VECTORS CALULUS FORMULAS TO KNOW AND LOVE
(from Chapter 17 in Stewart)

First, in all of the following:

- The notation \( r(t) = \overrightarrow{r}(t) \) indicates a position vector that specifies a curve \( C \). We assume that \( a \leq t \leq b \).
  - For example, \( r = 3 \cos(t) i + 3 \sin(t) j + 0k \), for \( 0 \leq t \leq 2\pi \).
- Then, \( ds = |r'(t)| dt \) is the arclength element along a curve, and \( dr = r'(t) dt \) is the vector element along a curve.
- The notation \( r(u, v) = \overrightarrow{r}(u, v) \) indicates a position vector that is a function of any two variables and which specifies a surface \( S \). We assume that the variables \( (u, v) \in D \) for some domain \( D \). For example, \( r = <r \cos(\theta), r \sin(\theta), 4 - r^2> \), for \( 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi \).
- Then \( dS = |r_u \times r_v| du dv \) is the surface area element on a surface, and \( dS = r_u \times r_v du dv \) is the oriented vector surface area element (with vector normal to the surface).
- The function \( f(x, y) \) or \( f(x, y, z) \) is a scalar function that returns a value: e.g., \( f(x, y, z) = 3xy + \sin(z) \).
- The vector function \( \mathbf{F}(x, y) \) or \( \mathbf{F}(x, y, z) \) is a vector field that returns a vector: e.g., \( \mathbf{F} = 2xy \mathbf{i} + x^2 \mathbf{j} \). We will also use the notation \( \mathbf{F} = P \mathbf{i} + Q \mathbf{j} \) when referring to two-dimensional vector fields. For example, in the preceding vector field, \( P = 2xy \) and \( Q = x^2 \).
- Because we write \( \mathbf{F} = P \mathbf{i} + Q \mathbf{j} \) for two-dimensional \( \mathbf{F} \), the following integral notations are equivalent: \( \int \mathbf{F} \cdot dr = \int P \, dx + Q \, dy \).

Make sure that the notations indicated above are obvious to you—*to the point of your making the assumption that if you see \( r(t) \) written somewhere you immediately think “this defines a curve \( C \)” and start thinking about what curve it is, etc.*

Theorems of Vector Calculus

- **The Fundamental Theorem of Line Integrals**
  \[
  \int_C \nabla f \cdot dr = f(r(b)) - f(r(a))
  \]
  Note that this applies for conservative \( \mathbf{F} \): \( \int_C \mathbf{F} \cdot dr = f(r(b)) - f(r(a)) \).

- **Green’s Theorem**
  \[
  \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA = \oint_C \mathbf{F} \cdot dr
  \]
  Here \( D \) is a region in the \( xy \)-plane, and \( C \) is the curve that bounds \( D \), oriented counterclockwise. Therefore: Green’s theorem is only applicable to closed curves \( C \) that are in the \( xy \)-plane. Also note that for two-dimensional functions \( \mathbf{F}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \text{curl} \mathbf{F} \cdot \mathbf{k} \)—and that \( \mathbf{k} \) is the normal vector for a region (surface) in the \( xy \)-plane.

- **Stokes’ Theorem**
  \[
  \iint_S \text{curl} \mathbf{F} \cdot dS = \oint_C \mathbf{F} \cdot dr
  \]
  Here \( S \) is any bounded surface \( S \), and \( C \) is the curve that bounds \( S \), oriented “counterclockwise.” Therefore: Stokes’ theorem is only applicable to the odd cases when we want to find \( \iint_S \text{curl} \mathbf{F} \cdot dS \), and/or cases when we want to find \( \oint_C \mathbf{F} \cdot dr \) and there is a “nice” surface \( S \) bounded by \( C \).

- **the Divergence Theorem (Gauss’ Theorem)**
  \[
  \iiint_E \text{div} \mathbf{F} \, dV = \iint_S \mathbf{F} \cdot dS
  \]
  Here \( S \) is the surface that bounds the volume \( E \). Therefore: the divergence theorem is most applicable to cases when we want to find the flux through a closed surface.