FINITE GROUP ACTIONS ON REDUCTIVE GROUPS AND
BUILDINGS AND TAMELY-RAMIFIED DESCENT IN
BRUHAT-TITS THEORY

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Dedicated to Guy Rousseau

Abstract. Let $K$ be a discretely valued field with Henselian valuation ring and separably closed (but not necessarily perfect) residue field of characteristic $p$, $H$ a connected reductive $K$-group, and $\Theta$ a finite group of automorphisms of $H$. We assume that $p$ does not divide the order of $\Theta$ and Bruhat-Tits theory is available for $H$ over $K$ with $B(H/K)$ the Bruhat-Tits building of $H(K)$. We will show that then Bruhat-Tits theory is also available for $G := (H^\Theta)^\circ$ and $B(H/K)^\Theta$ is the Bruhat-Tits building of $G(K)$. (In case the residue field of $K$ is perfect, this result was proved in [PY1] by a different method.) As a consequence of this result, we obtain that if Bruhat-Tits theory is available for a connected reductive $K$-group $G$ over a finite tamely-ramified extension $L$ of $K$, then it is also available for $G$ over $K$ and $B(G/K) = B(G/L)^{\text{Gal}(L/K)}$. Using this, we prove that if $G$ is quasi-split over $L$, then it is already quasi-split over $K$.

Introduction. This paper is a sequel to our recent paper [P2]. We will assume familiarity with that paper; we will freely use results, notions and notations introduced in it.

Let $\mathcal{O}$ be a discretely valued Henselian local ring with valuation $\omega$. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}$ and $K$ the field of fractions of $\mathcal{O}$. We will assume throughout that the residue field $\kappa$ of $\mathcal{O}$ is separably closed. Let $\hat{\mathcal{O}}$ denote the completion of $\mathcal{O}$ with respect to the valuation $\omega$ and $\hat{K}$ the completion of $K$. For any $\mathcal{O}$-scheme $\mathcal{X}$, $\mathcal{X}(\mathcal{O})$ and $\mathcal{X}(\hat{\mathcal{O}})$ will always be assumed to carry the Hausdorff-topology induced from the metric-space topology on $\mathcal{O}$ and $\hat{\mathcal{O}}$ respectively. It is known that if $\mathcal{X}$ is smooth, then $\mathcal{X}(\mathcal{O})$ is dense in $\mathcal{X}(\hat{\mathcal{O}})$, [GGM, Prop. 3.5.2]. Similarly, for any $K$-variety $\mathcal{X}$, $\mathcal{X}(K)$ and $\mathcal{X}(\hat{K})$ will be assumed to carry the Hausdorff-topology induced from the metric-space topology on $K$ and $\hat{K}$ respectively. In case $\mathcal{X}$ is a smooth $K$-variety, $\mathcal{X}(K)$ is dense in $\mathcal{X}(\hat{K})$, [GGM, Prop. 3.5.2].

Throughout this paper $H$ will denote a connected reductive $K$-group. In this introduction, and beginning with §2 everywhere, we will assume that Bruhat-Tits theory is available for $H$ over $K$ [P2, 1.9, 1.10]. Then Bruhat-Tits theory is also available for the derived subgroup $\mathcal{D}(H)$ of $H$ over $K$ [P2, 1.11]. Thus there is an affine building called the Bruhat-Tits building of $H(K)$, that is a polysimplicial complex given with a metric, and $H(K)$ acts on it by polysimplicial isometries.
This building is also the Bruhat-Tits building of $\mathcal{D}(H)(K)$ and we will denote it by $\mathcal{B}(\mathcal{D}(H)/K)$. It is known (cf. [P2, 3.11, 1.11]) that Bruhat-Tits theory is also available over $K$ for the centralizer of any $K$-split torus in $H$ and for the derived subgroup of such centralizers.

Let $\mathfrak{Z}$ be the maximal $K$-split torus in the center of $H$. Let $V(\mathfrak{Z}) = \mathbb{R} \otimes_{\mathbb{Z}} \text{Hom}_K(\text{GL}_1, \mathfrak{Z}_K)$. Then there is a natural action of $H(K)$ on this Euclidean space by translations, with $\mathcal{D}(H)(K)$ acting trivially. The enlarged Bruhat-Tits building $\mathcal{B}(H/K)$ of $H(K)$ is the direct product $V(\mathfrak{Z}) \times \mathcal{B}(\mathcal{D}(H)/K)$. The apartments of this building, as well as that of $\mathcal{B}(\mathcal{D}(H)/K)$, are in bijective correspondence with maximal $K$-split tori of $H$. Given a maximal $K$-split torus $T$ of $H$, the corresponding apartment of $\mathcal{B}(H/K)$ is an affine space under $V(T) := \mathbb{R} \otimes_{\mathbb{Z}} \text{Hom}_K(\text{GL}_1, T)$.

Given a nonempty bounded subset $\Omega$ of an apartment of $\mathcal{B}(\mathcal{D}(H)/K)$, there is a smooth affine $\mathcal{O}$-group scheme $\mathcal{H}_\Omega$ with generic fiber $H$, associated with $\Omega$, such that $\mathcal{H}_\Omega(0)$ is the subgroup $H(K)^{\mathcal{O}}$ of $H(K)$ consisting of elements that fix $V(\mathfrak{Z}) \times \Omega \subset \mathcal{B}(H/K)$ pointwise [P2, 1.9.1.10]. The neutral component $\mathcal{H}_\Omega^0$ of $\mathcal{H}_\Omega$ is an open affine $\mathcal{O}$-subgroup scheme of the latter; it is by definition the union of the generic fiber $H$ of $\mathcal{H}_\Omega$ and the identity component of its special fiber. The group scheme $\mathcal{H}_\Omega^0$ is called the Bruhat-Tits group scheme associated to $\Omega$. The special fiber of $\mathcal{H}_\Omega^0$ will be denoted by $\mathcal{H}_\Omega^0$.

Let $\Theta$ be a finite group of automorphisms of $H$. We assume that the order of $\Theta$ is not divisible by the characteristic of the residue field $\kappa$. Let $G = (H^\Theta)^0$. This group is also reductive, see [Ri, Prop. 10.1.5] or [PY1, Thm. 2.1]. The goal of this paper is to show that Bruhat-Tits theory is available for $G$ over $K$, and the enlarged Bruhat-Tits building of $G(K)$ can be identified with the subspace $\mathcal{B}(H/K)^\Theta$ of $\mathcal{B}(H/K)$ consisting of points fixed under $\Theta$ (see §3). These results have been inspired by the main theorem of [PY1], which implies that if the residue field $\kappa$ is algebraically closed (then every reductive $K$-group is quasi-split [P2, 1.7], so Bruhat-Tits theory is available for any such group over $K$), the enlarged Bruhat-Tits building of $G(K)$ is indeed $\mathcal{B}(H/K)^\Theta$.

In §4, we will use the above results to obtain “tamely-ramified descent”: (1) We will show that if a connected reductive $K$-group $G$ is quasi-split over a finite tamely-ramified extension $L$ of $K$, then it is quasi-split over $K$ (Theorem 4.4); this result has been proved by Philippe Gille in [Gi] by an entirely different method. (2) The enlarged Bruhat-Tits building $\mathcal{B}(G/K)$ of $G(K)$ can be identified with the subspace of points of the enlarged Bruhat-Tits building of $G(L)$ that are fixed under the action of the Galois group $\text{Gal}(L/K)$. This latter result was proved by Guy Rousseau in his unpublished thesis [Rou, Prop. 5.1.1]. It is a pleasure to dedicate this paper to him for his important contributions to Bruhat-Tits theory.

Acknowledgements. I thank Brian Conrad, Bas Edixhoven and Philippe Gille for their helpful comments. I thank the referee for carefully reading the paper and for her/his detailed comments and suggestions which helped me to improve the exposition. I was partially supported by NSF-grant DMS-1401380.
For a $K$-split torus $S$, let $X_*(S) = \text{Hom}(\text{GL}_1, S)$ and $V(S) := \mathbb{R} \otimes_{\mathbb{Z}} X_*(S)$. Then for a maximal $K$-split torus $T$ of $H$, the apartment $A(T)$ of $\mathcal{B}(H/K)$ corresponding to $T$ is an affine space under $V(T)$.

1. Passage to completion

We begin by proving the following well-known result.

**Proposition 1.1.** $K$-rank $H = \hat{K}$-rank $H$.

*Proof.* Let $T$ be a maximal $K$-split torus of $H$ and $Z$ be its centralizer in $H$. Let $Z_a$ be the maximal $K$-anisotropic connected normal subgroup of $Z$. Then

$$\hat{K} \text{-rank } H = \hat{K} \text{-rank } Z = \dim(T) + \hat{K} \text{-rank } Z_a = K \text{-rank } H + \hat{K} \text{-rank } Z_a.$$ 

So to prove the proposition, it suffices to show that $Z_a$ is anisotropic over $\hat{K}$. But according to Theorem 1.1 of [P2], $Z_a$ is anisotropic over $\hat{K}$ if and only if $Z_a(\hat{K})$ is bounded. The same theorem implies that $Z_a(K)$ is bounded. As $Z_a(K)$ is dense in $Z_a(\hat{K})$, we see that $Z_a(\hat{K})$ is bounded. \hfill $\square$

**Proposition 1.2.** Bruhat-Tits theory for $H$ is available over $K$ if and only if it is available over $\hat{K}$. Moreover, if Bruhat-Tits theory for $H$ is available over $K$, then the enlarged Bruhat-Tits buildings of $H(K)$ and $H(\hat{K})$ are equal.

It was shown by Guy Rousseau in his thesis that the enlarged Bruhat-Tits buildings of $H(K)$ and $H(\hat{K})$ coincide [Rou, Prop. 2.3.5]. Moreover, every apartment in the building of $H(K)$ is also an apartment in the building of $H(\hat{K})$; however, the latter may have many more apartments.

*Proof.* We assume first that Bruhat-Tits theory is available for $H$ over $K$ and let $\mathcal{B}(H/K)$ denote the enlarged Bruhat-Tits building of $H(K)$. We begin by showing that the action of $H(K)$ on $\mathcal{B}(H/K)$ extends to an action of $H(\hat{K})$ by isometries. For this purpose, we recall that $H(K)$ is dense in $H(\hat{K})$ and the isotropy at any point $x \in \mathcal{B}(H/K)$ is a bounded open subgroup of $H(K)$. Now let $\{h_i\}$ be a sequence in $H(K)$ which converges to a point $\hat{h} \in H(\hat{K})$, then given any open subgroup of $H(K)$, for all large $i$ and $j$, $h_i^{-1} h_j$ lies in this open subgroup. Thus for any point $x \in \mathcal{B}(H/K)$, the sequence $h_i \cdot x$ is eventually constant, i.e., there exists a positive integer $n$ such that $h_i \cdot x = h_n \cdot x$ for all $i \geq n$. We define $\hat{h} \cdot x = h_n \cdot x$. This gives a well-defined action of $H(\hat{K})$ on $\mathcal{B}(H/K)$ by isometries.

For a nonempty bounded subset $\Omega$ of an apartment of the Bruhat-Tits building $\mathcal{B}(\mathcal{O}(H)/K)$, let $\mathcal{H}_{\Omega}$ and $\mathcal{H}_{\Omega}^\circ$ be the smooth affine $\mathcal{O}$-group schemes as in the Introduction. Then as $\mathcal{H}_{\Omega}(\mathcal{O})$ is a closed and open subgroup of $H(\hat{K})$ containing $\mathcal{H}_{\Omega}(\mathcal{O})$ as a dense subgroup, we see that $\mathcal{H}_{\Omega}(\mathcal{O})$ equals the subgroup $H(\hat{K})^\Omega$ of $H(\hat{K})$ consisting of elements that fix $V(\mathcal{O}) \times \Omega$ pointwise.

Let $T$ be a maximal $K$-split torus of $H$, then by Proposition 1.1, $T_{\hat{K}}$ is a maximal $\hat{K}$-split torus of $H_{\hat{K}}$. Let $A$ be the apartment of $\mathcal{B}(H/K)$, or of $\mathcal{B}(\mathcal{O}(H)/K)$,
corresponding to $T$. Then every maximal $\hat{K}$-split torus of $H_\mathbb{R}$ is of the form $\hat{h}T_\mathbb{R}\hat{h}^{-1}$ for an $\hat{h} \in H(\hat{K})$, and we define the corresponding apartment to be $\hat{h} \cdot A$. We now declare $\mathcal{B}(H/K)$ (resp. $\mathcal{B}(\mathcal{D}(H)/K)$) to be the enlarged Bruhat-Tits building (resp. the Bruhat-Tits building) of $H(\hat{K})$ with these apartments.

Let $A$ be an apartment of the Bruhat-Tits building of $H(K)$ corresponding to a maximal $K$-split torus $T$ of $H$ and $\hat{h} \in H(\hat{K})$. Given a nonempty bounded subset $\hat{\Omega}$ of $\hat{A} := \hat{h} \cdot A$, the subset $\Omega := \hat{h}^{-1} \cdot \hat{\Omega}$ is contained in $A$. The closed and open subgroup $\hat{h}H(\hat{K})^{\Omega} \hat{h}^{-1} = \hat{h}\mathcal{H}_\Omega(\hat{\Omega})\hat{h}^{-1}$ of $H(\hat{K})$ is the subgroup $H(\hat{K})^{\hat{\Omega}}$ consisting of elements that fix $V(3) \times \hat{\Omega}$ pointwise. Now as $H(K)$ is dense in $H(\hat{K})$ and $H(\hat{K})^{\hat{\Omega}}$ is an open subgroup, $H(\hat{K}) = H(\hat{K})^{\hat{\Omega}} : H(K)$, so $\hat{h} = h' \cdot h$, with $h' \in H(\hat{K})^{\hat{\Omega}}$ and $h \in H(K)$. Thus the apartment $\hat{A} = \hat{h} \cdot A = h' \cdot hA$, and $hA$ is an apartment of the Bruhat-Tits building of $H(K)$. As $h' \in H(\hat{K})^{\hat{\Omega}}$, the apartment $hA$ contains $\hat{\Omega}$. This shows that any bounded subset $\hat{\Omega}$ of an apartment of the Bruhat-Tits building of $H(\hat{K})$ is contained in an apartment of the Bruhat-Tits building of $H(K)$. We define the $\hat{\Omega}$-group schemes $\mathcal{H}_\hat{\Omega}$ and $\mathcal{H}_\hat{\Omega}^2$ associated to $\hat{\Omega}$ to be the group schemes obtained from the corresponding $\hat{\Omega}$-group schemes (given by considering $\hat{\Omega}$ to be a nonempty bounded subset of an apartment of the building of $H(\hat{K})$) by extension of scalars $\mathbb{O} \hookrightarrow \hat{\mathbb{O}}$.

Let us assume now that Bruhat-Tits theory is available for $H$ over $\hat{K}$. Then Bruhat-Tits theory is also available for $\mathcal{D}(H)$ over $\hat{K}$ [P2, 1.11]. The action of $H(\hat{K})$ on its building $\mathcal{B}(\mathcal{D}(H)/\hat{K})$ restricts to an action of $H(K)$ by isometries. Let $T$ be a maximal $K$-split torus of $G$ and $A$ be the apartment of $\mathcal{B}(\mathcal{D}(H)/\hat{K})$ corresponding to $T_\mathbb{R}$. We consider the polysimplicial complex $\mathcal{B}(\mathcal{D}(H)/\hat{K})$, with apartments $\hat{h} \cdot A$, $h \in H(K)$, as the building of $H(K)$ and denote it by $\mathcal{B}(\mathcal{D}(H)/K)$.

Let $\hat{\Omega}$ be a nonempty bounded subset of the apartment $\hat{A} := \hat{h} \cdot A$, $\hat{h} \in H(\hat{K})$, in the building $\mathcal{B}(\mathcal{D}(H)/\hat{K})$. As $H(K)$ is dense in $H(\hat{K})$, the intersection $\mathcal{H}_\hat{\Omega}^{\hat{\Omega}}(\hat{\Omega})\hat{h} \cap H(K)$ is nonempty. For any $h$ in this intersection, $\hat{\Omega}$ is contained in the apartment $h \cdot A$ of $\mathcal{B}(\mathcal{D}(H)/\hat{K})$. This implies, in particular, that any two facets lie on an apartment of $\mathcal{B}(\mathcal{D}(H)/\hat{K})$. We now note that the $\hat{\Omega}$-group schemes $\mathcal{H}_\hat{\Omega}$ and $\mathcal{H}_\hat{\Omega}^2$ admit unique descents to smooth affine $\hat{\mathbb{O}}$-group schemes with generic fiber $H$, [BLR, Prop. D.4(b) in §6.1]; the affine rings of these descents are $K[H] \cap \hat{\mathcal{H}}_\hat{\Omega}$ and $K[H] \cap \hat{\mathcal{H}}^{\hat{\Omega}}$ respectively.

In view of the preceding proposition, we may (and do) replace $\hat{\mathbb{O}}$ and $K$ with $\hat{\mathcal{H}}_\hat{\Omega}$ and $\hat{K}$ respectively to assume in the rest of this paper that $\hat{\mathbb{O}}$ and $\hat{K}$ are complete.

2. Fixed points in $\mathcal{B}(H/K)$ under a finite automorphism group $\Theta$ of $H$

We will henceforth assume that Bruhat-Tits theory is available for $H$ over $K$. 

Let $G$ be a smooth affine $K$-group and $\mathcal{G}$ be a smooth affine $\mathcal{O}$-group scheme with generic fiber $G$. According to [BrT2, 1.7.1-1.7.2] $\mathcal{G}$ is “étalé” and hence by (ET) of [BrT2, 1.7.1] its affine ring has the following description:

$$\mathcal{O}[\mathcal{G}] = \{ f \in K[G] \mid f(\mathcal{G}(\mathcal{O})) \subset \mathcal{O} \}.$$ 

Let $\Omega$ be a nonempty bounded subset of an apartment of $\mathcal{B}(\mathcal{D}(H)/K)$. As the $\mathcal{O}$-group scheme $\mathcal{H}_{\Omega}$ is smooth and affine and its generic fiber is $H$, the affine ring of $\mathcal{H}_{\Omega}$ has thus the following description:

$$\mathcal{O}[\mathcal{H}_{\Omega}] = \{ f \in K[H] \mid f(H(K)^{\Omega}) \subset \mathcal{O} \}.$$ 

**Proposition 2.2.** Let $\Omega$ be a nonempty bounded subset of an apartment of $\mathcal{B}(\mathcal{D}(H)/K)$. Let $\mathcal{H}_{\Omega}$ and $\mathcal{H}_{\Omega}^{\circ}$ be as above. Let $G$ be a smooth connected $K$-subgroup of $H$ and $\mathcal{T}$ be a smooth affine $\mathcal{O}$-group scheme with generic fiber $G$ and connected special fiber. Assume that a subgroup $\mathcal{S}$ of $\mathcal{T}(\mathcal{O})$ of finite index fixes $\Omega$ pointwise (i.e., $\mathcal{S} \subset H(K)^{\Omega}$). Then there is a $\mathcal{O}$-group scheme homomorphism $\varphi : \mathcal{G} \to \mathcal{H}_{\Omega}^{\circ}$ that is the natural inclusion $G \hookrightarrow H$ on the generic fibers. So the subgroup $\mathcal{G}(\mathcal{O})$ of $G(K)$ is contained in $\mathcal{H}_{\Omega}^{\circ}(\mathcal{O})$ and hence it fixes $\Omega$ pointwise. If $F$ is a facet of $\mathcal{B}(\mathcal{D}(H)/K)$ that meets $\Omega$, then $\mathcal{G}(\mathcal{O})$ fixes $F$ pointwise.

Let $S$ be a $K$-split torus of $H$ and $\mathcal{F}$ the $\mathcal{O}$-torus with generic fiber $S$. If a subgroup of the maximal bounded subgroup $\mathcal{S}(\mathcal{O})$ of $S(K)$ of finite index fixes $\Omega$ pointwise, then there is a maximal $K$-split torus $T$ of $H$ containing $S$ such that $\Omega$ is contained in the apartment of $\mathcal{B}(\mathcal{D}(H)/K)$ corresponding to $T$.

**Proof.** Since the fibers of the smooth affine group scheme $\mathcal{G}$ are connected and the residue field $\kappa$ is separably closed, the subgroup $\mathcal{S}$ is Zariski-dense in $G$, and its image in $\mathcal{G}(\kappa)$ is Zariski-dense in the spacial fiber of $\mathcal{G}$. Using this observation, we easily see that the affine ring $\mathcal{O}[\mathcal{G}] (\subset K[G])$ of $\mathcal{G}$ has the following description (cf. [BrT2, 1.7.2]):

$$\mathcal{O}[\mathcal{G}] = \{ f \in K[G] \mid f(\mathcal{S}) \subset \mathcal{O} \}.$$ 

This description of $\mathcal{O}[\mathcal{G}]$ implies at once that the inclusion $\mathcal{S} \hookrightarrow H(K)^{\Omega}$ induces a $\mathcal{O}$-group scheme homomorphism $\varphi : \mathcal{G} \to \mathcal{H}_{\Omega}^{\circ}$ that is the natural inclusion $G \hookrightarrow H$ on the generic fibers. Since $\mathcal{G}$ has connected fibers, the homomorphism $\varphi$ factors through $\mathcal{H}_{\Omega}^{\circ}$.

Any facet $F$ of $\mathcal{B}(\mathcal{D}(H)/K)$ that meets $\Omega$ is stable under $\mathcal{G}(\mathcal{O}) (\subset H(K))$, so a subgroup of $\mathcal{G}(\mathcal{O})$ of finite index fixes it pointwise. Now applying the result of the preceding paragraph, for $F$ in place of $\Omega$, we see that there is a $\mathcal{O}$-group scheme homomorphism $\mathcal{G} \to \mathcal{H}_{F}^{\circ}$ that is the natural inclusion $G \hookrightarrow H$ on the generic fibers and hence $\mathcal{G}(\mathcal{O})$ fixes $F$ pointwise.

Now we will prove the last assertion of the proposition. It follows from what we have shown above that there is a $\mathcal{O}$-group scheme homomorphism $\iota : \mathcal{G} \to \mathcal{H}_{\Omega}^{\circ}$ that is the natural inclusion $S \hookrightarrow H$ on the generic fibers ($\iota$ is actually a closed immersion, see [PY2, Lemma 4.1]). Applying [P2, Prop. 2.1(i)] to the centralizer of $\iota(\mathcal{G})$ (in $\mathcal{H}_{\Omega}^{\circ}$) in place of $\mathcal{G}$, and $\mathcal{O}$ in place of $\mathcal{O}$, we see that there is a closed $\mathcal{O}$-torus.
The following is a simple consequence of the preceding proposition.

**Corollary 2.3.** Let $G$, $S$, $\mathcal{I}$, and $\mathcal{I}'$ be as in the preceding proposition. Then the set of points of $\mathcal{B}(\mathcal{I}(H)/K)$ that are fixed under $\mathcal{I}(\Theta)$ is the union of facets pointwise fixed under $\mathcal{I}(\Theta)$. The set of points of the enlarged building $\mathcal{B}(H/K)$ that are fixed under a finite-index subgroup $S$ of the maximal bounded subgroup $S(K)_b (= \mathcal{I}(\Theta))$ of $S(K)$ is the enlarged Bruhat-Tits building $\mathcal{B}(Z_H(S)/K)$ of the centralizer $Z_H(S)(K)$ of $S$ in $H(K)$.

2.4. Let $\Theta$ be a finite group of automorphisms of the reductive $K$-group $H$. There is a natural action of $\Theta$ on the Bruhat-Tits building $\mathcal{B}(\mathcal{I}(H)/K)$ of $H(K)$ by polysimplicial isometries such that for all $h \in H(K)$, $x \in \mathcal{B}(\mathcal{I}(H)/K)$ and $\theta \in \Theta$, we have $\theta(h \cdot x) = \theta(h) \cdot \theta(x)$.

Let $\Omega$ be a nonempty bounded subset of an apartment of $\mathcal{B}(\mathcal{I}(H)/K)$. Assume that $\Omega$ is stable under the action of $\Theta$ on $\mathcal{B}(\mathcal{I}(H)/K)$. Then $\mathcal{H}_\Omega(\Theta)$ is stable under the action of $\Theta$ on $H(K)$, so the affine ring $\mathcal{O}[\mathcal{H}_\Omega]$ is stable under the action of $\Theta$ on $K[H]$. This implies that $\Theta$ acts on the group scheme $\mathcal{H}_\Omega$ by $\Theta$-group scheme automorphisms. The neutral component $\mathcal{H}_\Omega^0$ of $\mathcal{H}_\Omega$ is of course stable under this action.

In the following we assume that the characteristic $p$ of the residue field $\kappa$ does not divide the order of $\Theta$. Then $G := (H^\Theta)^\circ$ is a reductive group, see [Ri, Prop. 10.1.5] or [PY1, Thm. 2.1]. We will prove that Bruhat-Tits theory is available for $G$ over $K$ and the enlarged Bruhat-Tits building of $G(K)$, as a metric space, can be identified with the subspace $\mathcal{B}(H/K)^{\Theta}$ of points of $\mathcal{B}(H/K)$ fixed under $\Theta$.

Let $C$ be the maximal $K$-split central torus of $G$ and $H'$ be the derived subgroup of the centralizer of $C$ in $H$. Then $H'$ is a connected semi-simple subgroup of $H$ stable under the group $\Theta$ of automorphisms of $H$; $(H'^\Theta)^\circ (\subset G)$ contains the derived subgroup of $G$ and its central torus is $K$-anisotropic. Replacing $H$ with $H'$ we assume in the sequel that $H$ is semi-simple and the central torus of $G$ is $K$-anisotropic (cf. [P2, 3.11, 1.11]).

For a subset $X$ of a set given with an action of $\Theta$, we denote by $X^\Theta$ the subset of points of $X$ that are fixed under $\Theta$. We will denote $\mathcal{B}(H/K)^{\Theta}$ by $\mathcal{B}$ in the sequel.

If a facet of $\mathcal{B}(H/K)$ is stable under the action of $\Theta$, then its barycenter is fixed under $\Theta$. Conversely, if a facet $F$ contains a point $x$ fixed under $\Theta$, then being the unique facet containing $x$, $F$ is stable under the action of $\Theta$.

2.5. We introduce the following partial order “$\prec$” on the set of nonempty subsets of $\mathcal{B}(H/K)$: Given two nonempty subsets $\Omega$ and $\Omega'$, $\Omega' \prec \Omega$ if the closure $\overline{\Omega}$ of $\Omega$ contains $\Omega'$. If $F$ and $F'$ are facets of $\mathcal{B}(H/K)$, with $F' \prec F$, or equivalently, $\mathcal{H}_{F'}(\Theta) \subset \mathcal{H}_F(\Theta)$, we say that $F'$ is a face of $F$. In a collection $\mathcal{C}$ of facets, thus a
facet is \textit{maximal} if it is not a proper face of any facet belonging to \( \mathcal{C} \), and a facet is \textit{minimal} if no proper face of it belongs to \( \mathcal{C} \).

Now let \( X \) be a convex subset of \( \mathcal{B}(H/K) \) and \( \mathcal{C} \) be the set of facets of \( \mathcal{B}(H/K) \), or facets lying in a given apartment \( A \), that meet \( X \). Then the following assertions are easy to prove (see Proposition 9.2.5 of [BrT1]): (1) All maximal facets in \( \mathcal{C} \) are of equal dimension and a facet \( F \in \mathcal{C} \) is maximal if and only if \( \dim(F \cap X) \) is maximal. (2) Let \( F \) be a facet lying in an apartment \( A \). Assume that \( F \) is maximal among the facets of \( A \) that meet \( X \), and let \( A_F \) be the affine subspace of \( A \) spanned by \( F \). Then every facet of \( A \) that meets \( X \) is contained in \( A_F \) and \( A \cap X \) is contained in the affine subspace of \( A \) spanned by \( F \cap X \).

The subset \( \mathcal{B} = \mathcal{B}(H/K)^{\Theta} \) of \( \mathcal{B}(H/K) \) is closed and convex. Hence the assertions of the preceding paragraph hold for \( \mathcal{B} \) in place of \( X \). We will show in this section that \( \mathcal{B} \) is an affine building with apartments described below. We begin with the following proposition which has been suggested by Proposition 1.1 of [PY1], and the proof given here is an adaptation of the proof of that proposition.

\textbf{Proposition 2.6.} Let \( A \) be an apartment of \( \mathcal{B}(H/K) \) and \( F \) a facet of \( A \) that meets \( \mathcal{B} \). Let \( \Omega \) be a nonempty bounded subset of the affine subspace \( A_F \) of \( A \) spanned by \( F \). We assume that \( \Omega \) contains \( F \) and is stable under the action of \( \Theta \) on \( \mathcal{B}(H/K) \). Let \( \mathcal{H} := \mathcal{H}_\Omega^\Theta \) be the Bruhat-Tits smooth affine \( \mathcal{O} \)-group scheme with generic fiber \( H \), and connected special fiber \( \mathcal{H}^\text{pred} := \mathcal{H} \big/ \mathcal{R}_u \kappa(\mathcal{H}) \), associated with \( \Omega \). Let \( \mathcal{H}^\text{pred} \) be the maximal pseudo-reductive quotient of \( \mathcal{H} \). Then there exist \( K \)-split tori \( S \subset T \) in \( H \) such that

(i) \( T \) is a maximal \( K \)-split torus of \( H \) and \( \Omega \) is contained in the apartment \( A(T) \) corresponding to \( T \);

(ii) \( S \) is stable under \( \Theta \) and the special fiber of the schematic closure \( \mathcal{I} \) of \( S \) in \( \mathcal{H} \) maps onto the central torus of \( \mathcal{H}^\text{pred} \).

\textbf{Proof.} Let \( \mathcal{I} \) be the set of maximal \( K \)-split tori \( T \) of \( H \) such that \( \Omega \subset A(T) \). Then the automorphism group \( \Theta \) clearly permutes \( \mathcal{I} \), and the subgroup \( \mathcal{P} := \mathcal{H}(\mathcal{O}) \) acts transitively on \( \mathcal{I} \) [P2, Prop. 2.2(i)]. Hence, for every \( T \in \mathcal{I} \), \( \Omega \) is contained in the affine subspace of \( A(T) \) spanned by the facet \( F \).

For \( T \in \mathcal{I} \), let \( S_T \) be the lift of the central torus of \( \mathcal{H}^\text{pred} \) in \( T \). It is clear that the pair \( (S, T) \) satisfy (i) and (ii) if \( S \) is \( \Theta \)-stable. We consider \( S := \{ S_T \mid T \in \mathcal{I} \} \); \( \Theta \) acts by permutation on \( S \) and \( \mathcal{P} \) acts transitively on it. We will find an element of \( S \) that is \( \Theta \)-stable. We first prove the following lemma.

\textbf{Lemma 2.7.} Let \( T \in \mathcal{I} \) and \( S := S_T \) be as above. Then

(i) The normalizer of \( S \) in \( \mathcal{P} \) centralizes \( S \).

(ii) \( \mathcal{P} = \mathcal{P}_S \cdot \mathcal{U} \), where \( \mathcal{P}_S \) is the centralizer of \( S \) in \( \mathcal{P} \) and \( \mathcal{U} \) is the kernel of the natural homomorphism \( \mathcal{H}(\mathcal{O}) \to \mathcal{H}^\text{pred}(\kappa) \).
Proof. (i) The affine subspace \( A(T)_F \) of \( A(T) \) spanned by \( F \) is an affine space under the \( \mathbb{R} \)-vector space \( V(S) \). So for any \( x \in F \), \( V(S) + x = A(T)_F \). Now let \( h \) be an element of \( \mathcal{P} \) that normalizes \( S \). Then \( h \) takes \( A(T)_F = V(S) + x (\subset A(T)) \) to \( V(S) + h \cdot x = V(S) + x (\subset A(hTh^{-1})) \) by an affine transformation whose derivative gives the action of \( h \) on \( V(S) \). As \( h \) fixes the open subset \( F \) of \( A(T)_F \) pointwise, its derivative acts trivially on \( V(S) \) and hence \( h \) centralizes \( S \).

(ii) Let \( I \) and \( T \) be the closed \( \Theta \)-tori in \( \mathcal{H} \) with generic fibers \( S \) and \( T \) respectively. Then the centralizer \( \mathcal{H}^I \) of \( I \) in \( \mathcal{H} \) is a smooth affine \( \Theta \)-subgroup scheme [CGP, Prop. A.8.10(2)]. Let \( \mathcal{F} \) be the special fiber of \( I \) and \( \mathcal{H}^\mathcal{F} \) be the centralizer of \( \mathcal{F} \) in the special fiber \( \mathcal{H} \) of \( \mathcal{H} \). Since \( \mathcal{O} \) is Henselian, the natural map \( (\mathcal{P}_S =) \mathcal{H}^I(0) \to \mathcal{H}^\mathcal{F}(\kappa) \) is surjective [EGAIV \( 1 \) 18.5.17]. As the image of \( \mathcal{F} \) in \( \mathcal{H}^\mathcal{P}_{\text{pred}} \) is central, the natural homomorphism \( \mathcal{H}^\mathcal{F} \to \mathcal{H}^\mathcal{P}_{\text{pred}} \) is surjective (see [Bo, Prop. 9.6]). On the other hand, \( \mathcal{R}_{u,\kappa}(\mathcal{H}) \cap \mathcal{H}^\mathcal{F} = \mathcal{R}_{u,\kappa}(\mathcal{H}^\mathcal{F}) \) ([CGP, Prop. A.8.14]; note that as \( \mathcal{F} \) is a torus, both \( \mathcal{H}^\mathcal{F} \) and \( (\mathcal{R}_{u,\kappa}(\mathcal{H}))^\mathcal{F} = \mathcal{R}_{u,\kappa}(\mathcal{H}) \cap \mathcal{H}^\mathcal{F} \) are smooth and connected). So the natural map \( \mathcal{H}^\mathcal{F} / \mathcal{R}_{u,\kappa}(\mathcal{H}^\mathcal{F}) \to \mathcal{H}^\mathcal{P}_{\text{pred}} \) is an isomorphism. Since \( \kappa \) is separably closed, this implies that \( \mathcal{H}^\mathcal{F}(\kappa) \to \mathcal{H}^\mathcal{P}_{\text{pred}}(\kappa) \) is surjective. Hence, the map \( \mathcal{P}_S \to \mathcal{H}^\mathcal{P}_{\text{pred}}(\kappa) \) is surjective too. From this we conclude that \( \mathcal{P} = \mathcal{P}_S \cdot \mathcal{U} \). \( \square \)

We will now complete the proof of Proposition 2.6. As in the preceding lemma, let \( \mathcal{U} \) be the kernel of the natural homomorphism \( \mathcal{H}(\mathcal{O}) \to \mathcal{H}^\mathcal{P}_{\text{pred}}(\kappa) \). Since \( \Omega \) has been assumed to be stable under the action of \( \Theta \) on \( \mathfrak{B}(H/K) \), the group \( \Theta \) acts on \( \mathcal{H} \) by \( \Theta \)-group scheme automorphisms. So \( \mathcal{U} \) is stable under the induced action of \( \Theta \) on \( \mathcal{P} = \mathcal{H}(\mathcal{O}) \). We will now describe a descending \( \Theta \)-stable filtration of the subgroup \( \mathcal{U} \). For a non-negative integer \( i \), let \( \mathcal{U}_i \) be the kernel of the homomorphism \( \mathcal{P} = \mathcal{H}(\mathcal{O}) \to \mathcal{H}(\mathcal{O}/\mathfrak{m}^{i+1}) \). Then each \( \mathcal{U}_i \) is a normal subgroup of \( \mathcal{P} \) and is stable under the action of \( \Theta \) on the latter, \( \mathcal{U}_i \supset \mathcal{U}_{i+1} \), and \( \mathcal{U}_i / \mathcal{U}_{i+1} \) is a \( \kappa \)-vector space for all \( i \geq 0 \) [CGP, Prop. A.5.12]. The quotient \( \mathcal{U} / \mathcal{U}_0 \) is isomorphic to \( \mathcal{R}_{u,\kappa}(\mathcal{H})(\kappa) \). If \( p = 0 \), we consider the ascending filtration of the nilpotent group \( \mathcal{R}_{u,\kappa}(\mathcal{H})(\kappa) \) given by its ascending central series, and if \( p \neq 0 \) we consider the ascending filtration of the unipotent group \( \mathcal{R}_{u,\kappa}(\mathcal{H})(\kappa) \) given by Corollary B.3.3 of [CGP] to obtain an ascending filtration of \( \mathcal{U} / \mathcal{U}_0 \). The inverse image in \( \mathcal{U} \) of this filtration of \( \mathcal{U} / \mathcal{U}_0 \) gives us a descending \( \Theta \)-stable filtration of the nilpotent group \( \mathcal{R}_{u,\kappa}(\mathcal{H})(\kappa) \). For all \( j \geq -n \), \( \mathcal{U}_j \) is a normal subgroup of \( \mathcal{P} \) that is stable under the action of \( \Theta \) on the latter, \( \mathcal{U}_j / \mathcal{U}_{j+1} \) is a commutative group of exponent \( p \) if \( p \neq 0 \), and is a vector space over \( \mathbb{Q} \) if \( p = 0 \). For convenience, we will denote \( \mathcal{U}_j \) by \( \mathcal{U}^{(j+n+1)} \) for all \( j \). Thus we have a decreasing filtration \( \mathcal{U} = \mathcal{U}^{(1)} \supset \mathcal{U}^{(2)} \supset \mathcal{U}^{(3)} \cdots \).

For \( S \in \mathfrak{S} \), let \( \mathcal{Z}^{(j)}_S \) be the centralizer of \( S \) in \( \mathcal{U}^{(j)} \). If for \( \theta \in \Theta \), there exists \( u(\theta) \in \mathcal{U}^{(j)} \) such that \( \theta(S) = u(\theta)^{-1} S u(\theta) \), then \( \mathcal{Z}^{(j)}_S \mathcal{U}^{(j+1)} \) is \( \Theta \)-stable. To
see this, let \( \theta \in \Theta \), and pick \( u(\theta) \in U^{(j)} \) such that \( \theta(S) = u(\theta)^{-1}Su(\theta) \). Then \( \theta(Z^{(j)}_S) = u(\theta)^{-1}Z^{(j)}_SU(\theta) \). So \( \theta(Z^{(j)}_S U^{(j+1)}) = u(\theta)^{-1}Z^{(j)}_SU(\theta)U^{(j+1)} = Z^{(j)}_SU^{(j+1)} \) since \( U^{(j)}/U^{(j+1)} \) is commutative. This shows that \( Z^{(j)}_S U^{(j+1)} \) is \( \Theta \)-stable. Now as \( \Theta \) is a finite group of order prime to \( p \) if \( p \neq 0 \), and \( U^{(j)}/Z^{(j)}_SU^{(j+1)} \) is a commutative divisible group if \( p = 0 \), we conclude that \( H^1(\Theta, U^{(j)}/Z^{(j)}_SU^{(j+1)}) = 0 \) for all \( p \).

Now we fix an \( S_0 \in \mathcal{S} \). Then for \( \theta \in \Theta \), clearly \( \theta(S_0) \in \mathcal{S} \), and since \( \mathcal{P} \) acts transitively on \( \mathcal{S} \), we see using Lemma 2.7(ii) (for \( S_0 \) in place of \( S \)) that \( \theta(S_0) = u_1(\theta)^{-1}S_0u_1(\theta) \) with \( u_1(\theta) \in U^{(1)}(= \mathcal{U}) \). As \( Z^{(1)}_{S_0} \) is the normalizer of \( S_0 \) in \( U^{(1)} \) (Lemma 2.7(i)), we see that \( \theta \mapsto u_1(\theta) \mod Z^{(1)}_{S_0} \) is a 1-cocycle on \( \Theta \) with values in \( U^{(1)}/Z^{(1)}_{S_0} U^{(2)} \), and hence it is a 1-coboundary. This means that there is a \( \theta \in U^{(1)} \) such that \( u_1(\theta) := v_1^{-1}u_1(\theta)\theta(v_1) \in Z^{(1)}_{S_0} U^{(2)} \) for all \( \theta \in \Theta \).

Let \( S_1 = v_1^{-1}S_0v_1 \). Then for \( \theta \in \Theta \), we have \( \theta(S_1) = u_1(\theta)^{-1}S_1u_1(\theta) \). Observe that \( u_1(\theta) \in Z^{(1)}_{S_0} U^{(2)} = v_1 Z^{(1)}_{S_1} v_1^{-1} U^{(2)} = Z^{(1)}_{S_1} U^{(2)} \) as \( U^{(1)}/U^{(2)} \) is commutative. So for each \( \theta \in \Theta \), there is an element \( u_2(\theta) \in U^{(2)} \) such that \( \theta(S_1) = u_2(\theta)^{-1}S_1u_2(\theta) \). Now, as above, using the fact that the normalizer of \( S_1 \) in \( U^{(2)} \) is the centralizer \( Z^{(2)}_{S_1} \), we see that \( \theta \mapsto u_2(\theta) \mod Z^{(2)}_{S_1} \) is a 1-cocycle on \( \Theta \) with values in \( U^{(2)}/Z^{(2)}_{S_1} U^{(3)} \), and hence it is a 1-coboundary. Therefore, there is a \( v_2 \in U^{(2)} \) such that \( u_2(\theta) := v_2^{-1}u_2(\theta)\theta(v_2) \in Z^{(2)}_{S_1} U^{(3)} \) for all \( \theta \in \Theta \).

Repeating the above argument, we construct a sequence \( \{S_i\} \) of tori in \( \mathcal{S} \), and a sequence of elements \( v_i \in U^{(i)} \), such that

\[ \bullet \quad S_i = v_i^{-1}S_{i-1}v_i \quad \text{and for each} \quad \theta \in \Theta \), there is an element \( u_{i+1}(\theta) \in U^{(i+1)} \) such that \( \theta(S_i) = u_{i+1}(\theta)^{-1}S_iu_{i+1}(\theta) \), and \( \theta \mapsto u_{i+1}(\theta) \mod Z^{(i+1)}_{S_i} U^{(i+2)} \) is a 1-cocycle on \( \Theta \) with values in \( U^{(i+1)}/Z^{(i+1)}_{S_i} U^{(i+2)} \).

For \( i \geq 1 \), let \( w_i = v_1v_2 \cdots v_i \). Then \( S_i = w_i^{-1}S_0w_i \). Since \( v_j \in U^{(j)} \), and \( \mathcal{O} \) has been assumed to be complete, \( w := \lim_{i \to \infty} w_i \) exists in \( U \). Let \( S = w^{-1}S_0w \). For \( \theta \in \Theta \), as \( \theta(S_i) = u_{i+1}(\theta)^{-1}S_iu_{i+1}(\theta) \), we see that \( u_1(\theta)\theta(w_i)u_{i+1}(\theta)^{-1}w_i^{-1} \) normalizes \( S_0 \). Since the normalizer of \( S_0 \) in \( H(K) \) is closed, taking \( i \to \infty \), we conclude that \( u_1(\theta)\theta(w)w^{-1} \) normalizes \( S_0 \). This implies that \( \theta(S) = S \) for all \( \theta \in \Theta \).

2.8. Let \( x, y \in \mathcal{B} = \mathcal{B}(H/K) \). Let \( F \) be a facet of \( \mathcal{B}(H/K) \) which contains \( x \) in its closure and is maximal among the facets that meet \( \mathcal{B} \), and let \( \Omega = F \cup \{y\} \). Let \( S \subset T \) be a pair of \( K \)-split tori with properties (i) and (ii) of Proposition 2.6, and \( S_G \) and \( T_G \) be the maximal subtori of \( S \) and \( T \) respectively contained in \( G \). Let \( A \) be the apartment of \( \mathcal{B}(H/K) \) corresponding to the maximal \( K \)-split torus \( T \) of \( H \). Then \( A \) contains \( y \) and the closure of \( F \), and so it also contains \( x \). Moreover, \( A \) is an affine space under \( V(T) \), the affine subspace \( V(S) + x \) of \( A \) contains \( F \) and is spanned by it. The affine subspaces \( V(S_G) + x \subset V(T_G) + x \) of \( A \) are clearly
contained in $\mathcal{B} = \mathcal{B}(H/K)^{\Theta}$. As $V(S)^{\Theta} = V(S_G)$ and $F \subset V(S) + x$, we see that $F^{\Theta}$ is contained in $V(S_G) + x$. But since the facet $F$ is maximal among the facets that meet $\mathcal{B}$, $A^{\Theta} (= A \cap \mathcal{B})$ is contained in the affine subspace of $A$ spanned by $F^{\Theta}$. Therefore, $A^{\Theta} = V(S_G) + x$. This implies that $V(S_G) + x = V(T_G) + x$ and hence $S_G = T_G$. We will now show that $S_G$ is a maximal $K$-split torus of $G$.

Let $S'$ be a maximal $K$-split torus of $G$ containing $S_G$. Then the centralizer $M := Z_H(S')$ of $S'$ in $H$ is stable under $\Theta$. The enlarged Bruhat–Tits building $\mathcal{B}(M/K)$ of $M(K)$ is identified with the union of apartments of $\mathcal{B}(H/K)$ that correspond to maximal $K$-split tori of $M$ (these are precisely the maximal $K$-split tori of $H$ that contain $S'$), cf. [P2, 3.11]. Let $z$ be a point of $\mathcal{B}(M/K)^{\Theta}$ and $T'$ be a maximal $K$-split torus of $M$ such that the corresponding apartment $A'$ of $\mathcal{B}(M/K)$ contains $z$. Then $A' = V(T') + z$ and hence $A'^{\Theta} = A' \cap \mathcal{B} = V(T')^{\Theta} + z = V(S') + z$ is an affine subspace of $A'$ of dimension $\dim(S')$. Let $F'$ be a facet of $A'$ that contains the point $z$ in its closure and is maximal among the facets of $A'$ meeting $\mathcal{B}$. Then $A'^{\Theta}$ is contained in the affine subspace of $A'$ spanned by $F'^{\Theta}$, so $\dim(F'^{\Theta}) = \dim(S') \geq \dim(S_G)$. But $\dim(F'^{\Theta}) = \dim(S_G) \geq \dim(F^{\Theta})$. This implies that $\dim(S_G) = \dim(S')$ and hence $S' = S_G$. So $S_G$ is a maximal $K$-split torus of $G$.

Thus we have established the following proposition:

**Proposition 2.9.** Given points $x, y \in \mathcal{B}$, there exists a maximal $K$-split torus $S_G$ of $G$, and a maximal $K$-split torus $T$ of $H$ containing $S_G$ and hence contained in $Z_H(S_G)$, such that the apartment $A$ of $\mathcal{B}(Z_H(S_G)/K)$ corresponding to $T$ contains $x$ and $y$. Moreover, $A^{\Theta} = A \cap \mathcal{B}$ is the affine subspace $V(S_G) + x$ of $A$ of dimension $\dim(S_G)$.

We will now derive the following proposition which will give us apartments in the Bruhat–Tits building of $G(K)$. In the sequel, we will use $S$, instead of $S_G$, to denote a maximal $K$-split torus of $G$. As $M := Z_H(S)$ is stable under $\Theta$, the enlarged Bruhat–Tits building $\mathcal{B}(M/K)$ of $M(K)$ contains a $\Theta$-fixed point.

**Proposition 2.10.** Let $S$ be a maximal $K$-split torus of $G$ and let $T$ be a maximal $K$-split torus of $H$ containing $S$ such that the apartment $A$ of $\mathcal{B}(H/K)$ corresponding to $T$ contains a $\Theta$-fixed point $x$. Then $\mathcal{B}(Z_H(S)/K)^{\Theta} = V(S) + x = A^{\Theta}$. So $\mathcal{B}(Z_H(S)/K)^{\Theta}$ is an affine space under the $\mathbb{R}$-vector space $V(S)$.

**Proof.** Let $C$ be the central torus of $Z_H(S)$ and $Z_H(S)'$ the derived subgroup. Then $C$, $Z_H(S)$ and $Z_H(S)'$ are stable under $\Theta$; $G' := (Z_H(S)^{\Theta})^0$ is anisotropic over $K$ since $S$ is a maximal $K$-split torus of $G$, and so also of $(Z_H(S)^{\Theta})^0 (\subset G)$. Now applying Proposition 2.9 to $Z_H(S)'$ in place of $H$, we see that the Bruhat–Tits building $\mathcal{B}(Z_H(S)'/K)$ of $Z_H(S)'(K)$ contains only one point fixed under $\Theta$. For if $y, z \in \mathcal{B}(Z_H(S)'/K)^{\Theta}$, then there is an apartment $A'$ of $\mathcal{B}(Z_H(S)'/K)$ that contains these points. Moreover, the dimension of the affine subspace $A'^{\Theta}$ of $A'$ is 0 as $G'$ is anisotropic over $K$. Therefore, $y = z$. This proves that $\mathcal{B}(Z_H(S)'/K)^{\Theta}$ consists of a single point. Hence, $\mathcal{B}(Z_H(S)/K)^{\Theta} = V(C)^{\Theta} + x = V(S) + x$, and so it is an affine space under $V(S)$. \qed
2.11. Let $S$ be a maximal $K$-split torus of $G$. Let $N := N_G(S)$ and $Z := Z_G(S)$ be respectively the normalizer and the centralizer of $S$ in $G$. As $N$ (in fact, the normalizer $N_H(S)$ of $S$ in $H$) normalizes the centralizer $Z_H(S)$ of $S$ in $H$, there is a natural action of $N(K)$ on $B(Z_H(S)/K)$ and $N(K)$ stabilizes $B(Z_H(S)/K)^\Theta$ under this action. For $n \in N(K)$, the action of $n$ carries an apartment $A$ of $B(Z_H(S)/K)$ to the apartment $n \cdot A$ by an affine transformation.

Now let $T$ be a maximal $K$-split torus of $Z_H(S)$ such that the corresponding apartment $A := A_T$ of $B(Z_H(S)/K)$ contains a $\Theta$-fixed point $x$. According to the previous proposition, $B(Z_H(S)/K)^\Theta = V(S) + x = A^\Theta$. So we can view $B(Z_H(S)/K)^\Theta$ as an affine space under $V(S)$. We will now show, using the proof of the lemma in 1.6 of [PY1], that $B(Z_H(S)/K)^\Theta$ has the properties required of an apartment corresponding to the maximal $K$-split torus $S$ in the Bruhat-Tits building of $G(K)$ if such a building exists. We need to check the following three conditions.

A1: The action of $N(K)$ on $B(Z_H(S)/K)^\Theta = A^\Theta$ is by affine transformations and the maximal bounded subgroup $Z(K)_b$ of $Z(K)$ acts trivially.

Let $\text{Aff}(A^\Theta)$ be the group of affine automorphisms of $A^\Theta$ and $\varphi : N(K) \to \text{Aff}(A^\Theta)$ be the action map.

A2: The group $Z(K)$ acts by translations, and the action is characterized by the following formula: for $z \in Z(K)$,

$$\chi(\varphi(z)) = -\omega(\chi(z))$$

for all $\chi \in X_K^*(Z) (\hookrightarrow X_K^*(S))$, here we regard the translation $\varphi(z)$ as an element of $V(S)$.

A3: For $g \in \text{Aff}(A^\Theta)$, denote by $dg \in \text{GL}(V(S))$ the derivative of $g$. Then the map $N(K) \to \text{GL}(V(S)), n \mapsto dg(n)$, is induced from the action of $N(K)$ on $X_*(S)$ (i.e., it is the Weyl group action).

Moreover, as the central torus of $G$ is $K$-anisotropic, these three conditions determine the affine structure on $B(Z_H(S)/K)^\Theta$ uniquely; see [T, 1.2].

**Proposition 2.12.** Conditions A1, A2 and A3 hold.

**Proof.** The action of $n \in N(K)$ on $B(Z_H(S)/K)$ carries the apartment $A = A_T$ via an affine isomorphism $f(n) : A \to A_{nTn^{-1}}$, to the apartment $A_{nTn^{-1}}$ corresponding to the torus $nTn^{-1}$ containing $S$. As $(A_{nTn^{-1}})^\Theta = B(Z_H(S)/K)^\Theta = A^\Theta$, we see that $f(n)$ keeps $A^\Theta$ stable and so $\varphi(n) := f(n)|_{A^\Theta}$ is an affine automorphism of $A^\Theta$.

The derivative $df(n) : V(T) \to V(nTn^{-1})$ is induced from the map

$$\text{Hom}_K(GL_1, T) = X_*(T) \to X_*(nTn^{-1}) = \text{Hom}_K(GL_1, nTn^{-1}),$$

$\lambda \mapsto \text{Int} n \cdot \lambda$, where $\text{Int} n$ is the inner automorphism of $H$ determined by $n \in N(K) \subset H(K)$. So, the restriction $d\varphi(n) : V(S) \to V(S)$ is induced from the homomorphism $X_*(S) \to X_*(S), \lambda \mapsto \text{Int} n \cdot \lambda$. This proves A3.

Condition A3 implies that $d\varphi$ is trivial on $Z(K)$. Therefore, $Z(K)$ acts by translations. The action of the bounded subgroup $Z(K)_b$ on $A^\Theta$ admits a fixed point.
by the fixed point theorem of Bruhat-Tits. Therefore, $Z(K)_b$ acts by the trivial translation. This proves A1.

Since the image of $S(K)$ in $Z(K)/Z(K)_b \simeq \mathbb{Z}^{\dim(S)}$ is a subgroup of finite index, to prove the formula in A2, it suffices to prove it for $z \in S(K)$. But for $z \in S(K)$, $zTz^{-1} = T$, and $f(z)$ is a translation of the apartment $A$ ($\varphi(z)$ is regarded as an element of $V(T)$) which satisfies (see 1.9 of [P2]):

$$\chi(f(z)) = -\omega(\chi(z)) \text{ for all } \chi \in X^*_K(T).$$

This implies the formula in A2, since the restriction map $X^*_K(T) \to X^*_K(S)$ is surjective and the image of the restriction map $X^*_K(Z) \to X^*_K(S)$ is of finite index in $X^*_K(S)$.

2.13. Apartments of $\mathcal{B}$. By definition, the apartments of $\mathcal{B}$ are the affine spaces $\mathcal{B}(Z_H(S)/K)\Theta$ under the $\mathbb{R}$-vector space $V(S)$ (of dimension = $K$-rank $G$) for maximal $K$-split tori $S$ of $G$. For any apartment $A$ of $\mathcal{B}(Z_H(S)/K)$ that contains a $\Theta$-fixed point, $\mathcal{B}(Z_H(S)/K)^\Theta = A^\Theta$ (Proposition 2.10). The subgroup $N_G(S)(K)$ of $G(K)$ acts by affine transformations on the apartment $\mathcal{B}(Z_H(S)/K)^\Theta$ and $Z_G(S)(K)$ acts on it by translations (Proposition 2.12). Conjugacy of maximal $K$-split tori of $G$ under $G(K)$ implies that this group acts transitively on the set of apartments of $\mathcal{B}$.

Propositions 2.9 and 2.10 imply the following proposition at once:

**Proposition 2.14.** Given any two points of $\mathcal{B}$, there is a maximal $K$-split torus $S$ of $G$ such that the corresponding apartment of $\mathcal{B}$ contains these two points.

**Proposition 2.15.** Let $A$ be an apartment of $\mathcal{B}$. Then there is a unique maximal $K$-split torus $S$ of $G$ such that $A = \mathcal{B}(Z_H(S)/K)^\Theta$. So the stabilizer of $A$ in $G(K)$ is $N_G(S)(K)$.

**Proof.** We fix a maximal $K$-split torus $S$ of $G$ such that $A = \mathcal{B}(Z_H(S)/K)^\Theta$. We will show that $S$ is uniquely determined by $A$. For this purpose, we observe that the subgroup $N_G(S)(K)$ of $G(K)$ acts on $A$ and the maximal bounded subgroup $Z_G(S)(K)_b$ of $Z_G(S)(K)$ acts trivially (Proposition 2.12). So the subgroup $\mathcal{Z}$ of $G(K)$ consisting of elements that fix $A$ pointwise is a bounded subgroup of $G(K)$, normalized by $N_G(S)(K)$, and it contains $Z_G(S)(K)_b$. Now, using the Bruhat decomposition of $G(K)$ with respect to $S$, we see that every bounded subgroup of $G(K)$ that is normalized by $N_G(S)(K)$ is a normal subgroup of the latter. Hence the identity component of the Zariski-closure of $\mathcal{Z}$ is $Z_G(S)$. As $S$ is the unique maximal $K$-split torus of $G$ contained in $Z_G(S)$, both the assertions follow. □

2.16. The affine Weyl group of $G$. Let $G(K)^+$ denote the (normal) subgroup of $G(K)$ generated by $K$-rational elements of the unipotent radicals of parabolic $K$-subgroups of $G$. Let $S$ be a maximal $K$-split torus of $G$, $N$ and $Z$ respectively be the normalizer and centralizer of $S$ in $G$. Let $N(K)^+ := N(K) \cap G(K)^+$. Then $N(K)^+$ maps onto the Weyl group $W := N(K)/Z(K)$ of $G$ (this can be seen using, for example, [CGP, Prop. C.2.24(i)]).
Let $A$ be the apartment of $B$ corresponding to $S$. As in 2.11, let $\varphi : N(K) \to \text{Aff}(A)$ be the action map, then the affine Weyl group $W_{\text{aff}}$ of $G/K$ is by definition the subgroup $\varphi(N(K)^+)$ of $\text{Aff}(A)$.

3. Bruhat-Tits theory for $G$ over $K$

3.1. Bruhat-Tits group schemes $H^\Theta_{\Omega}$. Let $\Omega$ be a nonempty $\Theta$-stable bounded subset of an apartment of $B(H/K)$. Let $H^\Theta_{\Omega}$ be the smooth affine $\Theta$-group scheme associated to $\Omega$ in 2.1. There is a natural action of $\Theta$ on $H^\Theta_{\Omega}$ by $\Theta$-group scheme automorphisms (2.4). Define the functor $H^\Theta_{\Omega}$ of $\Theta$-fixed points that associates to a commutative $\Theta$-algebra $C$ the subgroup $H^\Theta_{\Omega}(C)$ of $H^\Theta_{\Omega}$ consisting of elements fixed under $\Theta$. The functor $H^\Theta_{\Omega}$ is represented by a closed smooth $\Theta$-subgroup scheme of $H^\Theta_{\Omega}$ (see Propositions 3.1 and 3.4 of [E], or Proposition A.8.10 of [CGP]); we will denote this closed smooth $\Theta$-subgroup scheme also by $H^\Theta_{\Omega}$. Its generic fiber is $H^\Theta$, and so the identity component of the generic fiber is $G$. The neutral component $(H^\Theta_{\Omega})^\circ$ of $H^\Theta_{\Omega}$ is by definition the union of the identity components of its generic and special fibers; it is an open (so smooth) affine $\Theta$-subgroup scheme [PY2, §3.5] with generic fiber $G$. The index of the subgroup $(H^\Theta_{\Omega})^\circ(0)$ in $H^\Theta_{\Omega}(0)$ is known to be finite [EGA IV, Cor. 15.6.5]. It is obvious that $(H^\Theta_{\Omega})^\circ = (H^\Theta_{\Omega})^\circ$. We will denote $(H^\Theta_{\Omega})^\circ$ by $H_{\Omega}^\circ$ in the sequel and call it the Bruhat-Tits $\Theta$-group scheme associated to $G$ and $\Omega$. The special fiber of $H_{\Omega}^\circ$ will be denoted $H^\circ_{\Omega}$. As $H_{\Omega}^\circ(0) \subset H_{\Omega}(0)$, $H^\circ_{\Omega}(0)$ fixes $\Omega$ pointwise.

3.2. Let $\Omega' \prec \Omega$ be nonempty bounded subsets of an apartment of $B(H/K)$. We assume that both $\Omega$ and $\Omega'$ are stable under the action of $\Theta$ on $B(H/K)$. The $\Theta$-group scheme homomorphism $H_{\Omega} \to H_{\Omega'}$ of [P2, 1.10] restricts to a homomorphism $\rho_{\Omega, \Omega'} : H_{\Omega} \to H_{\Omega'}$, and by [E, Prop. 3.5], or [CGP, Prop. A.8.10(2)], it induces a $\Theta$-group scheme homomorphism $H_{\Omega}^\Theta \to H_{\Omega'}^\Theta$. The last homomorphism gives a $\Theta$-group scheme homomorphism $\rho_{\Omega, \Omega'} : (H_{\Omega}^\Theta)^\circ = H_{\Omega}^\circ \to H_{\Omega'}^\circ = (H_{\Omega'}^\Theta)^\circ$ that is the identity homomorphism on the generic fiber $G$.

3.3. Let $A$ be the apartment of $B$ corresponding to a maximal $K$-split torus $S$ of $G$ and $\Omega$ be a nonempty bounded subset of $A$. The apartment $A$ is contained in an apartment $A$ of $B(H/K)$ that corresponds to a maximal $K$-split torus $T$ of $H$ containing $S$ and $A = A \cap B = A^\Theta$ (2.13). So $\Omega$ is a bounded subset of $A$. The group scheme $H_{\Omega}$ contains a closed split $\Theta$-torus $\mathcal{T}$ with generic fiber $T$, see [P2, 1.9]. Let $\mathcal{J}$ be the $\Theta$-subtorus of $\mathcal{T}$ whose generic fiber is $S$ ($\mathcal{J}$ is the schematic closure of $S$ in $\mathcal{T}$). The automorphism group $\Theta$ of $H_{\Omega}$ acts trivially on the $\Theta$-torus $\mathcal{T}$ (since $S \subset G \subset H^\Theta$) and hence this torus is contained in $H_{\Omega}^\circ$. The special fiber $\mathcal{T}$ of $\mathcal{J}$ is a maximal torus of $\mathcal{T}$ since $S$ is a maximal $K$-split torus of $G$.

Proposition 3.4. Let $A$ and $A'$ be apartments of $B$ and $\Omega$ a nonempty bounded subset of $A \cap A'$. Then there exists an element $g \in H^\circ_{\Omega}(0)$ that maps $A$ onto $A'$. Any such element fixes $\Omega$ pointwise.
Thus the subgroup $P$ is the maximal $K$-split tori of $G$ corresponding to the apartments $\mathcal{A}$ and $\mathcal{A}'$ respectively and $\mathcal{F}$ and $\mathcal{F}'$ be the $0$-tori of $\mathcal{G}$ with generic fibers $S$ and $S'$ respectively. The special fibers $\overline{\mathcal{F}}$ and $\overline{\mathcal{F}'}$ of $\mathcal{F}$ and $\mathcal{F}'$ are maximal split tori of $\overline{\mathcal{F}}$, and hence according to a result of Borel and Tits there is an element $\overline{g}$ of $\overline{\mathcal{F}}(\kappa)$ which conjugates $\overline{\mathcal{F}}$ onto $\overline{\mathcal{F}'}$ [CGP, Thm. C.2.3]. Now [P2, Prop. 2.1(ii)] implies that there exists a $g \in \mathcal{G}(\mathcal{O})$ lying over $\overline{g}$ that conjugates $\mathcal{F}$ onto $\mathcal{F}'$. This element fixes $\Omega$ pointwise and conjugates $S$ onto $S'$ and hence maps $\mathcal{A}$ onto $\mathcal{A}'$.

3.5. Given a point $x \in \mathcal{B}$, for simplicity we will denote $\mathcal{G}_{x}^\Theta$, $\mathcal{H}_{x}^\Theta$, $\mathcal{H}_{x}^\Theta$ and $\mathcal{H}_{\{x\}}^\Theta$ by $\mathcal{G}_{x}^\Theta$, $\mathcal{H}_{x}^\Theta$, $\mathcal{H}_{x}^\Theta$ and $\mathcal{H}_{\{x\}}^\Theta$ respectively, and the special fibers of these group schemes will be denoted by $\overline{\mathcal{G}}_{x}$, $\overline{\mathcal{H}}_{x}$, $\overline{\mathcal{H}}_{x}$ and $\overline{\mathcal{H}}_{\{x\}}$ respectively. The subgroup of $H(K)$ (resp. $G(K)$) consisting of elements that fix $x$ will be denoted by $H(K)^x$ (resp. $G(K)^x$). The subgroup $\mathcal{G}_{x}^\Theta(\mathcal{O}) (\subset G(K)^x)$ is of finite index in $G(K)^x$.

3.6. Parahoric subgroups of $G(K)$. For $x \in \mathcal{B}$, $\mathcal{G}_{x}^\Theta$ and $P_x := \mathcal{G}_{x}^\Theta(\mathcal{O})$ will respectively be called the Bruhat-Tits parahoric $\mathcal{O}$-group scheme and the parahoric subgroup of $G(K)$ associated with the point $x$. Let $S$ be a maximal $K$-split torus of $G$ such that $x$ lies in the apartment $\mathcal{A}$ of $\mathcal{B}$ corresponding to $S$. Then the group scheme $\mathcal{G}_{x}^\Theta$ contains a closed split $0$-torus $\mathcal{F}$ whose generic fiber is $S(3.3)$. The parahoric subgroups of $G(K)$ are by definition the subgroups $P_x$ for $x \in \mathcal{B}$. For a given parahoric subgroup $P_x$, the associated Bruhat-Tits parahoric $\mathcal{O}$-group scheme is $\mathcal{G}_{x}^\Theta$.

(i) Let $P$ be a parahoric subgroup of $G(K)$, $\mathcal{G}^\Theta$ the associated Bruhat-Tits parahoric $\mathcal{O}$-group scheme, $\overline{\mathcal{G}}^\Theta$ the special fiber of $\mathcal{G}^\Theta$, and $\mathcal{P}$ be a subgroup of $P$ of finite index. Then the image of $\mathcal{P}$ in $\overline{\mathcal{G}}^\Theta(\kappa)$ is Zariski-dense in the connected group $\overline{\mathcal{G}}^\Theta$, so the affine ring of $\mathcal{G}^\Theta$ is:

$$\mathcal{O}[\mathcal{G}^\Theta] = \{f \in K[G] \mid f(\mathcal{P}) \subset \mathcal{O}\}.$$ 

Thus the subgroup $\mathcal{P}$ “determines” the group scheme $\mathcal{G}^\Theta$, and hence $P$ is the unique parahoric subgroup of $G(K)$ containing $\mathcal{P}$ as a subgroup of finite index.

(ii) Let $P$ and $\mathcal{G}^\Theta$ be as in the preceding paragraph. Let $\Omega$ be a nonempty $\Theta$-stable bounded subset of an apartment of $\mathcal{B}(H/K)$ and $\mathcal{G}^\Theta_\Omega$ be as in 3.1. We assume that $\Omega$ is fixed pointwise by $P$. Then the inclusion of $P$ in $H(K)^\Omega (= \mathcal{H}_{\Omega}(\mathcal{O}))$ gives a $\mathcal{O}$-group scheme homomorphism $\mathcal{G}^\Theta \to \mathcal{H}_{\Omega}^\Theta$ (Proposition 2.2). This homomorphism obviously factors through $\mathcal{G}^\Theta_\Omega$ to give a $\mathcal{O}$-group scheme homomorphism $\mathcal{G}^\Theta \to \mathcal{G}^\Theta_\Omega$ that is the identity on the generic fiber $G$.

Suppose $x, y \in \mathcal{B}(H/K)$ are fixed by $P$, and $[xy]$ is the geodesic joining $x$ and $y$. Then $P$ fixes every point $z$ of $[xy]$. Let $\mathcal{G}^\Theta_{[xy]}$ be as in 3.1 (for $\Omega = [xy]$). There are $\mathcal{O}$-group scheme homomorphisms $\mathcal{G}^\Theta \to \mathcal{G}^\Theta_{[xy]}$ and $\mathcal{G}^\Theta \to \mathcal{G}^\Theta_z$ that are the identity on the generic fiber $G$. 

Proof. We will use Proposition 2.1(ii) of [P2], with $\mathcal{O}$ in place of $\mathcal{O}$, and denote $\mathcal{G}^\Theta_{\Omega}$ by $\mathcal{G}$, and its special fiber by $\overline{\mathcal{G}}$, in this proof. Let $S$ and $S'$ be the maximal $K$-split tori of $G$ corresponding to the apartments $\mathcal{A}$ and $\mathcal{A}'$ respectively and $\mathcal{F}$ and $\mathcal{F}'$ be the $0$-tori of $\mathcal{G}$ with generic fibers $S$ and $S'$ respectively. The special fibers $\overline{\mathcal{F}}$ and $\overline{\mathcal{F}'}$ of $\mathcal{F}$ and $\mathcal{F}'$ are maximal split tori of $\overline{\mathcal{F}}$, and hence according to a result of Borel and Tits there is an element $\overline{g}$ of $\overline{\mathcal{F}}(\kappa)$ which conjugates $\overline{\mathcal{F}}$ onto $\overline{\mathcal{F}'}$ [CGP, Thm. C.2.3]. Now [P2, Prop. 2.1(ii)] implies that there exists a $g \in \mathcal{G}(\mathcal{O})$ lying over $\overline{g}$ that conjugates $\mathcal{F}$ onto $\mathcal{F}'$. This element fixes $\Omega$ pointwise and conjugates $S$ onto $S'$ and hence maps $\mathcal{A}$ onto $\mathcal{A}'$. 

□
3.7. Polysimplicial structure on $B$. Let $P$ be a parahoric subgroup of $G(K)$ and $\mathcal{O}$ be the Bruhat-Tits parahoric $\mathcal{O}$-group scheme associated with $P$ (3.6). Let $B(H/K)^P$ denote the set of points of $B(H/K)$ fixed by $P$. According to Corollary 2.3, $B(H/K)^P$ is the union of facets pointwise fixed by $P$. Let $\mathcal{F}_P := B(H/K)^P \cap B$. This closed convex subset is by definition the closed facet of $B$ associated with the parahoric subgroup $P$. The $\mathcal{O}$-group scheme $\mathcal{O}$ contains a closed split $\mathcal{O}$-torus whose generic fiber $S$ is a maximal $K$-split torus of $G(3.3)$. The subgroup $\mathcal{J}(O)$ (of $S(K)$) is the maximal bounded subgroup of $S(K)$ and it is contained in $P(= \mathcal{O}(O))$, so, according to Corollary 2.3, $\mathcal{F}_P$ is contained in the enlarged building $B(Z_H(S)/K)$ of $Z_H(S)(K)$. This implies that the closed facet $\mathcal{F}_P$ is contained in the apartment $A := B(Z_H(S)/K)^{\Theta} (= B(Z_H(S)/K) \cap B)$ of $B$ corresponding to the maximal $K$-split torus $S$ of $G$.

Let $\mathcal{F}_P$ be the subset of points of $\mathcal{F}_P$ that are not fixed by any parahoric subgroup of $G(K)$ larger than $P$. Then $\mathcal{F}_P = \mathcal{F}_P - \bigcup_{Q \supset P} \mathcal{F}_Q$. Given another parahoric subgroup subgroup $Q$ of $G(K)$, if $\mathcal{F}_Q = \mathcal{F}_P$, then $Q = P$. (To see this, we choose points $x, y \in B$ such that $\mathcal{F}_x^\circ(O) = P$ and $\mathcal{F}_y^\circ(O) = Q$. Then $y \in \mathcal{F}_Q = \mathcal{F}_P$. So $P$ fixes $y$. Now using 3.6 (ii) we see that $P \subset Q$. We similarly see that $Q \subset P$.) Hence if $Q \supset P$, then $\mathcal{F}_Q$ is properly contained in $\mathcal{F}_P$. By definition, $\mathcal{F}_P$ is the facet of $B$ associated with the parahoric subgroup $P$ of $G(K)$, and as $P$ varies over the set of parahoric subgroups of $G(K)$, these are are all the facets of $B$. We will show below (Propositions 3.11 and 3.13) that $\mathcal{F}_P$ is convex and bounded.

For a parahoric subgroup $Q$ of $G(K)$ containing $P$, obviously, $\mathcal{F}_Q \subset \mathcal{F}_P \subset \mathcal{F}_P$, thus $\mathcal{F}_Q \prec \mathcal{F}_P$ and hence $\mathcal{F}_P$ is a maximal facet if and only if $P$ is a minimal parahoric subgroup of $G(K)$. The maximal facets of $B$ are called the chambers of $B$. It is easily seen using the observations contained in 2.5 that all the chambers are of equal dimension. We say that a facet $\mathcal{F}'$ of $B$ is a face of a facet $\mathcal{F}$ if $\mathcal{F}' \prec \mathcal{F}$, i.e., if $\mathcal{F}'$ is contained in the closure of $\mathcal{F}$.

In the following three lemmas (3.8, 3.9 and 3.10), $k$ is any field of characteristic $p \geq 0$. We will use the notation introduced in [CGP, §2.1].

**Lemma 3.8.** Let $\mathcal{K}$ be a smooth connected affine algebraic $k$-group and $Q$ be a pseudo-parabolic $k$-subgroup of $\mathcal{K}$. Let $S$ be a $k$-torus of $Q$ whose image in the maximal pseudo-reductive quotient $M := Q/\mathcal{R}_{u,k}(Q)$ of $Q$ contains the maximal central torus of $M$. Then any 1-parameter subgroup $\lambda : GL_1 \to \mathcal{K}$ such that $Q = P_{\lambda}(\lambda)\mathcal{R}_{u,k}(\mathcal{K})$ has a conjugate under $\mathcal{R}_{u,k}(Q)(k)$ with image in $S$.

**Proof.** Let $\lambda : GL_1 \to \mathcal{K}$ be a 1-parameter subgroup such that $Q = P_{\lambda}(\lambda)\mathcal{R}_{u,k}(\mathcal{K})$. The image $\mathcal{T}$ of $\lambda$ is contained in $Q$ and it maps into the central torus of $M$. Therefore, $\mathcal{T}$ is contained in the solvable subgroup $S\mathcal{R}_{u,k}(Q)$ of $Q$. Note that as $S$ is commutative, the derived subgroup of $S\mathcal{R}_{u,k}(Q)$ is contained in $\mathcal{R}_{u,k}(Q)$, so the maximal $k$-tori of $S\mathcal{R}_{u,k}(Q)$ are conjugate to each other under $\mathcal{R}_{u,k}(Q)(k)$ [Bo, Thm. 19.2]. Hence, there is a $u \in \mathcal{R}_{u,k}(Q)(k)$ such that $u\mathcal{T}u^{-1} \subset S$. Then the image of the 1-parameter subgroup $\mu : GL_1 \to S$, defined as $\mu(t) = u\lambda(t)u^{-1}$, is contained in $S$. 

[Box]
Lemma 3.9. Let $H$ be a smooth connected affine algebraic $k$-group given with an action by a finite group $\Theta$ and $U$ be a smooth connected $\Theta$-stable unipotent normal $k$-subgroup of $H$. We assume that $p$ does not divide the order of $\Theta$. Let $\bar{S}$ be a $\Theta$-stable $k$-torus of $\bar{H} := H/U$. Then there exists a $\Theta$-stable $k$-torus $S$ in $H$ that maps isomorphically onto $\bar{S}$. In particular, there exists a $\Theta$-stable $k$-torus in $H$ that maps isomorphically onto the maximal central torus of $\bar{H}$.

Proof. Let $T$ be a $k$-torus of $H$ that maps isomorphically onto $\bar{S}(\subset \bar{H})$. Considering the $\Theta$-stable solvable subgroup $TU$; using conjugacy under $U(k)$ of maximal $k$-tori of this solvable group [Bo, Thm. 19.2], we see that for $\theta \in \Theta$, $\theta(T) = u(\theta)^{-1}TU(\theta)$ for some $u(\theta) \in U(k)$. Let $U(k) := U_0 \supset U_1 \supset U_2 \cdots \supset U_n = \{1\}$ be the descending central series of the nilpotent group $U(k)$. Each subgroup $U_i$ is $\Theta$-stable and $U_i/U_{i+1}$ is a commutative $p$-group if $p \neq 0$, and a $\mathbb{Q}$-vector space if $p = 0$. Now let $i \leq n$, be the largest integer such that there exists a $k$-torus $S$ in $TU$ that maps onto $\bar{S}$, and for every $\theta \in \Theta$, there is a $u(\theta) \in U_i$ such that $\theta(S) = u(\theta)^{-1}SU(\theta)$. Let $N_i$ be the normalizer of $S$ in $U_i$. Then, for $\theta \in \Theta$, $\theta(N_i) = u(\theta)^{-1}N_iU(\theta)$ and hence as $U_i/U_{i+1}$ is commutative, we see that $\theta(U_i/U_{i+1}) = N_iU_{i+1}$, i.e., $N_iU_{i+1}$ is $\Theta$-stable. It is easy to see that $\theta \mapsto u(\theta)$ mod $(N_iU_{i+1})$ is a 1-cocycle on $\Theta$ with values in $U_i/N_iU_{i+1}$. But $H^1(\Theta, U_i/N_iU_{i+1})$ is trivial since the finite group $\Theta$ is of order prime to $p$. So there exists $u \in U_i$ such that for all $\theta \in \Theta$, $u^{-1}u(\theta)\theta(u)$ lies in $N_iU_{i+1}$. Now let $S' = u^{-1}S$. Then the normalizer of $S'$ in $U_i$ is $u^{-1}N_iu$ and again as $U_i/U_{i+1}$ is commutative, $u^{-1}N_iu \cdot U_{i+1} = N_iU_{i+1}$. For $\theta \in \Theta$, we choose $u(\theta) \in U_{i+1}$ such that $u^{-1}u(\theta)\theta(u) \in u^{-1}N_iu \cdot u'(\theta)$. Then $\theta(S') = u'(\theta)^{-1}SU'(\theta)$ for all $\theta \in \Theta$. This contradicts the maximality of $i$ unless $i = n$. \hfill $\Box$

Lemma 3.10. Let $H$ be a smooth connected affine algebraic $k$-group given with an action by a finite group $\Theta$. We assume that $p$ does not divide the order of $\Theta$. Let $S = (H^\Theta)^\circ$. Then

(i) $\mathcal{R}_{u,k}(S) = (S \cap \mathcal{R}_{u,k}(H))^\circ = (\mathcal{R}_{u,k}(H)^\Theta)^\circ$; moreover, $S/(S \cap \mathcal{R}_{u,k}(H))$ is pseudo-reductive, and if $k$ is perfect then $S \cap \mathcal{R}_{u,k}(H) = \mathcal{R}_{u,k}(S)$.

(ii) Given a $\Theta$-stable pseudo-parabolic $k$-subgroup $Q$ of $H$, $P := S \cap Q$ is a pseudo-parabolic $k$-subgroup of $S$, so $P$ is connected and it equals $(S^\Theta)^\circ$.

(iii) Conversely, given a pseudo-parabolic $k$-subgroup $P$ of $S$, and a maximal $k$-torus $S \subset P$, there is a $\Theta$-stable pseudo-parabolic $k$-subgroup $Q$ of $H$, $Q$ containing the centralizer $Z_H(S)$ of $S$ in $H$, such that $P = S \cap Q = (S^\Theta)^\circ$.

Proof. The first assertion of (i) immediately follows from [CGP, Prop. A.8.14(2)]. Now we observe that as $\mathcal{R}_{u,k}(S) = (S \cap \mathcal{R}_{u,k}(H))^\circ$, $(S \cap \mathcal{R}_{u,k}(H))/\mathcal{R}_{u,k}(S)$ is a finite étale (unipotent) normal subgroup of the pseudo-reductive quotient $S/\mathcal{R}_{u,k}(S)$ of $S$ so it is central. Thus the kernel of the quotient map $\pi : S/\mathcal{R}_{u,k}(S) \to S/(S \cap \mathcal{R}_{u,k}(H))$ is an étale unipotent central subgroup. Hence, $S/(S \cap \mathcal{R}_{u,k}(H))$ is pseudo-reductive as $S/\mathcal{R}_{u,k}(S)$ is. Moreover, if $k$ is perfect then every pseudo-reductive $k$-group is
Proposition 3.11. Let $\mathfrak{g}$ have proved (iii).

Since $\mathfrak{g}_u,k(\mathfrak{g}) \subset \mathfrak{g} \cap \mathfrak{g}_u,k(\mathfrak{k}) \subset \mathfrak{g} \cap \mathfrak{k}$, to prove (ii), we can replace $\mathfrak{k}$ by its pseudo-reductive quotient $\mathfrak{k}/\mathfrak{g}_u,k(\mathfrak{k})$ and assume that $\mathfrak{k}$ is pseudo-reductive. Then $\mathfrak{g}$ is also pseudo-reductive by (i). Let $U = \mathfrak{g}_u,k(\mathfrak{g})$ be the $k$-unipotent radical of $\mathfrak{g}$; $U$ is $\Theta$-stable. Let $S$ be a $\Theta$-stable $k$-torus in $\mathfrak{g}$ that maps isomorphically onto the maximal central torus of the pseudo-reductive quotient $\mathfrak{g} = \mathfrak{g}/U$ (Lemma 3.9). By Lemma 3.8, there exists a 1-parameter subgroup $\lambda : GL_1 \to S$ such that $\lambda = P_3(\lambda)$.

Let $\mu = \sum_{\theta \in \Theta} \theta \cdot \lambda$. Then $\mu$ is invariant under $\Theta$ and so it is a 1-parameter subgroup of $\mathfrak{g}$. We will now show that $\mathfrak{g} = P_3(\mu)$. Let $\Phi$ (resp. $\Psi$) be the set of weights in the Lie algebra of $\mathfrak{g}$ (resp. $P_3(\mu)$) with respect to the adjoint action of $S$. Then since $\mathfrak{g}$, $P_3(\mu)$ and $S$ are $\Theta$-stable, the subsets $\Phi$ and $\Psi$ (of $X(S)$) are stable under the action of $\Theta$ on $X(S)$. Hence, for all $a \in \Phi$, as $\langle a, \lambda \rangle \geq 0$, we conclude that $\langle a, \mu \rangle \geq 0$. Therefore, $\Phi \subset \Psi$. On the other hand, for $b \in \Psi$, $\langle b, \mu \rangle > 0$. If $b \in \Psi$ does not belong to $\Phi$, then for $\theta \in \Theta$, $\theta \cdot b \notin \Phi$, so for all $\theta \in \Theta$, $\langle \theta \cdot b, \lambda \rangle < 0$, which implies that $\langle b, \mu \rangle < 0$. This is a contradiction. Therefore, $\Phi = \Psi$ and so $\mathfrak{g} = P_3(\mu)$.

Now observe that $(\mathfrak{g}^\Theta)^\circ \subset \mathfrak{g} \cap \mathfrak{g} \subset \mathfrak{g}^\Theta$. As $\mathfrak{g}^\Theta$ is a smooth subgroup ([E, Prop.3.4] or [CGP, Prop. A.8.10(2)]), $\mathfrak{g} \cap \mathfrak{g}$ is a smooth $k$-subgroup, and since it contains the pseudo-parabolic $k$-subgroup $P_3(\mu)$, it is a pseudo-parabolic $k$-subgroup of $\mathfrak{g}$ [CGP, Prop. 3.5.8], hence in particular it is connected. Therefore, $\mathfrak{g} \cap \mathfrak{g} = (\mathfrak{g}^\Theta)^\circ$.

Now we will prove (iii). Let $\lambda : GL_1 \to S$ be a 1-parameter subgroup such that $P = P_3(\lambda)\mathfrak{g}_u,k(\mathfrak{g})$. Then $\mathfrak{g} := P_3(\lambda)\mathfrak{g}_u,k(\mathfrak{k})$ is a pseudo-parabolic $k$-subgroup of $\mathfrak{k}$ that is $\Theta$-stable (since $\lambda$ is $\Theta$-invariant) and it contains $P$ as well as $Z_3(S)$. According to (ii), $\mathfrak{g} \cap \mathfrak{g} = (\mathfrak{g}^\Theta)^\circ$ is a pseudo-parabolic $k$-subgroup of $\mathfrak{g}$ containing $P$. The Lie algebras of $P$ and $(\mathfrak{g}^\Theta)^\circ$ are clearly equal. This implies that $P = \mathfrak{g} \cap \mathfrak{g} = (\mathfrak{g}^\Theta)^\circ$ and we have proved (iii). \qed

**Proposition 3.11.** Let $P$ be a parahoric subgroup of $G(K)$ and $\mathfrak{T}_P$ and $\overline{\mathfrak{T}}_P$ be as in 3.7.

(i) Given $x \in \mathfrak{T}_P$ and $y \in \overline{\mathfrak{T}}_P$, for every point $z$ of the geodesic $[xy]$, except possibly for $z = y$, $\mathcal{G}_z(\mathfrak{O}) = P$.

(ii) Let $F$ be a facet of $\mathcal{B}(H/K)$ that meets $\mathfrak{T}_P$ and is maximal among such facets. Then $\mathcal{G}_F(\mathfrak{O}) = P$. Thus $F \cap \mathfrak{B} \subset \mathfrak{T}_P$.

The first assertion of this proposition implies that $\mathfrak{T}_P$ is convex. The second assertion implies that $\mathfrak{T}_P$ is an open-dense subset of $\overline{\mathfrak{T}}_P$, hence the closure of $\mathfrak{T}_P$ is $\overline{\mathfrak{T}}_P$.

**Proof.** To prove the first assertion, let $[xy]$ be the geodesic joining $x$ and $y$. Let $F_0, F_1, \ldots, F_n$ be the facets of $\mathcal{B}(H/K)$ containing a segment of positive length of the geodesic $[xy]$ (so each $F_i$ is $\Theta$-stable and is fixed pointwise by $P$, hence $P \subset \mathcal{G}_{F_i}(\mathfrak{O})$, cf. 3.6(ii)). Then $[xy] \subset \bigcup_i F_i$. We assume the facets $\{F_i\}$ indexed so that $x$ lies in $F_0$, $y$ lies in $F_n$, and for each $i < n$, $F_i \cap F_{i+1}$ is nonempty. Let $z_0 = x$. For every
positive integer $i \leq n$, $F_{i-1} \cap F_i$ contains a unique point of $[xy]$; we will denote this point by $z_i$.

To prove the second assertion of the proposition along with the first, we take $x$ to be a point of $\mathcal{B}$ such that $\mathcal{G}_x^0(\mathcal{O}) = P$ (so $x \in \mathcal{F}_P$) and take $y$ to be any point of $F \cap \mathcal{B}$. Let $[xy]$, and for $i \leq n$, $F_i$ and $z_i$ be as in the preceding paragraph. Then $F_n = F$.

Since $x \in F_0$, there is a $\mathcal{O}$-group scheme homomorphism $\mathcal{G}_F^0 \to \mathcal{G}_x^0$ that is the identity on the generic fiber $G$. Thus, $\mathcal{G}_F^0(\mathcal{O}) \subset P$. But $P \subset \mathcal{G}_F^0(\mathcal{O})$, so $\mathcal{G}_F^0(\mathcal{O}) = \mathcal{G}_F^0(\mathcal{O}) = P$. Let $j \leq n$ be a positive integer such that for all $i < j$, $\mathcal{G}_{z_i}^0(\mathcal{O}) = \mathcal{G}_F^0(\mathcal{O}) = P$. The inclusion of $\{z_j\}$ in $F_{j-1} \cap F_j$ gives rise to $\mathcal{O}$-group scheme homomorphisms $\mathcal{H}_{F_{j-1}} \xrightarrow{\sigma_j} \mathcal{H}_{z_j} \leftarrow \mathcal{H}_F$ that are the identity on the generic fiber $H$. The images of the induced homomorphisms $\mathcal{H}_{F_{j-1}} \xrightarrow{\pi_j} \mathcal{H}_{z_j} \xrightarrow{\overline{\sigma}_j} \mathcal{H}_F$ are pseudo-parabolic $\kappa_\ast$-subgroups of $\mathcal{H}_{z_j}$ ([P2, 1.10(2)]). We conclude by Lie algebra consideration that $\overline{\sigma}_j(\mathcal{H}_{F_{j-1}}) = (\overline{\sigma}_j(\mathcal{H}_{F_{j-1}})^{\Theta})^\circ$ and $\overline{\pi}_j(\mathcal{H}_F) = (\overline{\pi}_j(\mathcal{H}_F)^{\Theta})^\circ$, and Lemma 3.10(ii) implies that both of these subgroups are pseudo-parabolic subgroups of $\mathcal{H}_{z_j}$. As $\mathcal{G}_{F_{j-1}}^0(\mathcal{O}) = P$, whereas, $P \subset \mathcal{G}_F^0(\mathcal{O}) \subset \mathcal{G}_{z_j}^0(\mathcal{O})$, we see that $\overline{\sigma}_j(\mathcal{H}_{F_{j-1}})$ is contained in $\overline{\pi}_j(\mathcal{H}_F)$. Let $\overline{Q}$ and $\overline{Q}'$ respectively be the images of $\overline{\sigma}_j(\mathcal{H}_{F_{j-1}})$ and $\overline{\pi}_j(\mathcal{H}_F)$ in the maximal pseudo-reductive quotient $\mathcal{G}_{z_j}^\text{pred} := \mathcal{H}_{z_j}^\circ / \mathcal{H}_{u,\kappa_\ast}(\mathcal{H}_{z_j})$ of $\mathcal{H}_{z_j}$. Then $\overline{Q} \subset \overline{Q}'$, and both of them are pseudo-parabolic subgroups of $\mathcal{G}_{z_j}^\text{pred}$.

Now let $S$ be a maximal $K$-split torus of $G$ such that the apartment of $\mathcal{B}$ corresponding to $S$ contains the geodesic $[xy]$ and let $v \in V(S)$ so that $v + x = y$. Then for all sufficiently small positive real number $\epsilon$, $-\epsilon v + z_j \in F_{j-1}$ and $\epsilon v + z_j \in F_j$. Using [P2, 1.10(3)] we infer that the images of the pseudo-parabolic subgroups $\overline{\sigma}_j(\mathcal{H}_{F_{j-1}})$ and $\overline{\pi}_j(\mathcal{H}_F)$ of $\mathcal{H}_{z_j}$ in the maximal pseudo-reductive quotient $\overline{\mathcal{H}}_{z_j}^\text{pred} := \mathcal{H}_{z_j}^\circ / \mathcal{H}_{u,\kappa_\ast}(\mathcal{H}_{z_j})$ of $\mathcal{H}_{z_j}$ are opposite pseudo-parabolic subgroups. Therefore, the image $\mathcal{H}$ of $\overline{\sigma}_j(\mathcal{H}_{F_{j-1}}) \cap \overline{\pi}_j(\mathcal{H}_F)$ in $\overline{\mathcal{H}}_{z_j}^\text{pred}$ is pseudo-reductive. Proposition A.8.14(2) of [CGP] implies then that $(\mathcal{H})^{\circ}$ is pseudo-reductive. It is obvious that under the natural homomorphism $\pi : \mathcal{G}_{z_j}^\text{pred} \to \overline{\mathcal{H}}_{z_j}^\text{pred}$, the image of $\overline{Q} = \overline{Q} \cap \overline{Q}'$ is $(\mathcal{H})^{\circ}$. As the kernel of the homomorphism $\pi$ is a finite (étale unipotent) subgroup (Lemma 3.10(i)), and $(\mathcal{H})^{\circ}$ is pseudo-reductive, we see that $\overline{Q}$ is a pseudo-reductive subgroup of $\mathcal{G}_{z_j}^\text{pred}$. But since $\overline{Q}$ is a pseudo-parabolic subgroup of the latter, we must have $\overline{Q} = \mathcal{G}_{z_j}^\text{pred}$, and hence, $\overline{Q}' = \mathcal{G}_{z_j}^\text{pred}$. So, $\overline{\sigma}_j(\mathcal{H}_{F_{j-1}}) = \mathcal{G}_{z_j}^\circ = \overline{\pi}_j(\mathcal{H}_F)$.

Since the natural homomorphism $\mathcal{G}_{F_{j-1}}^0(\mathcal{O}) \to \overline{\mathcal{H}}_{F_{j-1}}^\ast(\kappa)$ is surjective (as $\mathcal{O}$ is henselian and $\mathcal{G}_{F_{j-1}}^0$ is smooth, [EGA IV, 18.5.17]), and $\overline{\sigma}_j(\mathcal{H}_{F_{j-1}}) = \mathcal{G}_{z_j}^\circ$, the image
of $G^0_{F_{j-1}}(\mathcal{O})$ ($\subset G^0_j(\mathcal{O})$) in $\mathcal{F}^\circ_j(\kappa)$ is Zariski-dense in $\mathcal{F}^\circ_j$. From this we see that
\[ \mathcal{O}[G^0_j] = \{ f \in K[G] \mid f(G^0_{F_{j-1}}(\mathcal{O})) \subset \mathcal{O} \} = \mathcal{O}[G^0_{F_{j-1}}], \]
cf. [BrT2, 1.7.2] and 2.1. Therefore, $\sigma_j|_{G^0_{F_{j-1}}}: G^0_{F_{j-1}} \to G^0_j$ is a $\mathcal{O}$-group scheme isomorphism. We similarly see that $\rho_j|_{G^0_{F_{j-1}}}: G^0_{F_{j-1}} \to G^0_j$ is a $\mathcal{O}$-group scheme isomorphism. Now since $G^0_{F_{j-1}}(\mathcal{O}) = P$, we conclude that $P = G^0_j(\mathcal{O}) = G^0_{F_j}(\mathcal{O})$. By induction it follows that $P = G^0_i(\mathcal{O}) = G^0_{F_i}(\mathcal{O})$ for all $i \leq n$. In particular, for all $z \in [xy]$, except possibly for $z = y$, $G^0_z(\mathcal{O}) = P$, and $G^0_{F_0}(\mathcal{O}) = P$. □

For parahoric subgroups $P$ and $Q$ of $G(K)$, if $\mathcal{F}_P \cap \mathcal{F}_Q$ is nonempty, then for any $z$ in this intersection, $P = G^0_z(\mathcal{O}) = Q$ (Proposition 3.11). Thus every point of $\mathcal{B}$ is contained in a unique facet.

We will use the following simple lemma in the proof of the next proposition.

**Lemma 3.12.** Let $S$ be a maximal $K$-split torus of $G$, $A$ the corresponding apartment of $\mathcal{B}$, and $\mathcal{C}$ be a noncompact closed convex subset of $A$. Then for any point $x \in \mathcal{C}$, there is an infinite ray originating at $x$ and contained in $\mathcal{C}$.

**Proof.** Recall that $A$ is an affine space under the vector space $V(S) = \mathbb{R} \otimes_{\mathbb{Z}} X_*(S)$. We identify $A$ with $V(S)$ using translations by elements in the latter, with $x$ identified with the origin 0, and use a positive definite inner product on $V(S)$ to get a norm on $A$. With this identification, $\mathcal{C}$ is a closed convex subset of $V(S)$ containing 0. Since $\mathcal{C}$ is noncompact, there exist unit vectors $v_i \in V(S)$, $i \geq 1$, and positive real numbers $s_i \to \infty$ such that $s_i v_i$ lies in $\mathcal{C}$. After replacing $\{v_i\}$ by a subsequence, we may (and do) assume that the sequence $\{v_i\}$ converges to a unit vector $v$. We will now show that for every nonnegative real number $t$, $tv$ lies in $\mathcal{C}$, and this will prove the lemma. To see that $tv$ lies in $\mathcal{C}$, it suffices to observe that for a given $t$, the sequence $\{t v_i\}$ converges to $tv$, and for all sufficiently large $i$ (so that $s_i \geq t$), $tv_i$ lies in $\mathcal{C}$. □

**Proposition 3.13.** For any parahoric subgroup $P$ of $G(K)$, the associated closed facet $\mathcal{F}_P$ of $\mathcal{B}$, and so also the associated facet $\mathcal{F}_P(\subset \mathcal{F}_P)$, is bounded.

**Proof.** Let $S$ be a maximal $K$-split torus of $G$ such that the corresponding apartment of $\mathcal{B}$ contains $\mathcal{F}_P$ (3.7). Assume, if possible, that $\mathcal{F}_P$ is noncompact and fix a point $x$ of $\mathcal{F}_P$. Then, according to the preceding lemma, there is an infinite ray $\mathcal{R} := \{tv + x \mid t \in \mathbb{R}_{\geq 0}\}$, for some $v \in V(S)$, originating at $x$ and contained in $\mathcal{F}_P$. It is obvious from Proposition 3.11(i) that this ray is actually contained in $\mathcal{F}_P$. Hence, for every point $z \in \mathcal{R}$, $G^0_z(\mathcal{O}) = P$.

As the central torus of $G$ has been assumed to be $K$-anisotropic, there is a non-divisible root $a$ of $G$, with respect to $S$, such that $\langle a, v \rangle > 0$. Let $S_a$ be the identity component of the kernel of $a$ and $G_a$ (resp. $H_a$) be the derived subgroup of the centralizer of $S_a$ in $G$ (resp. $H$). Fix $t \in \mathbb{R}_{\geq 0}$, and let $y = tv + x \in \mathcal{R}$. Let $\mathcal{I}$ be the closed 1-dimensional $\mathcal{O}$-split torus of $G^0_y$ whose generic fiber is the maximal $K$-split torus of $G_a$ contained in $S$ and let $\lambda : GL_1 \to \mathcal{I}$ ($\hookrightarrow G^0_y \hookrightarrow \mathcal{I}_y$) be the $\mathcal{O}$-isomorphism such that $\langle a, \lambda \rangle > 0$. Let $c = \langle a, v \rangle / \langle a, \lambda \rangle$. Then $\langle a, v - c \lambda \rangle = 0$. 


Let \( Y \) be the \( \mathcal{O} \)-subgroup scheme of \( \mathcal{H}_y \) representing the functor
\[
R \rightsquigarrow \{ h \in \mathcal{H}_y(R) \mid \lim_{t \to 0} \lambda(t) h(t)^{-1} = 1 \},
\]
cf. [CGP, Lemma 2.1.5]. Using the last assertion of 2.1.8(3), and the first assertion of 2.1.8(4), of [CGP] (with \( k \), which is an an arbitrary commutative ring in these assertions, replaced by \( \mathcal{O} \), and \( G \) replaced by \( \mathcal{H}_y \)), we see that \( Y \) is a closed smooth unipotent \( \mathcal{O} \)-subgroup scheme of \( \mathcal{H}_y \) with connected fibers; the generic fiber of \( Y \) is \( U_H(\lambda) \), where \( U_H(\lambda) \) is as in [CGP, Lemma 2.1.5] with \( G \) replaced by \( H \). We consider the smooth closed \( \mathcal{O} \)-subgroup scheme \( Y^\Theta \) of \( Y \). As \( Y^\Theta \) is clearly normalized by \( \mathcal{F} \), it has connected fibers, and hence it is contained in \( (\mathcal{H}_y^\Theta)^\circ = G_\mathcal{F}^\circ \). The generic fiber of \( Y^\Theta \) is \( U_H(\lambda)^\Theta \) that contains the root group \( U_a(= U_{G_a}(\lambda)) \) of \( G \) corresponding to the root \( a \).

As \( \bigcup_{z \in \mathcal{R}} \mathcal{U}_z(0) \supset U_{H_a}(\lambda) \mathcal{K} \supset U_a(\mathcal{K}) \), we see that \( \bigcup_{z \in \mathcal{R}} \mathcal{U}_z^\Theta(0) \supset U_a(\mathcal{K}) \). Now since \( \mathcal{U}_z^\Theta \supset \mathcal{U}_z^\circ \), we conclude that \( \bigcup_{z \in \mathcal{R}} \mathcal{U}_z^\Theta(0) \supset U_a(\mathcal{K}) \). But for all \( z \in \mathcal{R} \), \( \mathcal{U}_z^\Theta(0) = P \), so the parahoric subgroup \( P \) contains the unbounded subgroup \( U_a(\mathcal{K}) \).

This is a contradiction.

Proposition 3.13 implies that each closed facet of \( \mathcal{B} \) is a compact polyhedron. Considering the facets lying on the boundary of a maximal closed facet of \( \mathcal{B} \), we see that \( \mathcal{B} \) contains facets of every dimension \( \leq K\text{-rank} G \).

3.14. Let \( P \) be a parahoric subgroup of \( G(\mathcal{K}) \) and \( \mathcal{F} := \mathcal{F}_P \) be the facet of \( \mathcal{B} \) associated to \( P \) in 3.7. Then for any \( x \in \mathcal{F} \), since \( P \subset \mathcal{F}_x^\mathcal{O}(0) \subset \mathcal{F}_x^\circ(0) = P \) (3.6(ii)), \( \mathcal{F}_x^\mathcal{O}(0) = P \) and hence the natural \( \mathcal{O} \)-group scheme homomorphism \( \mathcal{F}_x^\mathcal{O} \to \mathcal{F}_x^\circ \) is an isomorphism. In particular, for any facet \( F \) of \( \mathcal{B}(H/\mathcal{K}) \) that meets \( \mathcal{F} \), \( \mathcal{F}_x^\mathcal{O} = \mathcal{F}_x^\circ \).

Proposition 3.15. Let \( \mathcal{F} \) be a facet of \( \mathcal{B} \). Then the \( \kappa \)-unipotent radical \( \mathcal{R}_{a,\kappa}^{}(\mathcal{F}_x^\circ) \) of \( \mathcal{F}_x^\circ \) equals \( (\mathcal{F}_x^\circ \cap \mathcal{R}_{a,\kappa}(\mathcal{H}_x^\circ)) \).

Let \( \mathcal{F} \) and \( \mathcal{F}' \) be two facets of \( \mathcal{B} \), with \( \mathcal{F}' \prec \mathcal{F} \). Then:

(i) The kernel of the induced homomorphism \( \mathcal{p}_{\mathcal{F}'}: \mathcal{F}_x^\circ \to \mathcal{F}_x^\circ \) between the special fibers is a smooth unipotent \( \kappa \)-subgroup of \( \mathcal{F}_x^\circ \) and the image \( \mathcal{p}(\mathcal{F}'/\mathcal{F}) \) is a pseudo-parabolic \( \kappa \)-subgroup of \( \mathcal{F}_x^\circ \).

(ii) If \( F \) and \( F' \) are facets of \( \mathcal{B}(H/\mathcal{K}) \), \( F' \prec F \), that meet \( \mathcal{F} \) and \( \mathcal{F}' \) respectively, then \( \mathcal{p}(\mathcal{F}'/\mathcal{F}) = (\mathcal{F}_x^\circ) \), where \( \mathcal{F}_x^\circ \) is the image of \( \mathcal{p}_{F',F}: \mathcal{H}_x^\circ \to \mathcal{H}_x^\circ \).

(iii) The inverse image of the subgroup \( \mathcal{p}(\mathcal{F}'/\mathcal{F})(\kappa) \) of \( \mathcal{F}_x^\circ(\kappa) \), under the natural surjective homomorphism \( \mathcal{F}_x^\circ(\kappa) \to \mathcal{F}_x^\circ(\kappa) \), is \( \rho_{\mathcal{F}',\mathcal{F}}^{}(\mathcal{F}_x^\circ(\kappa)) \subset \mathcal{F}_x^\circ(\kappa) \).

Given a pseudo-parabolic \( \kappa \)-subgroup \( \mathcal{F} \) of \( \mathcal{F}_x^\circ \), there is a facet \( \mathcal{F} \) of \( \mathcal{B} \) with \( \mathcal{F}' \prec \mathcal{F} \) such that the image of the homomorphism \( \mathcal{p}_{\mathcal{F}'}: \mathcal{F}_x^\circ \to \mathcal{F}_x^\circ \) equals \( \mathcal{F} \).

Proof. The first assertion of the proposition follows immediately from Lemma 3.10(i).

To prove (i), we fix \( x \in \mathcal{F} \) and let \( F' \) be the facet of \( \mathcal{B}(H/\mathcal{K}) \) containing \( x \). As the closure of \( \mathcal{F} \) contains \( x \), there is a facet \( F \) of \( \mathcal{B}(H/\mathcal{K}) \) that meets \( \mathcal{F} \) and
contains $x$ in its closure. Then $F' \subset F$, i.e., $F' \prec F$, and $F$ and $F'$ meet $\mathcal{F}$ and $\mathcal{F}'$ respectively. Hence, $\mathcal{G}_{\mathcal{F}} = \mathcal{G}_{F'} = (\mathcal{H}_F^\Theta)^\circ$ and $\mathcal{G}_{\mathcal{F}'} = \mathcal{G}_{F''} = (\mathcal{H}_{F''}^\Theta)^\circ$ (3.14). Now we will prove assertions (i) and (ii) together. The kernel $\mathcal{K}$ of the homomorphism $\overline{p}_{F',F} : \mathcal{H}_F^\Theta \to \mathcal{H}_{F'}^\Theta$ is a smooth unipotent $\kappa$-subgroup, and the image $\mathcal{V}$ is a pseudo-parabolic $\kappa$-subgroup of $\mathcal{H}_{F'}^\Theta$ [P2, 1.10 (1), (2)]. The pseudo-parabolic subgroup $\mathcal{V}$ is clearly $\Theta$-stable as the facets $F$ and $F'$ are $\Theta$-stable. The kernel of $\overline{p}_{F',F}$ is $\mathcal{H} \cap \mathcal{V}$, and its image is contained in $(\mathcal{V}^\Theta)^\circ$. Therefore, the kernel of $\overline{p}_{F',F}$ contains $(\mathcal{H}^\Theta)^\circ$ and is contained in $\mathcal{H}^\Theta$. As $\mathcal{H}^\Theta$ is a smooth subgroup of $\mathcal{H}$, we see that the kernel of $\overline{p}_{F',F}$ is smooth.

Since the image of the Lie algebra homomorphism $L(\mathcal{F}^\Theta) \to L(\mathcal{V}^\Theta)$ induced by $\overline{p}_{F',F}$ is $L(\mathcal{V})^\Theta$, the containment $p(F' / \mathcal{F}) = \overline{p}_{F',F}(\mathcal{F}) \subset (\mathcal{V}^\Theta)^\circ$ is equality. According to Lemma 3.10(ii), $(\mathcal{V}^\Theta)^\circ$ is a pseudo-parabolic $\kappa$-subgroup of $\mathcal{V}^\Theta$.

To prove (iii), let $F' \prec F$ be as in the proof of (i) above and $\mathcal{V}$ be the image of $\overline{p}_{F',F} : \mathcal{H}_F^\Theta \to \mathcal{H}_{F'}^\Theta$. Then, as we saw above, $\mathcal{V}$ is a $\Theta$-stable pseudo-parabolic $\kappa$-subgroup of $\mathcal{H}_F^\Theta$ and $p(F' / \mathcal{F}) = \overline{p}_{F',F}(\mathcal{F}) = (\mathcal{V}^\Theta)^\circ$. The inverse image of the subgroup $\mathcal{V}(\kappa)$ of $\mathcal{H}_F^\Theta(\kappa)$ under the natural surjective homomorphism $\mathcal{H}_F^\Theta(\Omega) \to \mathcal{H}_F^\Theta(\kappa)$ equals $\rho_{F',F}(\mathcal{H}_F^\Theta(\Omega)) \subset \mathcal{H}_F^\Theta(\kappa)$, see [P2, 1.10 (4)]. Let $\mathcal{G}_F = (\mathcal{H}_F^\Theta)^\Theta$ and $\mathcal{G}_{F'} = (\mathcal{H}_{F'}^\Theta)^\Theta$. We will denote the $\Theta$-group scheme homomorphism $\mathcal{G}_F \to \mathcal{G}_{F'}$ induced by $\rho_{F',F}$ by $\overline{p}_{F',F}$; the corresponding homomorphism $\mathcal{F}_F \to \mathcal{F}_{F'}$ between the special fibers of $\mathcal{G}_F$ and $\mathcal{G}_{F'}$ will be denoted by $\overline{p}_{F',F}$. The neutral components of $\mathcal{G}_F$ and $\mathcal{G}_{F'}$ are $\mathcal{G}_F^\Theta$ and $\mathcal{G}_{F'}^\Theta$ respectively (3.14). Let $\mathcal{G}_F^\Theta(\kappa) \subset \mathcal{G}_F(\kappa)$ be the inverse image of $\mathcal{G}_F^\Theta$ in $\mathcal{G}_F$ under $\rho_{F',F}$. Since the homomorphism $\rho_{F',F}$ is the identity on the generic fiber $H$, we infer that $h \in \mathcal{H}_F^\Theta(\Omega)$ is fixed under $\Theta$ if and only if $\rho_{F',F}(h)$, and as the generic fiber of both $\mathcal{G}_F$ and $\mathcal{G}_{F'}$ is $G$, the generic fiber of $\mathcal{G}_F^\Theta$ is also $G$. It is easily seen now that the inverse image of the subgroup $p(F' / \mathcal{F})(\kappa)$ of $\mathcal{H}_F^\Theta(\kappa)$, under the natural surjective homomorphism $\mathcal{G}_F^\Theta(\Omega) \to \mathcal{G}_F^\Theta(\kappa)$, is $\rho_{F',F}(\mathcal{G}_F^\Theta(\Omega))$. We will presently show that the last group equals $\rho_{F',F}(\mathcal{G}_{F'}^\Theta(\Omega))$, this will prove (iii).

$\mathcal{G}_F^\Theta$ is the union of its generic fiber $G$ and its special fiber $\mathcal{F}_F$; and the identity component of $\mathcal{F}_F^\Theta$ is clearly $\mathcal{F}_F^\Theta$. We have shown above that the image $\mathcal{F}$ of $\mathcal{F}_F^\Theta$ under the homomorphism $\overline{p}_{\mathcal{F}_F^\Theta,F}(\mathcal{F}_F^\Theta(\kappa)) = \mathcal{F}(\kappa)$. So, according to [CGP, Thm. C.2.23], there is a pseudo-parabolic $\kappa$-subgroup $\mathcal{V}$ of $\mathcal{F}_F^\Theta$, that contains $\mathcal{F}$, such that $\overline{p}_{\mathcal{F}_F^\Theta,F}(\mathcal{V}(\kappa)) = \mathcal{F}(\kappa)$. But since $\kappa$ is infinite, $\mathcal{F}(\kappa)$/ $\mathcal{F}(\kappa)$ is infinite unless $\mathcal{F} = \mathcal{F}$. So we conclude that $\mathcal{F} = \mathcal{V}$, and then $\overline{p}_{F',F}(\mathcal{V}(\kappa)) = \mathcal{F}(\kappa) = \rho_{F',F}(\mathcal{G}_{\mathcal{F}_F^\Theta}(\kappa))$. Now using this, and the fact that the natural homomorphism $\mathcal{G}_{\mathcal{F}_F^\Theta}(\Omega) \to \mathcal{F}_F^\Theta(\kappa)$ is surjective (since $\Omega$ is henselian and
\[ \mathcal{F} \] is smooth, \([\text{EGA IV}_4,\ 18.5.17]\) and the kernel of this homomorphism equals the kernel of the natural surjective homomorphism \( \mathcal{F}_F(0) \to \mathcal{F}_F(\kappa) \), we see that \( \rho^G_{\mathcal{F},F}(\mathcal{F}_F(0)) = \rho^G_{\mathcal{F},F}(\mathcal{F}_F(0)) \). This proves (iii).

Finally, to prove the last assertion of the proposition, we fix a facet \( F' \) of \( \mathcal{B}(H/K) \) that meets \( \mathcal{F}' \). Then \( \mathcal{F}_F = \mathcal{F}_F' \) (3.14). Using Lemma 3.10(iii) for \( \kappa \) in place of \( k \) and \( \mathcal{F}_F' \) in place of \( \mathcal{F} \), we find a \( \Theta \)-stable pseudo-parabolic \( \kappa \)-subgroup \( \mathcal{D} \) of \( \mathcal{F}_F' \), such that \( \mathcal{F} = (\mathcal{D}^\circ) \circ \). Let \( (F' \prec)F \) be the facet of \( \mathcal{B}(H/K) \) corresponding to \( \mathcal{D} \). Then \( F \) is stable under \( \Theta \)-action. As \( F' \prec F \), there is a natural \( \Theta \)-group scheme homomorphism \( \rho_{F',F} : \mathcal{F}_F \to \mathcal{F}_F \), that restricts to a \( \Theta \)-group scheme homomorphism \( \rho^\circ_{F',F} : \mathcal{F}_F \to \mathcal{F}_F' \). Let \( \mathcal{D} \) be the image of the former. Then according to (ii), the image of the latter is \((\mathcal{D}^\circ)\circ = \mathcal{F} \). Let \( P = \mathcal{F}_F(0) \subset \mathcal{F}_F(0) =: \mathcal{Q} \), and \( \mathcal{F} = \mathcal{F}_P \). Then \( P \subset Q \) are parahoric subgroups of \( G(K) \), \( \mathcal{F}' = \mathcal{F}_Q \subset \mathcal{F}_Q \subset \mathcal{F}_P = \mathcal{F} \), thus \( \mathcal{F}' \prec \mathcal{F} \). As \( F \) and \( F' \) meet \( \mathcal{F} \) and \( \mathcal{F}' \) respectively, \( \mathcal{F}_F = \mathcal{F}_F' \) and \( \mathcal{F}_F = \mathcal{F}_F' \) (3.14), and hence the image of the homomorphism \( \rho^G_{\mathcal{F},F} : \mathcal{F}_F \to \mathcal{F}_F' \) equals \( \mathcal{F} \).

Proposition 3.15 and \([\text{CGP}, \text{Propositions 2.2.10 and 3.5.1}]\) imply the following.

(Recall that the residue field \( \kappa \) of \( K \) has been assumed to be separably closed!)

Corollary 3.16. (i) A facet \( \mathcal{F} \) of \( \mathcal{B} \) is a chamber (=maximal facet) if and only if \( \mathcal{F}_\mathcal{F} \) does not contain a proper pseudo-parabolic \( \kappa \)-subgroup. Equivalently, \( \mathcal{F} \) is a chamber if and only if the pseudo-reductive quotient \( \mathcal{G}_\mathcal{F}^{\text{pred}} \) is commutative (this is the case if and only if \( \mathcal{G}_\mathcal{F}^{\text{pred}} \) contains a unique maximal \( \kappa \)-torus, or, equivalently, every torus of this pseudo-reductive group is central).

(ii) The codimension of a facet \( \mathcal{F} \) of \( \mathcal{B} \) equals the \( \kappa \)-rank of the derived subgroup of the pseudo-split pseudo-reductive quotient \( \mathcal{G}_\mathcal{F}^{\text{pred}} := \mathcal{G}_\mathcal{F}^{\text{pred}} / \mathcal{G}_{u,\rho}(\mathcal{F}_\mathcal{F}) \) of \( \mathcal{G}_\mathcal{F}^{\text{pred}} \).

We will now establish the following analogues of Propositions 3.5–3.7 of [P2].

Proposition 3.17. Let \( \mathcal{A} \) be an apartment of \( \mathcal{B} \), and \( \mathcal{C}, \mathcal{C}' \) two chambers in \( \mathcal{A} \). Then there is a gallery joining \( \mathcal{C} \) and \( \mathcal{C}' \) in \( \mathcal{A} \), i.e., there is a finite sequence

\[ \mathcal{C} = \mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_m = \mathcal{C}' \]

of chambers in \( \mathcal{A} \) such that for \( i \) with \( 1 \leq i \leq m \), \( \mathcal{C}_{i-1} \) and \( \mathcal{C}_i \) share a face of codimension 1.

Proof. Let \( \mathcal{A}_2 \) be the codimension 2-skelton of \( \mathcal{A} \), i.e., the union of all facets in \( \mathcal{A} \) of codimension at least 2. Then \( \mathcal{A}_2 \) is a closed subset of \( \mathcal{A} \) of codimension 2, so \( \mathcal{A} - \mathcal{A}_2 \) is a connected open subset of the affine space \( \mathcal{A} \). Hence \( \mathcal{A} - \mathcal{A}_2 \) is arcwise connected. This implies that given points \( x \in \mathcal{C} \) and \( x' \in \mathcal{C}' \), there is a piecewise linear curve in \( \mathcal{A} - \mathcal{A}_2 \) joining \( x \) and \( x' \). Now the chambers in \( \mathcal{A} \) that meet this curve make a gallery joining \( \mathcal{C} \) to \( \mathcal{C}' \). \( \square \)
As the central torus of $G$ is $K$-anisotropic, the dimension of any apartment, or any chamber, in $B$ is equal to the $K$-rank of $G$. A panel in $B$ is by definition a facet of codimension 1.

**Proposition 3.18.** $B$ is thick, that is any panel is a face of at least three chambers, and every apartment of $B$ is thin, that is any panel lying in an apartment is a face of exactly two chambers of the apartment.

**Proof.** Let $F$ be a facet of $B$ that is not a chamber, and $C$ be a chamber of which $F$ is a face. Then there is an $O$-group scheme homomorphism $\rho_{G,F}^C : G_C \rightarrow G_F$ (3.2). The image of $G_C$ in $G_F$, under the induced homomorphism of special fibers, is a minimal pseudo-parabolic $\kappa$-subgroup of $G_F$, and conversely, any minimal pseudo-parabolic $\kappa$-subgroup of the latter determines a chamber of $B$ with $F$ as a face (Corollary 3.16). Now as $\kappa$ is infinite, $G_F$ contains infinitely many minimal pseudo-parabolic $\kappa$-subgroups. We conclude that $F$ is a face of infinitely many chambers.

The second assertion follows at once from the following well-known result in algebraic topology: In any simplicial complex whose geometric realization is a topological manifold without boundary (such as an apartment $A$ of $B$), any simplex of codimension 1 is a face of exactly two chambers (i.e., maximal dimensional simplices). □

**Proposition 3.19.** Let $A$ be an apartment of $B$ and $S$ be the maximal $K$-split torus of $G$ corresponding to this apartment. (Then $A = B(Z_H(S)/K)^\Theta$.) The group $N_G(S)(K)$ acts transitively on the set of chambers of $A$.

**Proof.** According to Proposition 3.17, given any two chambers in $A$, there exists a minimal gallery in $A$ joining these two chambers. So to prove the proposition by induction on the length of a minimal gallery joining two chambers, it suffices to prove that given two different chambers $\mathcal{C}$ and $\mathcal{C}'$ in $A$ which share a panel $F$, there is an element $n \in N_G(S)(K)$ such that $n \cdot \mathcal{C} = \mathcal{C}'$. Let $\mathcal{G} := G_F$ be the Bruhat-Tits smooth affine $O$-group scheme associated with the panel $F$, and $\mathcal{G} \subset \mathcal{G}$ be the closed $O$-torus with generic fiber $S$. Let $\mathcal{F}$ be the special fiber of $\mathcal{G}$ and $\mathcal{F}'$ the special fiber of $\mathcal{G}'$. Then $\mathcal{F}$ is a maximal torus of $\mathcal{F}$. The chambers $\mathcal{C}$ and $\mathcal{C}'$ correspond to minimal pseudo-parabolic $\kappa$-subgroups of $\mathcal{F}$ and $\mathcal{F}'$ of $\mathcal{F}$ (Corollary 3.16). Both of these minimal pseudo-parabolic $\kappa$-subgroups contain $\mathcal{F}$ since the chambers $\mathcal{C}$ and $\mathcal{C}'$ lie in $A$. But then by Theorems C.2.5 and C.2.3 of [CGP], there is an element $\bar{n} \in \mathcal{F}(\kappa)$ that normalizes $\mathcal{F}$ and conjugates $\mathcal{F}$ onto $\mathcal{F}'$. Now from Proposition 2.1(iii) of [P2] we conclude that there is an element $n \in N_{G}(\mathcal{F})(\kappa)(\mathcal{F})$ lying over $\bar{n}$. It is clear that $n$ normalizes $S$ and hence it lies in $N_G(S)(K)$; it fixes $F$ pointwise and $n \cdot \mathcal{C} = \mathcal{C}'$. □

Now in view of Propositions 2.14, 3.4, 3.17 and 3.18, Theorem 3.11 of [Ro] (cf. also [P2, 1.8]) implies that $B$ is an affine building if for any maximal $K$-split torus $S$ of $G$, $B(Z_H(S)/K)^\Theta$ is taken to be the corresponding apartment, and $B$ is given the polysimplicial structure described in 3.7. Thus we obtain the following:
Theorem 3.20. $\mathcal{B} = \mathcal{B}(H/K)^\Theta$ is an affine building. Its apartments are the affine spaces $\mathcal{B}(Z_H(S)/K)^\Theta$ under $V(S) := \mathbb{R} \otimes_{\mathbb{Z}} X_*(S)$ for maximal $K$-split tori $S$ of $G$. Its facets are as in 3.7. The group $G(K)$ acts on $\mathcal{B}$ by polysimplicial isometries.

From Propositions 2.15 and 3.19 we obtain the following.

Proposition 3.21. $G(K)$ acts transitively on the set of ordered pairs $(A, C)$ consisting of an apartment $A$ of $\mathcal{B}$ and a chamber $C$ of $A$.

Remark 3.22. (i) As in [P2, 3.16], using the preceding proposition we can obtain Tits systems in suitable subgroups of $G(K)$.

(ii) As in [P2, §5], we can obtain filtration of root groups and a valuation of root datum for $G/K$.

§4. Tamely-ramified descent

We begin by proving the following proposition:

Proposition 4.1. Let $k$ be a field of characteristic $p \geq 0$. Let $\mathcal{H}$ be a noncommutative pseudo-reductive $k$-group, $\theta$ a $k$-automorphism of $\mathcal{H}$ of finite order not divisible by $p$, and $\mathfrak{G} := (\mathcal{H}(\theta))^\circ$. Then

(i) No maximal torus of $\mathfrak{G}$ is central in $\mathcal{H}$.

(ii) The centralizer in $\mathcal{H}$ of any maximal torus of $\mathfrak{G}$ is commutative.

(iii) Given a maximal $k$-torus $\mathfrak{S}$ of $\mathfrak{G}$, there is a $\theta$-stable maximal $k$-torus of $\mathcal{H}$ containing $\mathfrak{S}$.

(iv) If $k$ is separably closed, then $\mathcal{H}$ contains a $\theta$-stable proper pseudo-parabolic $k$-subgroup.

Proof. We fix an algebraic closure $\bar{k}$ of $k$. Let $\mathcal{H}'$ be the maximal reductive quotient of $\mathcal{H}_{\bar{k}}$. As $\mathcal{H}$ is noncommutative, $\mathcal{H}'$ is also noncommutative (see [CGP, Prop. 1.2.3]).

The automorphism $\theta$ induces a $\bar{k}$-automorphism of $\mathcal{H}'$ which we will denote again by $\theta$. According to a theorem of Steinberg [St, Thm. 7.5], $\mathcal{H}_{\bar{k}}$ contains a $\theta$-stable Borel subgroup $\mathcal{B}$, and this Borel subgroup contains a $\theta$-stable maximal torus $\mathfrak{T}$. The natural quotient map $\pi : \mathcal{H}_{\bar{k}} \to \mathcal{H}'$ carries $\mathfrak{T}$ isomorphically onto a maximal torus of $\mathcal{H}'$. We endow the root system of $\mathcal{H}'$ with respect to the maximal torus $\mathfrak{T}' := \pi(\mathfrak{T}) \cap \mathfrak{Z}(\mathcal{H}')$ of the derived subgroup $\mathfrak{D}(\mathcal{H}')$ of $\mathcal{H}'$ with the ordering determined by the Borel subgroup $\pi(\mathcal{B})$. Let $a$ be the sum of all positive roots. Then as $\pi(\mathcal{B})$ is $\theta$-stable, $a$ is fixed under $\theta$ acting on the character group $X(\mathfrak{T}')$ of $\mathfrak{T}'$. Therefore, $X(\mathfrak{T}')$ admits a nontrivial torsion-free quotient on which $\theta$ acts trivially. This implies that $\mathfrak{T}$ contains a nontrivial subtorus $\mathfrak{F}$ that is fixed pointwise under $\theta$ and is mapped by $\pi$ into $\mathfrak{T}'$ (\subset $\mathfrak{D}(\mathcal{H}')$). The subtorus $\mathfrak{F}$ is therefore contained in $\mathfrak{G}_{\bar{k}}$. Since the center of the semi-simple group $\mathfrak{D}(\mathcal{H}')$ does not contain a nontrivial smooth connected subgroup, we infer that $\mathfrak{F}$ is not central in $\mathcal{H}_{\bar{k}}$. Thus the subgroup $\mathfrak{G}_{\bar{k}}$ contains a noncentral torus of $\mathcal{H}_{\bar{k}}$. Now by conjugacy of maximal tori in $\mathfrak{G}_{\bar{k}}$, we see that no maximal torus of this group can be central in $\mathcal{H}_{\bar{k}}$. This proves (i).
To prove (ii), let $S$ be a maximal torus of $\mathcal{G}$. Then the centralizer $Z_{\mathcal{G}}(S)$ of $S$ in $\mathcal{G}$ is a $\theta$-stable pseudo-reductive subgroup of $\mathcal{G}$, and $(Z_{\mathcal{G}}(S)^{\theta})^\circ = Z_{\mathcal{G}}(S)$. As $S$ is a maximal torus of $Z_{\mathcal{G}}(S)$ that is central in $Z_{\mathcal{G}}(S)$, if $Z_{\mathcal{G}}(S)$ were noncommutative, we could apply (i) to this subgroup in place of $\mathcal{G}$ to get a contradiction.

To prove (iii), we consider the centralizer $Z_{\mathcal{H}}(S)$ of $S$ in $\mathcal{H}$. This centralizer is $\theta$-stable and commutative according to (ii). The unique maximal $k$-torus of it contains $S$ and is a $\theta$-stable maximal torus of $\mathcal{H}$.

To prove (iv), we assume now that $k$ is separably closed and let $S$ be a maximal torus of $\mathcal{G}$. Then $S$ is $k$-split, and in view of (i), there is a 1-parameter subgroup $\lambda : \text{GL}_1 \to S$ whose image is not central in $\mathcal{H}$. Then $P_\mathcal{H}(\lambda)$ is a $\theta$-stable proper pseudo-parabolic $k$-subgroup of $\mathcal{H}$.

In the following proposition we will use the notation introduced in §§1, 2. As in 2.4, we will assume that $H$ is semi-simple and the central torus of $G$ is $K$-anisotropic. We will further assume that $H$ is $K$-isotropic, $\Theta$ is a finite cyclic group of automorphisms of $H$, and $p$ does not divide the order of $\Theta$.

**Proposition 4.2.** The Bruhat-Tits building $B(\mathcal{H}/K)$ of $H(K)$ contains a $\Theta$-stable chamber.

**Proof.** Let $F$ be a $\Theta$-stable facet of $B(\mathcal{H}/K)$ that is maximal among the $\Theta$-stable facets. Let $\mathcal{H} := \mathcal{H}_F^\circ$ be the Bruhat-Tits smooth affine $\Theta$-group scheme with generic fiber $H$, and connected special fiber $\mathcal{H}$, corresponding to $F$. Let $\mathcal{H} := \mathcal{H} / R_{\text{un},\kappa}(\mathcal{H})$ be the maximal pseudo-reductive quotient of $\mathcal{H}$. In case $\mathcal{H}$ is commutative, $\mathcal{H}$ does not contain a proper pseudo-parabolic $\kappa$-subgroup and so $F$ is a chamber of $B(\mathcal{H}/K)$. We assume, if possible, that $\mathcal{H}$ is not commutative. As $F$ is stable under the action of $\Theta$, there is a natural action of this finite cyclic group on $\mathcal{H}$ by $\Theta$-group scheme automorphisms (2.4). This action induces an action of $\Theta$ on $\mathcal{H}$, and so also on its pseudo-reductive quotient $\mathcal{H}$. Now taking $\theta$ to be a generator of $\Theta$, and using the preceding proposition for $\mathcal{H}/\kappa$, we conclude that $\mathcal{H}$ contains a $\Theta$-stable proper pseudo-parabolic $\kappa$-subgroup. The inverse image $P$ in $\mathcal{H}$ of any such pseudo-parabolic subgroup of $\mathcal{H}$ is a $\Theta$-stable proper pseudo-parabolic $\kappa$-subgroup of $\mathcal{H}$. The facet $F'$ corresponding to $P$ is $\Theta$-stable and $F < F'$. This contradicts the maximality of $F$. Hence, $\mathcal{H}$ is commutative and $F$ is a chamber. \qed

To prove the next theorem (Theorem 4.4), we will use the following:

**Proposition 4.3.** Let $\mathfrak{K}$ be a field complete with respect to a discrete valuation and with separably closed residue field. Let $\mathfrak{G}$ be a connected absolutely simple $\mathfrak{K}$-group of inner type $A$ that splits over a finite tamely-ramified field extension $\mathcal{L}$ of $\mathfrak{K}$. Then $\mathfrak{G}$ is $\mathfrak{K}$-split.

**Proof.** We may (and do) assume that $\mathfrak{G}$ is simply connected. Then $\mathfrak{G}$ is $\mathfrak{K}$-isomorphic to $\text{SL}_n \mathfrak{D}$, where $\mathfrak{D}$ is a finite dimensional division algebra with center $\mathfrak{K}$ that splits over the finite tamely-ramified field extension $\mathcal{L}$ of $\mathfrak{K}$. By Propositions 4 and 12 of [S, Ch. II] the degree of $\mathfrak{D}$ is a power of $p$, where $p$ is the characteristic of the residue
field of $\mathfrak{R}$. But a noncommutative division algebra of degree a power of $p$ cannot split over a field extension of degree prime to $p$. So, $\mathfrak{D} = \mathfrak{R}$, hence $\mathfrak{G} \simeq \text{SL}_n$ is $\mathfrak{R}$-split.

**Theorem 4.4.** A semi-simple $K$-group $G$ that is quasi-split over a finite tamely-ramified field extension of $K$ is already quasi-split over $K$.

This theorem has been proved by Philippe Gille in [Gi] by an entirely different method.

**Proof.** We assume that all field extensions appearing in this proof are contained in a fixed separable closure of $K$. To prove the theorem, we may (and do) replace $G$ by its simply-connected central cover and assume that $G$ is simply connected. Let $S$ be a maximal $K$-split torus of $G$. Then $G$ is quasi-split over a (separable) extension $L$ of $K$ if and only if the derived subgroup $Z_G(S)'$ of the centralizer $Z_G(S)$ of $S$ is quasi-split over $L$. Moreover, $G$ is quasi-split over $K$ if and only if $Z_G(S)'$ is trivial. Therefore, to prove the theorem we need to show that any semi-simple simply connected $K$-anisotropic $K$-group that is quasi-split over a finite tamely-ramified field extension of $K$ is necessarily trivial. Let $G$ be any such group.

There exists a finite indexing set $I$, and for each $i \in I$, a finite separable field extension $K_i$ of $K$ and an absolutely almost simple simply connected $K_i$-anisotropic $K_i$-group $G_i$ such that $G = \prod_{i \in I} R_{K_i/K}(G_i)$. Now $G$ is quasi-split over a finite separable field extension $L$ of $K$ if and only if for each $i$, $R_{K_i/K}(G_i)$ is quasi-split over $L$. But $R_{K_i/K}(G_i)$ is quasi-split over $L$ if and only if $G_i$ is quasi-split over the compositum $L_i := K_i L$. For $i \in I$, the finite extension $K_i$ of $K$ is complete and its residue field is separably closed, and if $L$ is a finite tamely-ramified field extension of $K$, then $L_i$ is a finite tamely-ramified field extension of $K_i$. So to prove the theorem, we may (and do) replace $K$ by $K_i$ and $G$ by $G_i$ to assume that $G$ is an absolutely almost simple simply connected $K$-anisotropic $K$-group that is quasi-split over a finite tamely-ramified field extension of $K$. We will show that such a group $G$ is trivial.

Let $L$ be a finite tamely-ramified field extension of $K$ of minimal degree over which $G$ is quasi-split. Since the residue field $\kappa$ of $K$ is separably closed, $L$ is a cyclic extension of $K$. Let $\Theta$ be the Galois group of $L/K$. Then $\Theta$ is a finite cyclic group of order not divisible by $p (= \text{char}(\kappa))$.

As $G_L$ is quasi-split, Bruhat-Tits theory is available for $G$ over $L$ [BrT2, §4]. The Galois group $\Theta$ acts on $G(L)$ by continuous automorphisms and so it acts on the Bruhat-Tits building $\mathcal{B}(G/L)$ of $G(L)$ by polysimplicial isometries. Let $H = R_{L/K}(G_L)$. Then $H$ is quasi-split over $K$ and hence Bruhat-Tits theory is also available for $H$ over $K$. Let $\mathcal{B}(H/K)$ be the Bruhat-Tits building of $H(K) (= G(L))$. Elements of $\Theta$ act by $K$-automorphisms on $H$ and so on $\mathcal{B}(H/K)$ by polysimplicial isometries; moreover, $G = H^\Theta$. There is a natural $\Theta$-equivariant identification of the building $\mathcal{B}(H/K)$ with the building $\mathcal{B}(G/L)$. (Note that $K$-rank $H = L$-rank $G_L$, and there is a natural bijective correspondence between the set of maximal $K$-split
tori of $H$ and the set of maximal $L$-split tori of $G_L$, see [CGP, Prop. A.5.15(2)]. This correspondence will be used below.) The results of §3 imply that Bruhat-Tits theory is available for $G$ over $K$ and $\mathcal{B} := \mathcal{B}(H/K)^\Theta (= \mathcal{B}(G/L)^\Theta)$ is the Bruhat-Tits building of $G(K)$.

Since $G$ is $K$-anisotropic, the building of $G(K)$ consists of a single point, hence $\Theta$ fixes a unique point of $\mathcal{B}(G/L)$. Let $C$ be the facet of $\mathcal{B}(G/L)$ that contains this point. Then $C$ is stable under $\Theta$. According to Proposition 4.2, $C$ is a chamber. Let $\mathcal{H} := \mathcal{H}_C$ be the Bruhat-Tits smooth affine $\Theta$-group scheme associated to $C$ with generic fiber $H$ and connected special fiber $\overline{\mathcal{H}}$. As $C$ is a chamber, the maximal pseudo-reductive quotient $\overline{\mathcal{H}}^{\text{pred}}$ of $\overline{\mathcal{H}}$ is commutative [P2, 1.10].

Now using Proposition 2.6 for $\Omega = C = F$ we obtain a $\Theta$-stable maximal $K$-split torus $T$ of $H$ such that $C$ lies in the apartment $A(T)$ corresponding to $T$ (and the special fiber of the schematic closure of $T$ in $\overline{\mathcal{H}}$ maps onto the maximal torus of $\overline{\mathcal{H}}^{\text{pred}}$). Let $T'$ be the image of $T_L$ under the natural surjective homomorphism $q : H_L = R_{L/K}(G_L)_L \to G_L$. Then $T'$ is a $L$-torus of $G_L$ and according to [CGP, Prop. A.5.15(2)] it is the unique maximal $L$-split torus of $G_L$ such that $R_{L/K}(T')( \subset R_{L/K}(G_L) = H)$ contains the maximal $K$-split torus $T$ of $H$.

We identify $H(K)$ with $G(L)$. Then for $x \in H(K) (\subset H(L))$ and $\theta \in \Theta$, we have $q(\theta(x)) = \theta(x)$. Since $T(K)$ is $\Theta$-stable for $t \in T(K)$ and $\theta \in \Theta$, $\theta(t)$ lies in $T'(L)$. Now as $T(K)$ is Zariski-dense in $T$, its image in $T'(L)$ is Zariski-dense in $T'$. Since this image is stable under the action of $\Theta = \text{Gal}(L/K)$ on $G(L)$, from the Galois criterion [Bo, Ch. AG, Thm. 14.4(3)] we infer that $T'$ descends to a $K$-torus of $G$, i.e., there is a $K$-torus $T$ of $G$ such that $T' = T_L$. In the natural identification of $\mathcal{B}(H/K)$ with $\mathcal{B}(G/L)$, the apartment $A(T)$ of the former is $\Theta$-equivariantly identified with the apartment $A(T')$ of the latter. We will view the chamber $C$ as a $\Theta$-stable chamber in $A(T')$.

Let $\Delta$ be the basis of the affine root system of the absolutely almost simple, simply connected quasi-split $L$-group $G_L$ with respect to $T'$ ($= T_L$), determined by the $\Theta$-stable chamber $C$ [BrT2, §4]. Then $\Delta$ is stable under the action of $\Theta$ on the affine root system of $G_L$ with respect to $T'$. There is a natural $\Theta$-equivariant bijective correspondence between the set of vertices of $C$ and $\Delta$. Since $\mathcal{B}$, and hence $C^{\Theta}$, consists of a single point, $\Theta$ acts transitively on the set of vertices of $C$ so it acts transitively on $\Delta$. Now from the classification of irreducible affine root systems [BrT1, §1.4.6], we see that $G_L$ is a split group of type $A_n$ for some $n$. Proposition 4.3 implies that $G$ cannot be of inner type $A_n$ over $K$. On the other hand, if $G$ is of outer type $A_n$, then over a quadratic Galois extension $K'(\subset L)$ of $K$ it is of inner type. Now, according to Proposition 4.3, $G$ splits over $K'$. We conclude that $L = K'$ and hence $\#\Theta = 2$. As $\Theta$ acts transitively on $\Delta$ and $\#\Delta = n + 1$, we infer that $n + 1 = 2$, i.e., $n = 1$, and then $G$ is of inner type, a contradiction. $\Box$
4.5. Now let $k$ be a field endowed with a nonarchimedean discrete valuation. We assume that the valuation ring of $k$ is Henselian. Let $K$ be the maximal unramified extension of $k$, and $L$ be a finite tamely-ramified field extension of $K$ with Galois group $\Theta := \text{Gal}(L/K)$. Let $G$ be a connected reductive $k$-group that is quasi-split over $K$ and $H = R_{L/K}(G_L)$. Then $G = H^\Theta$, and by Theorem 3.20, the Bruhat-Tits building $\mathcal{B}(G/K)$ of $G(K)$ can be identified with the subspace of points in the Bruhat-Tits building of $G(L)$ ($= H(K)$) that are fixed under $\Theta$ (with polysimplicial structure on $\mathcal{B}(G/K)$ as in 3.7). Now by “unramified descent” [P2], Bruhat-Tits theory is available for $G$ over $k$ and the Bruhat-Tits building of $G(k)$ is $\mathcal{B}(G/K)^{\text{Gal}(k/K)}$.

References


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