1. Recall that the Taylor series for \( \sin(x) \) is \( \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \). Find the Taylor series for \( \text{Si}(x) = \int_0^x \frac{\sin(t)}{t} \, dt \).

\[ \text{Solution:} \] We know that \( \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \), so \( \frac{\sin(t)}{t} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n+1)!} \). Integrating, we have

\[ \text{Si}(x) = \int_0^x \frac{\sin(t)}{t} \, dt = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n+1)!} \, dt. \]

We assume that we can integrate term-by-term, to get

\[ \text{Si}(x) = \sum_{n=0}^{\infty} \int_0^x \frac{(-1)^n t^{2n}}{(2n+1)!} \, dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}. \]

We could also work this out with an expanded form of the series: \( \text{Si}(x) = \int_0^x (1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots + (-1)^n \frac{x^{2n}}{(2n+1)!} + \cdots) \, dt = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)(2n+1)!} + \cdots \). Note that because \( \text{Si}(x) \) is defined as the integral from \( t = 0 \) to \( t = x \) we don’t have a constant of integration in this problem.

2. Suppose that we know that \( \frac{dy}{dx} = f(y) \) for some function \( f(y) \). Also suppose that we approximate the solution to this differential equation, with initial condition \( y(0) = 0 \), with Euler’s method and \( \Delta x = 0.5 \).

(a) If we find \( y(0.5) \approx 1 \), \( y(1) \approx 1.5 \), \( y(1.5) \approx 1.75 \), and \( y(2) \approx 1.875 \),

\[ \text{Solution:} \] We know that in general Euler’s method gives \( y(x + \Delta x) = y(x) + \Delta x f(y(x)) \) (that is, \( y_{n+1} = y_n + \Delta x f(y_n) \)). Thus we know that \( y(0.5) = 1 = 0 + 0.5 \cdot f(0) \), and thus \( f(0) = \frac{dy}{dx} \big|_{x=0} = 2 \). Similarly, with \( y(1) = 1.5 \), we have \( y(1) = 1.5 = 1 + 0.5 \cdot f(0.5) \), so \( f(0.5) = \frac{dy}{dx} \big|_{x=0.5} = 1 \). And finally, we have \( y(1.5) = 1.75 \), so \( y(1.5) = 1.75 = 1.5 + 0.5 \cdot f(1) \), and \( f(1) = \frac{dy}{dx} \big|_{x=1} = 0.5 \). Then we know that the slope of solutions to the differential equation are the same at any given \( y \) value, so that we have the slope field shown to the right.

(b) Give a rough sketch of the slope field of this differential equation.

3. Find all solutions to the differential equation \( \frac{1}{t} \frac{dx}{dt} + p = 2 \).

\[ \text{Solution:} \] Rearranging the equation, we have \( \frac{dx}{dt} = t(2 - p) \). Note that \( p = 2 \) is therefore a solution. If \( p \neq 2 \), then \( \frac{dx}{2p} = t \, dt \), so that \( -\ln|2 - p| = \frac{1}{2} t^2 + C \). Solving for \( p \), we have \( p = 2 - Ae^{-t^2/2} \). Thus solutions are \( p = 2 \) or \( p = 2 - Ae^{-t^2/2} \).