an orange-and-chartreuse-clothed math professor gradually increases in the course of a class period, and is given (in meters/second) by \( v(t) = 2e^{t^2} \) (where \( t \) is in hours). Use a Riemann sum with \( \Delta t = 0.5 \) hr to estimate the total distance travelled by the professor during an hour and a half class period.

**Solution:** Note that the velocity is in m/s, while time is given in hours (3600 sec). We know that the distance travelled is
\[
D = \int_0^{1.5} v(t) \, dt \approx (0.5)(v(0) + v(0.5) + v(1))
\]
\[
= (0.5)(2 + 2.57 + 5.44) = 5.01 \text{ m/s · hr}.
\]
Converting hours to seconds, this is 5.01 · 3600 = 18,036 m. We could, of course, use a right-hand sum instead, to get \( D \approx (0.5)(v(0.5) + v(1) + v(1.5)) = 13.50 \times 3600 \) m.

3. (3 points) Find the average value of the function shown to the right, for the domain shown. The arc in the figure is a semi-circle. Be sure it is clear how you obtain your answer.

**Solution:** The average value of the function, which we’ll call \( f(x) \), is \( \frac{1}{b-a} \int_a^b f(x) \, dx \). Here \( a = -1 \) and \( b = 4 \), and the value of the integral is equal to the area under the curve \( y = f(x) \) (treating area below the \( x \)-axis as negative). This area is shown in the figure, with negative area hatched instead of shaded. Thus, taking each geometrically distinct section of \( f \) in turn, \( \int_{-1}^4 f(x) \, dx = -\frac{1}{2} + \frac{1}{4}\pi + 1 + \frac{3}{2} = 2 + \frac{\pi}{2} \). The average value is therefore \( \frac{2}{5} + \frac{\pi}{10} \).