It may or may not be useful to note that:

\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - x^7 + \frac{x^9}{9!} - \cdots \]
\[ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots \]
\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \]
\[ (1 + x)^p = 1 + px + \frac{p(p-1)x^2}{2!} + \frac{p(p-1)(p-2)x^3}{3!} + \cdots \]

1. What is the radius of convergence of \( \sum_{n=0}^{\infty} \frac{3^n x^n}{n+2} \)? (3 points)

**Solution:** We find the radius of convergence by using the ratio test. We need \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1 \) for convergence. This limit is, in this case, \( \lim_{n \to \infty} \left| \frac{\frac{3^{n+1} x^{n+1}}{n+3} \cdot \frac{n+2}{3^{n+1} x^{n+1}}} \right| = \lim_{n \to \infty} \left| \frac{3x \cdot n+2}{n+3} \right| \). As \( n \to \infty \), the ratio \( \frac{n+2}{n+3} \to 1 \), so the limit is \( 3|x| \). We therefore need \( 3|x| < 1 \), and the radius of convergence is \( R = \frac{1}{3} \).

2. Suppose that the Taylor series for a function \( f(x) \) is given to be \( f(x) = 2x + \frac{8x^3}{3!} + \frac{32x^5}{4!} + \frac{128x^7}{6!} + \cdots \). What are \( f(0) \)? \( f'''(0) \)? \( f^{(19)}(0) \)? (3 points)

**Solution:** The Taylor series around \( x = 0 \) for any function \( f(x) \) is given by \( f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \cdots \). Thus \( f(0) = 0 \) and \( f'''(0) = 3! \left( \frac{8}{3!} \right) = 24 \). The \( n \)th term in the series is \( \frac{x^n}{(n-1)!} \), so the 19th derivative is given by \( f^{(19)}(0) = 19! \left( \frac{2^{19}}{18!} \right) = 19 \cdot 2^{19} \).

3. A wandering polar weasel meditates for 2.718 minutes and then sketches the graph to the right, which shows three functions for values of \( x \) near 0. Astonishingly, one of these turns out to be exactly \( \frac{1}{1-x^2} \), one \( 2 - \cos(x) \), while the third is another function that remains anonymous to protect its identity. Which of the graphs correspond to each of the two functions specified? (4 points)

**Solution:** We know that the geometric series \( \frac{1}{1-x} = 1 + x + x^2 + \cdots \), so \( \frac{1}{1-x^2} = 1 + x^2 + x^4 + \cdots \). Similarly, \( \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \), so that \( 2 - \cos(x) = 1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots \).

We note that both of these functions are even, so the dash-dotted graph, which is odd, is the anonymous function. Further, looking at the \( x^2 \) terms of both of these, we see that \( \frac{1}{1-x^2} \approx 1 + x^2 > 1 + \frac{x^2}{2!} \approx 2 - \cos(x) \), so near \( x = 0 \) we know that \( \frac{1}{1-x^2} > 2 - \cos(x) \). Thus the solid curve must be \( \frac{1}{1-x^2} \) and the dashed one \( 2 - \cos(x) \).