A NEW APPROACH TO UNRAMIFIED DESCENT IN BRUHAT-TITS THEORY

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Dedicated to my wife Indu Prasad with gratitude

Abstract. We give a new approach to unramified descent in Bruhat-Tits theory of reductive groups over a discretely valued field k, with Henselian local ring and perfect residue field, which appears to be conceptually simpler, and more geometric, than the original approach of Bruhat and Tits. We are able to derive the main results of the theory over k from the theory over the maximal unramified extension K of k. Even in the most interesting case for number theory and representation theory, where k is a locally compact nonarchimedean field, the geometric approach described in this paper appears to be considerably simpler than the original approach.

Introduction. Let k be a discretely valued field whose local ring is Henselian. Throughout this paper, we will assume that the residue field of k is perfect. The Bruhat-Tits theory of reductive groups over k has two parts. The first part is the theory over the maximal unramified extension K of k; because of our assumption on k, any reductive k-group G is quasi-split over K (see 1.7) and hence $G(K)$ has a rather simple structure. This part of the theory is due to Iwahori-Matsumoto, Hijikata and Bruhat and Tits. The second part, called the "unramified descent" (or "étale descent"), is due to Bruhat and Tits. This part gives us the Bruhat-Tits theory over k, and also the Bruhat-Tits building of $G(k)$, from the Bruhat-Tits theory over K and the Bruhat-Tits building $\mathcal{B}(G/K)$ of $G(K)$, using descent of valuation of root datum from K to k. This second part is somewhat technical; see [BrT1, §9] and [BrT2, §5].

The purpose of this paper is to present an alternative approach to unramified descent which appears to be conceptually simpler, and more geometric, than the approach in [BrT1], [BrT2] in that it does not use descent of valuation of root datum from K to k to show that $\mathcal{B}(G/K)^{\Gamma}$, where Γ is the Galois group of K/k, is an affine building. In this approach, we will use the Bruhat-Tits theory, and the buildings, only over the maximal unramified extension K of k and derive the main results of the theory for reductive groups over k. In §4, we will describe a natural filtration of the root groups $U_a(k)$ and also a valuation of the root datum of G/k relative to a maximal k-split torus S, using the geometric results of §§2, 3 that provide the Bruhat-Tits building of $G(k)$. The approach described here appears to be considerably simpler than the original approach even for reductive groups over locally compact nonarchimedean fields (i.e., discretely valued complete fields with finite residue field). In §5, we prove results over discretely valued fields with

Henselian local ring and perfect residue field of dimension $\leqslant 1$; of these, Theorems 5.1 and 5.2 may be new.

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1. Preliminaries. We assume below that k is a field given with a discrete valuation ω and the local ring of k is Henselian. It is known that the local ring of k is Henselian if and only if the valuation ω extends uniquely to any algebraic extension of k. The local ring of a discretely valued complete field is Henselian. (For various equivalent definitions of the Henselian property, see [Be, §§2.3-2.4].) Let K be the maximal unramified extension of k contained in a fixed algebraic closure $\overline{k}$ of k. We will denote the unique valuation on the algebraic closure $\overline{k}\,(\supset K)$, extending the given valuation on k, also by ω. We will denote the residue field of k by κ and the local rings of k and K by $\mathfrak{o}$ and $\mathcal{O}$ respectively. We will assume in this paper that κ is perfect. Let G be a connected reductive group defined over k and $\mathrm{X}_k^*(G)$ be the group of k-rational characters on G.

The following theorem is due to Bruhat, Tits and Rousseau. An elementary proof was given in [P] which we recall here for the reader's convenience.

Theorem 1.1. *$G(k)$ is bounded if and only if G is anisotropic over k.*

Remark 1.2. Thus if k is a nondiscrete locally compact field, then $G(k)$ is compact if and only if G is k-anisotropic.

We fix a faithful k-rational representation of G on a finite dimensional k-vector space V and view G as a k-subgroup of $\mathrm{GL}(V)$. To prove the above theorem we will use the following two lemmas:

Lemma 1.3. *If $f : X \to Y$ is a finite $\overline{k}$-morphism between affine $\overline{k}$-schemes of finite type and B is a bounded subset of $Y(\overline{k})$ then the subset $f^{-1}(B)$ of $X(\overline{k})$ is bounded.*

Proof. Since $\overline{k}[X]$ is module-finite over $\overline{k}[Y]$, we can pick a finite set of generators of $\overline{k}[X]$ as a $\overline{k}[Y]$-module (so also as a $\overline{k}[Y]$-algebra), and each satisfies a monic polynomial over $\overline{k}[Y]$. Hence, this realizes X as a closed subscheme of the closed subscheme $Z \subset Y \times \mathbb{A}^n$ defined by n monic 1-variable polynomials $f_1(t_1), \dots , f_n(t_n)$ over $\overline{k}[Y]$, so it remains to observe that when one has a bound on the coefficients of a monic 1-variable polynomial over $\overline{k}$ of known degree (e.g., specializing any f_j at a $\overline{k}$-point of Y) then one gets a bound on its possible $\overline{k}$-rational roots depending only on the given coefficient bound and the degree of the monic polynomial. $\square$

Lemma 1.4. *Let $\mathcal{G}$ be an unbounded subgroup of $G(k)$ which is dense in G in the Zariski-topology. Then $\mathcal{G}$ contains an element g which has an eigenvalue α with $\omega(\alpha) < 0$.*

Proof. Let

$$\overline{k}\otimes_k V =: V_0 \supset V_1 \supset \cdots \supset V_s \supset V_{s+1} = \{0\}$$

be a flag of $G_{\overline{k}}$-invariant subspaces such that for $0 \leqslant i \leqslant s$, the natural representation ρ_i of $G_{\overline{k}}$ on $W_i := V_i/V_{i+1}$ is irreducible. Let $\rho = \bigoplus_i \rho_i$ be the representation of $G_{\overline{k}}$ on $\bigoplus_i W_i$. The kernel of ρ is obviously a unipotent normal $\overline{k}$-subgroup scheme of the reductive group $G_{\overline{k}}$, and hence it is finite. Now as $\mathcal{G}$ is an unbounded subgroup of $G(k)$, Lemma 1.3 implies that $\rho(\mathcal{G})$ is an unbounded subgroup of $\rho(G(\overline{k}))$. Hence, there is a non-negative integer $a \leqslant s$ such that $\rho_a(\mathcal{G})$ is unbounded.

Since W_a is an irreducible $G_{\overline{k}}$-module, and $\mathcal{G}$ is dense in G in the Zariski-toplogy, $\rho_a(\mathcal{G})$ spans $\mathrm{End}_{\overline{k}}(W_a)$. We fix $\{g_i\} \subset \mathcal{G}$ so that $\{\rho_a(g_i)\}$ is a basis of $\mathrm{End}_{\overline{k}}(W_a)$. Let $\{f_i\} \subset \mathrm{End}_{\overline{k}}(W_a)$ be the basis which is dual to the basis $\{\rho_a(g_i)\}$ with respect to the trace-form. Then $\mathrm{Tr}(f_i \cdot \rho_a(g_j)) = \delta_{ij}$, where δ_{ij} is the Kronecker's delta. Now assume that the eigenvalues of all the elements of $\mathcal{G}$ lie in the valuation ring $\mathfrak{o}_{\overline{k}}$ of $\overline{k}$. Then for all $x \in \mathcal{G}$, $\mathrm{Tr}(\rho_a(x))$ is contained in $\mathfrak{o}_{\overline{k}}$. For $g \in \mathcal{G}$, if $\rho_a(g) = \sum c_i f_i$, with $c_i \in \overline{k}$, then $\mathrm{Tr}(\rho_a(g \cdot g_j)) = \sum_i c_i \mathrm{Tr}(f_i \cdot \rho_a(g_j)) = c_j$. As $\mathrm{Tr}(\rho_a(g \cdot g_j)) \in \mathfrak{o}_{\overline{k}}$, we conclude that c_j belongs to the ring of integers $\mathfrak{o}_{\overline{k}}$ for all j (and all $g \in \mathcal{G}$). This implies that $\rho_a(\mathcal{G})$ is bounded, a contradiction. □

Proof of Theorem 1.1. As $\mathrm{GL}_1(k) = k^{\times}$ is unbounded, we see that if G is k-isotropic, then $G(k)$ is unbounded. We will now assume that $G(k)$ is unbounded and prove the converse.

It is well known that $G(k)$ is dense in G in the Zariski-topology [Bo, 18.3], hence according to Lemma 1.4, there is an element $g \in G(k)$ which has an eigenvalue α with $\omega(\alpha) \neq 0$. Now in case k is of positive characteristic, after replacing g by a suitable power, we assume that g is semi-simple. On the other hand, in case k is of characteristic zero, let $g = s{\cdot}u = u{\cdot}s$ be the Jordan decomposition of g with $s \in G(k)$ semi-simple and $u \in G(k)$ unipotent. Then the eigenvalues of g are same as that of s. So, after replacing g with s, we may (and do) again assume that g is semi-simple. There is a maximal k-torus T of G such that $g \in T(k)$ (see [BoT], Proposition 10.3 and Theorem 2.14(a); note that according to Theorem 11.10 of [Bo], g is contained in a maximal torus of G). Since any absolutely irreducible representation of a torus is 1-dimensional, there exists a finite Galois extension $\mathfrak{K}$ of k and a character χ of $T_{\mathfrak{K}}$ such that $\chi(g) = \alpha$. Then

$$\omega\big(\big(\sum_{\gamma\in\mathrm{Gal}(\mathfrak{K}/k)} {}^{\gamma}\chi\big)(g)\big) = m\omega(\chi(g)) = m\omega(\alpha) \neq 0;$$

where $m = [\mathfrak{K} : k]$. Thus the character $\sum_{\gamma\in\mathrm{Gal}(\mathfrak{K}/k)} {}^{\gamma}\chi$ is nontrivial. On the other hand, this character is obviously defined over k. Hence, T admits a nontrivial character defined over k and therefore it contains a nontrivial k-split subtorus. This proves that if $G(k)$ is unbounded, then G is isotropic over k. □

Proposition 1.5. *We assume that the derived subgroup (G, G) of G is k-anisotropic. Then $G(k)$ contains a unique maximal bounded subgroup $G(k)_b$; it has the following* *description:*

$$G(k)_b = \{\, g \in G(k) \mid \chi(g) \in \mathfrak{o}^\times \text{ for all } \chi \in \mathrm{X}_k^*(G)\}.$$

Proof. Let S be the maximal k-split central torus of G and G_a be the maximal connected normal k-anisotropic subgroup of G. Then G_a contains (G, G) and $G = S \cdot G_a$ (almost direct product). Let $C = S \cap G_a$; C is a finite central k-subgroup scheme, so G_a/C is k-anisotropic. Let $f : G \to G/C = (S/C) \times (G_a/C)$ be the natural homomorphism. The image of the induced homomorphism $f^* : \mathrm{X}_k^*((S/C) \times (G_a/C)) \to \mathrm{X}_k^*(G)$ is of finite index. It is obvious that as $(G_a/C)(k)$ is bounded (by Theorem 1.1), the proposition is true for the direct product $(S/C) \times (G_a/C)$. Now using Lemma 1.3 we conclude that the proposition holds for G. □

1.6. Let S be a maximal k-split torus of G, $Z(S)$ its centralizer in G and $Z(S)' = (Z(S), Z(S))$ the derived subgroup of $Z(S)$. Then $Z(S)'$ is a connected semi-simple group which is anisotropic over k since S is a maximal k-split torus of G. Hence, by Theorem 1.1, $Z(S)'(k)$ is bounded, and according to Proposition 1.5, $Z(S)(k)$ contains a unique maximal bounded subgroup $Z(S)(k)_b$. This maximal bounded subgroup admits the following description:

$$Z(S)(k)_b = \{z \in Z(S)(k) \mid \chi(z) \in \mathfrak{o}^\times \ \text{ for all } \ \chi \in \mathrm{X}_k^*(Z(S))\}.$$

The restriction map $\mathrm{X}_k^*(Z(S)) \to \mathrm{X}_k^*(S)$ is injective and its image is of finite index in $\mathrm{X}_k^*(S)$. Let $\mathrm{X}_*(S) = \mathrm{Hom}_k(\mathrm{GL}_1, S)$ and $V(S) = \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{X}_*(S)$. Let the homomorphism $\nu : Z(S)(k) \to V(S)$ be defined by:

$$\chi(\nu(z)) = -\omega(\chi(z)) \ \text{ for } z \in Z(S)(k) \text{ and } \chi \in \mathrm{X}_k^*(Z(S)) \, (\hookrightarrow \mathrm{X}_k^*(S)).$$

Then $Z(S)(k)_b$ is the kernel of ν. As the image of ν is isomorphic to $\mathbb{Z}^r$, $r = \dim S$, we conclude that $Z(S)(k)/Z(S)(k)_b$ is isomorphic to $\mathbb{Z}^r$.

1.7. Fields of dimension $\leqslant 1$ and a theorem of Steinberg. A field $\mathfrak{F}$ is said to be of dimension $\leqslant 1$ if finite dimensional central simple algebras with center a finite separable extension of $\mathfrak{F}$ are matrix algebras [S, Ch. II, §3.1]. For example, every finite field is of dimension $\leqslant 1$.

We now recall the following theorem of Steinberg: *For a smooth connected linear algebraic group $\mathcal{G}$ defined over a field $\mathfrak{F}$ of dimension $\leqslant 1$, the Galois cohomology $\mathrm{H}^1(\mathfrak{F}, \mathcal{G})$ is trivial if either $\mathfrak{F}$ is perfect or $\mathcal{G}$ is reductive* [S, Ch. III, Thm. 1′ and Remark (1) in §2.3]. This vanishing theorem implies that *if $\mathfrak{F}$ is of dimension $\leqslant 1$, then every such $\mathfrak{F}$-group $\mathcal{G}$ is quasi-split, i.e., it contains a Borel subgroup defined over* $\mathfrak{F}$ [S, Ch. III, Thm. 1 in §2.2].

Let $\widehat{K}$ denote the completion of the maximal unramified extension $K \, (\subset \overline{k})$ of k. The discrete valuation on K extends uniquely to the completion $\widehat{K}$ and the residue field $\overline{\kappa}$ of K is also the residue field of $\widehat{K}$; since κ has been assumed to be perfect, $\overline{\kappa}$ is the algebraic closure of κ. Hence, by Lang's theorem, $\widehat{K}$ is a (C_1)-field [S, Ch. II, Example 3.3(c) in §3.3], so it is of dimension $\leqslant 1$ [S, Ch. II, Corollary in §3.2]. According to a well-known result (see, for example, Proposition 3.5.3(2) of [GGM] whose proof simplifies considerably in the smooth affine case), for any smooth algebraic K-group $\mathcal{G}$, the natural map $\mathrm{H}^1(K, \mathcal{G}) \to \mathrm{H}^1(\widehat{K}, \mathcal{G})$ is bijective. This result, combined with the above theorem of Steinberg, implies that for any connected reductive K-group $\mathcal{G}$, $\mathrm{H}^1(K, \mathcal{G})$ is trivial, and any such group is quasi-split.

Notation. Given a smooth connected linear algebraic group H defined over a *perfect* field F, we will denote its unipotent radical, i.e., the maximal smooth connected normal unipotent F-subgroup, by $\mathscr{R}_u(H)$. It is known that (as F is perfect) $\mathscr{R}_u(H)$ is F-split [Bo, Cor. 15.5(ii)], and $H/\mathscr{R}_u(H)$ is reductive. We will denote the Galois group $\mathrm{Gal}(K/k) = \mathrm{Gal}(\overline{\kappa}/\kappa)$ by Γ. It is obvious that descent of the Bruhat-Tits theory from K to k for a connected reductive k-group follows from the descent of the theory for its semi-simple derived subgroup. So, henceforth, G will denote a connected semi-simple algebraic group defined over k.

1.8. *What we will use from the Bruhat-Tits theory over K:* The K-group G_K is quasi-split (1.7) and there is an affine building $\mathcal{B}(G/K)$, called the Bruhat-Tits building of G/K. Its apartments are affine spaces which are in bijective correspondence with maximal K-split tori of G_K. Facets of $\mathcal{B}(G/K)$ of maximal dimension are called *chambers*.

Let T_K be a maximal K-split torus of G_K. As G_K is quasi-split, the centralizer $Z(T_K)$ of T_K in G_K is a torus. Let $N(T_K)$ denote the normalizer of T_K in G_K, and A be the apartment of $\mathcal{B}(G/K)$ corresponding to T_K. Then A is an affine space under $V(T_K) := \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{X}_*(T_K)$, where $\mathrm{X}_*(T_K) = \mathrm{Hom}_K(\mathrm{GL}_1, T_K)$, and there is an action of $N(T_K)(K)$ on A by affine transformations which we will describe now. Let $\mathrm{Aff}(A)$ be the group of affine automorphisms of A and $\nu : N(T_K)(K) \to \mathrm{Aff}(A)$ be the action map. For $n \in N(G)(K)$, the derivative $d\nu(n) : V(T_K) \to V(T_K)$ is the map induced by the action of n on $\mathrm{X}_*(T_K)$ (i.e., the Weyl group action). So for $z \in Z(T_K)(K)$, $d\nu(z)$ is the identity, hence $\nu(z)$ is a translation; this translation is described by the following formula:

$$\chi(\nu(z)) = -\omega(\chi(z)) \text{ for all } \chi \in \mathrm{X}_K^*(Z(T_K))\,(\hookrightarrow \mathrm{X}_K^*(T_K)),$$

here we regard the translation $\nu(z)$ as an element of $V(T_K)$. The above formula shows that the maximal bounded subgroup $Z(T_K)(K)_b$ of $Z(T_K)(K)$ acts trivially on A.

Given two points x and y of $\mathcal{B}(G/K)$ there is a unique geodesic $[xy]$ joining them and this geodesic lies in every apartment which contains x and y [BrT1, Prop. 2.5.4]. A subset of the building is called *convex* if for any x, y in the set, the geodesic $[xy]$ is contained in the set.

According to the proposition in [BrT2, 4.2.19], if $\mathcal{G}$ is a bounded subgroup of $G(K)$ (such as $Z(S)(k)_b$ in 1.6), then for any point $x \in \mathcal{B}(G/K)$, the orbit $\mathcal{G} \cdot x$ of x under $\mathcal{G}$ is bounded in the metric on $\mathcal{B}(G/K)$. Therefore, by the Bruhat-Tits fixed point theorem (Proposition 3.2.4 of [BrT1]), any closed convex subset of $\mathcal{B}(G/K)$ that is stable under the action of $\mathcal{G}$ contains a point fixed by $\mathcal{G}$.

Given a nonempty bounded subset Ω of an apartment A, Bruhat and Tits constructed a smooth affine $\mathcal{O}$-group scheme $\mathscr{G}_\Omega^\circ$ with generic fiber G_K, and with connected special fiber, whose group of $\mathcal{O}$-rational points, considered as a subgroup of $G(K)$, is bounded and of finite index in the subgroup of $G(K)$ consisting of elements which fix Ω pointwise (see [BrT2, §4]; for a simpler treatment of the existence and smoothness of such "Bruhat-Tits group schemes", see [Y]). If the above apartment A corresponds to the maximal K-split torus T_K of G_K, then there is a split maximal $\mathcal{O}$-torus $\mathscr{T}$ in $\mathscr{G}_\Omega^\circ$ with generic fiber T_K; note that $\mathscr{T}(\mathcal{O})$ is the maximal bounded subgroup of $T_K(K)$. If G is simply connected, then $\mathscr{G}_\Omega^\circ(\mathcal{O})$ is the stabilizer in $G(K)$ of the subset Ω. If Ω is a nonempty subset of a facet F of $\mathcal{B}(G/K)$, then $\mathscr{G}_\Omega^\circ = \mathscr{G}_F^\circ$. If Ω consists of a single point x, we will for simplicity denote $\mathscr{G}_\Omega^\circ$ by $\mathscr{G}_x^\circ$.

An apartment A' corresponding to the maximal K-split torus T'_K of G_K contains Ω if and only if there is a split maximal $\mathcal{O}$-torus $\mathscr{T}'$ in $\mathscr{G}_\Omega^\circ$ with generic fiber T'_K. This condition is equivalent to the condition that $\mathscr{G}_\Omega^\circ(\mathcal{O})\cap T'_K(K)$ is the maximal bounded subgroup of $T'_K(K)$, for if this condition is satisfied then by Lemma 4.1 of [PY2] the schematic closure of T'_K in $\mathscr{G}_\Omega^\circ$ is a $\mathcal{O}$-torus. Also, according to [BrT2, Proposition 4.6.28(iii)], $\mathscr{G}_\Omega^\circ(\mathcal{O})$ acts transitively on the set of apartments of the building $\mathcal{B}(G/K)$ containing Ω.

1.9. The Galois group Γ of K/k acts on $\mathcal{B}(G/K)$ through a finite quotient. For a subset X of $\mathcal{B}(G/K)$, $\overline{X}$ will denote the closure of X and by X^Γ the subset consisting of all $x \in X$ which are fixed under Γ.

If a nonempty bounded subset Ω of an apartment A corresponding to a maximal K-split torus T_K of G_K is stable under the action of Γ, then $\mathscr{G}_\Omega^\circ(\mathcal{O})$ is stable under Γ and so is the affine ring $\mathcal{O}[\mathscr{G}_\Omega^\circ]$ of $\mathscr{G}_\Omega^\circ$. In such cases (i.e., when Ω is stable under the action of Γ), the $\mathcal{O}$-group scheme $\mathscr{G}_\Omega^\circ$ admits a unique descent to a smooth affine $\mathfrak{o}$-group scheme; the affine ring of this descent is $(\mathcal{O}[\mathscr{G}_\Omega^\circ])^\Gamma$. As it is not likely to cause confusion, in the sequel whenever Ω is stable under Γ, we will use $\mathscr{G}_\Omega^\circ$ to denote this smooth affine $\mathfrak{o}$-group scheme; its generic fiber is G and the special fiber is a connected algebraic κ-group that will be denoted by $\overline{\mathscr{G}}_\Omega^\circ$. The unique maximal reductive quotient of $\overline{\mathscr{G}}_\Omega^\circ$ will be denoted by $\overline{G}_\Omega^{\mathrm{red}}$. For a point $x \in \mathcal{B}(G/K)$ fixed under Γ, we will denote $\overline{\mathscr{G}}_{\{x\}}^\circ$ and $\overline{G}_{\{x\}}^{\mathrm{red}}$ by $\overline{\mathscr{G}}_x^\circ$ and $\overline{G}_x^{\mathrm{red}}$ respectively.

If T_K is obtained from a k-torus T by base change to K, and Ω is stable under the action of Γ, then the $\mathcal{O}$-torus $\mathscr{T}$ of 1.8 admits a unique descent to a $\mathfrak{o}$-torus of $\mathscr{G}_\Omega^\circ$; in the sequel we will denote this $\mathfrak{o}$-torus also by $\mathscr{T}$. The generic fiber of $\mathscr{T}$ is T and $\mathscr{G}_\Omega^\circ(\mathfrak{o})\cap T(k)$ is the maximal bounded subgroup of $T(k)$. If the k-torus T contains a maximal k-split torus S of G, then the generic fiber of the maximal $\mathfrak{o}$-split subtorus $\mathscr{S}$ of $\mathscr{T}$ is S and the special fiber $\overline{\mathscr{S}}$ is a maximal κ-split torus of $\overline{\mathscr{G}}_\Omega^\circ$.

1.10. We introduce the following partial order on the set of facets of $\mathcal{B}(G/K)$: Given two facets F and F', $F' \prec F$ if $\overline{F}$ contains F', i.e., F' is a face of F, or, equivalently, $\mathscr{G}_F^\circ(\mathcal{O}) \subset \mathscr{G}_{F'}^\circ(\mathcal{O})$. In a collection $\mathcal{C}$ of facets, thus a facet is *maximal* if it is not a proper face of any facet belonging to $\mathcal{C}$, and a facet is *minimal* if no proper face of it belongs to $\mathcal{C}$.

Now let X be a convex subset of $\mathcal{B}(G/K)$ and $\mathcal{C}$ be the set of facets of $\mathcal{B}(G/K)$, or facets lying in a given apartment A, which meet X. Then it is easy to see (Proposition 9.2.5 (i), (ii), of [BrT1]) that all maximal facets in $\mathcal{C}$ are of equal dimension and for any two maximal facets F and F' in $\mathcal{C}$, $\dim F \cap X = \dim F' \cap X$.

1.11. Let $\mathcal{B} = \mathcal{B}(G/K)^{\Gamma}$. It is obvious that $\mathcal{B}$ is convex and is stable under the action of $G(k)$ on $\mathcal{B}(G/K)$. Therefore, the assertions in 1.10 for $\mathcal{B}$ in place of X hold. We shall prove that $\mathcal{B}$ is a "thick" affine building. To establish this, it would clearly suffice to prove it for absolutely almost simple groups since any semi-simple k-group is isogenous to the product of restriction of scalars of absolutely almost simple groups over finite separable extensions of k and the Bruhat-Tits building $\mathcal{B}(G/K)$ of $G(K)$ is the product of the buildings of the corresponding absolutely almost simple groups over finite separable extensions of K.

Now in case G is an absolutely almost simple k-group, $\mathcal{B}(G/K)$ is a simplicial complex (in general, $\mathcal{B}(G/K)$ is a polysimplicial complex) and we will show that $\mathcal{B}$ is also a simplicial complex, its simplices being the intersections with $\mathcal{B}$ of simplices of $\mathcal{B}(G/K)$ that are stable under Γ. The dimension of $\mathcal{B}$ is $r := k$-rank G. Simplices of dimension r are called *chambers*, and we will show that every simplex of $\mathcal{B}$ is a face of a chamber. The apartments of $\mathcal{B}$ are, by definition, the subcomplexes which are intersections of *special k-apartments* of $\mathcal{B}(G/K)$ (see below) with $\mathcal{B}$. We will show that the apartments of $\mathcal{B}$ are affine spaces of dimension r and they are in bijective correspondence with maximal k-split tori of G. To show that $\mathcal{B}$, considered as a simplicial complex, is a building, we will verify that the following four conditions defining a building in [T1, 3.1] hold:

(B1) $\mathcal{B}$ is *thick*, that is, any simplex of codimension 1 (i.e., of dimension $r-1$) is a face of at least three chambers.

(B2) The apartments are *thin chamber complexes*[1].

(B3) Any two simplices of $\mathcal{B}$ lie on an apartment.

(B4) If simplices $\mathcal{F}_1$ and $\mathcal{F}_2$ are contained in the intersection of two apartments $\mathcal{A}$ and $\mathcal{A}'$ of $\mathcal{B}$, then there is an isomorphism $\mathcal{A} \to \mathcal{A}'$ which fixes $\mathcal{F}_1$ and $\mathcal{F}_2$ pointwise.

1.12. Special k-tori and special k-apartments. According to the Bruhat-Tits theory [BrT2, Cor. 5.1.12] there exists a k-torus T in G that contains a maximal k-split torus S of G and T_K is a maximal K-split torus of G_K. Such a k-torus T will be called a *special k-torus* and the apartment in $\mathcal{B}(G/K)$ corresponding to T_K will henceforth be called a *special k-apartment* corresponding to the (special) k-torus T. It is clear from the definition that every special k-apartment is stable under the

[1]A simplicial complex Δ of dimension r is called a *chamber complex* if every simplex of Δ is a face of a chamber (i.e., a simplex of dimension r) and any two chambers of Δ can be joined by a *gallery* (see Proposition 3.5 for the definition). A chamber complex is *thin* if any simplex of codimension 1 is a face of exactly two chambers.

action of the Galois group Γ. If $x \neq y$ are two points of a special k-apartment A which are fixed under Γ, then the whole straight line in A passing through x and y is pointwise fixed under Γ.

It is obvious from the Bruhat-Tits fixed point theorem (Proposition 3.2.4 of [BrT1]) that a facet is Γ-stable if and only if it contains a point fixed under Γ, i.e., the facet meets $\mathcal{B}$. A facet in the building $\mathcal{B}(G/K)$ which meets $\mathcal{B}$ will be called a *k-facet*. Maximal k-facets in $\mathcal{B}(G/K)$ will be called *k-chambers*. Note that a k-chamber may not be a chamber (i.e., it may not be facet of maximal dimension); see, however, Proposition 2.1.

If $F' \prec F$ are two k-facets of $\mathcal{B}(G/K)$, then there is a natural $\mathfrak{o}$-group scheme homomorphism $\mathscr{G}_F^\circ \to \mathscr{G}_{F'}^\circ$. The image of the induced homomorphism $\mathscr{G}_F^\circ(\mathcal{O}) \to \overline{G}_{F'}^{\mathrm{red}}(\overline{\kappa})$ is the group of $\overline{\kappa}$-rational points of a parabolic κ-subgroup $\overline{P}$ of $\overline{G}_{F'}^{\mathrm{red}}$. Moreover, F is a maximal k-facet (i.e., it is a k-chamber) if and only if $\overline{P}$ is a minimal parabolic κ-subgroup. Thus, a k-facet C is a k-chamber if and only if the reductive κ-group $\overline{G}_C^{\mathrm{red}}$ does not contain a proper parabolic κ-subgroup, or, equivalently, this reductive group contains a unique maximal κ-split torus (this torus is central so it is contained in every maximal torus of $\overline{G}_C^{\mathrm{red}}$).

2. Six basic propositions

Proposition 2.1. *Every special k-apartment of $\mathcal{B}(G/K)$ contains a k-chamber. If κ is of dimension $\leqslant 1$, then every k-chamber is a chamber in $\mathcal{B}(G/K)$.*

Proof. Let A be a special k-apartment and T be the corresponding special k-torus. Then T contains a maximal k-split torus S of G and T_K is a maximal K-split torus of G_K. As A is stable under the action of Γ, by the Bruhat-Tits fixed point theorem, it contains a point x which is fixed under Γ. Let F be the facet lying on A which contains x. Then, by definition, F is a k-facet. Let $\mathscr{G}_F^\circ$ be the smooth affine $\mathfrak{o}$-group scheme, with connected fibers, associated to the k-facet F in 1.9. Let $\overline{\mathscr{G}}_F^\circ$ be the special fiber of $\mathscr{G}_F^\circ$ and $\overline{G}_F^{\mathrm{red}}$ the maximal reductive quotient of $\overline{\mathscr{G}}_F^\circ$. Let $\mathscr{T}$ be the maximal $\mathfrak{o}$-torus of $\mathscr{G}_F^\circ$ with generic fiber T, and let $\mathscr{S}$ be the maximal $\mathfrak{o}$-split subtorus of $\mathscr{T}$ (cf. 1.9). Then the generic fiber of $\mathscr{S}$ is S. Let $\overline{S}$ and $\overline{T}$ be the images in $\overline{G}_F^{\mathrm{red}}$ of the special fibers of $\mathscr{S}$ and $\mathscr{T}$ respectively; $\overline{S}$ is a maximal κ-split torus of $\overline{G}_F^{\mathrm{red}}$, and $\overline{T}$ is a maximal torus containing $\overline{S}$. We fix a minimal parabolic κ-subgroup $\overline{P}$ of $\overline{G}_F^{\mathrm{red}}$ containing $\overline{S}$, then $\overline{P}$ contains the centralizer of $\overline{S}$, and so it contains $\overline{T}$. Let P be the inverse image of $\overline{P}(\overline{\kappa})$ in $\mathscr{G}_F^\circ(\mathcal{O})\,(\subset G(K))$ under the natural homomorphism $\mathscr{G}_F^\circ(\mathcal{O}) \to \overline{G}_F^{\mathrm{red}}(\overline{\kappa})$. Then P is a parahoric subgroup of $G(K)$ contained in the parahoric subgroup $\mathscr{G}_F^\circ(\mathcal{O})$; P contains $\mathscr{T}(\mathcal{O})$ and is clearly stable under the action of Γ on $G(K)$. Let C be the facet of the Bruhat-Tits building $\mathcal{B}(G/K)$ fixed by P. Then C contains F in its closure and is stable under Γ, i.e., it is a k-facet; it is a k-chamber since $\overline{P}$ is a minimal parabolic κ-subgroup of $\overline{G}_F^{\mathrm{red}}$.

Moreover, as P contains the maximal bounded subgroup $\mathscr{T}(\mathcal{O})$ of $T(K)$, C lies on the apartment A.

If κ is of dimension $\leqslant 1$, $\overline{G}_F^{\,\mathrm{red}}$ is quasi-split (1.7) and hence the minimal parabolic subgroup $\overline{P}$ is a Borel subgroup of $\overline{G}_F^{\,\mathrm{red}}$. So, in this case, C is a chamber of the building $\mathcal{B}(G/K)$. □

Remark 2.2. Let A be a special k-apartment. According to Proposition 2.1, there is a k-chamber lying on A, so among the facets of A which meet $\mathcal{B}$, the maximal ones are k-chambers (1.10-1.11). Therefore, any k-facet in A is a face of a k-chamber contained in A.

Proposition 2.3. *Given a k-chamber C in the building $\mathcal{B}(G/K)$ which lies in a special k-apartment A, and a point $x \in \mathcal{B}$, there is a special k-apartment which contains C and x. Thus, in particular, every point of $\mathcal{B}$ lies on a special k-apartment.*

Proof. Let T be the special k-torus corresponding to the apartment A. Then T contains a maximal k-split torus S of G. We fix a point y of $C \cap \mathcal{B}$, then $\mathscr{G}_y^\circ = \mathscr{G}_C^\circ$. Let $\mathscr{S}$ be the maximal $\mathfrak{o}$-split torus in $\mathscr{G}_C^\circ$ with generic fiber S and $\overline{S}$ be the corresponding maximal κ-split torus of $\overline{G}_C^{\,\mathrm{red}}$ (see 1.9). As C is a k-chamber, $\overline{S}$ is central and so every maximal torus of $\overline{G}_y^{\,\mathrm{red}} = \overline{G}_C^{\,\mathrm{red}}$ contains it (1.12). By the uniqueness of the geodesic $[xy]$, every point on it is fixed under Γ, i.e., $[xy] \subset \mathcal{B}$. The composite map $\overline{\rho} : \overline{\mathscr{G}}_{[xy]}^\circ \to \overline{\mathscr{G}}_y^\circ \to \overline{G}_y^{\,\mathrm{red}} (= \overline{G}_C^{\,\mathrm{red}})$, induced by the inclusion of $\{y\}$ in $[xy]$, restricts to an isomorphism of any maximal κ-torus of $\overline{\mathscr{G}}_{[xy]}^\circ$ onto a maximal κ-torus of $\overline{G}_C^{\,\mathrm{red}}$. We fix a maximal κ-torus $\overline{\mathscr{T}}_{[xy]}$ of $\overline{\mathscr{G}}_{[xy]}^\circ$. Then the maximal κ-split subtorus of $\overline{\mathscr{T}}_{[xy]}$ is isomorphic to $\overline{S}$. Let $\mathscr{T}$ be the $\mathfrak{o}$-torus with character group isomorphic to the character group of $\overline{\mathscr{T}}_{[xy]}$ as a Γ-module. According to a result of Grothendieck [SGA3$_{\mathrm{II}}$, Exp. XI, Cor. 4.2], $\underline{\mathrm{Hom}}_{\mathrm{Spec}(\mathfrak{o})-\mathrm{gr}}(\mathscr{T}, \mathscr{G}_{[xy]}^\circ)$ is representable by a smooth $\mathfrak{o}$-scheme $\mathscr{X}$. Now since $\mathfrak{o}$ is Henselian, the natural map $\mathscr{X}(\mathfrak{o}) \to \mathscr{X}(\kappa)$ is surjective. As $\mathscr{T}_\kappa$ is clearly isomorphic to $\overline{\mathscr{T}}_{[xy]}\,(\subset \overline{\mathscr{G}}_{[xy]}^\circ)$, we infer that $\mathscr{G}_{[xy]}^\circ$ contains a $\mathfrak{o}$-torus isomorphic to $\mathscr{T}$ whose special fiber (as a κ-subgroup of $\overline{\mathscr{G}}_{[xy]}^\circ$) is $\overline{\mathscr{T}}_{[xy]}$. The generic fiber $T_{[xy]}$ of $\mathscr{T}_{[xy]}$ is then a k-torus of G that contains a maximal k-split torus of G (and it splits over K) so it is a special k-torus. The special k-apartment of $\mathcal{B}(G/K)$ determined by $T_{[xy]}$ contains $[xy]$ and hence also C and x. □

Proposition 2.4. *Given points x, y in $\mathcal{B}$, there is a special k-apartment in $\mathcal{B}(G/K)$ which contains both x and y. Therefore, given any two k-facets (which may not be different) in $\mathcal{B}(G/K)$, there is a special k-apartment containing them.*

Proof. Let F be the k-facet of $\mathcal{B}(G/K)$ which contains the point y. Let C be a k-facet which contains F in its closure, meets $\mathcal{B}$, and is maximal among the facets with these two properties. Then C is a k-chamber (Remark 2.2). Let $z \in C \cap \mathcal{B}$. Then according to the previous proposition there is a special k-apartment which contains z, and hence also C. Now the same proposition implies that there is a special k-apartment which contains C and x. This apartment then contains $\overline{C}$, and hence also F, and so it contains y. □

Proposition 2.5. *If G is anisotropic over k, then $\mathcal{B} = \mathcal{B}(G/K)^\Gamma$ consists of a single point.*

Proof. To prove the proposition we will use Proposition 2.4. If $\mathcal{B}$ contains points $x \neq y$, then according to that proposition there is a special k-apartment A in $\mathcal{B}(G/K)$ which contains both x and y. Let T be the special k-torus corresponding to A. As G is anisotropic over k, T is k-anisotropic. Hence, A^Γ consists of a single point. A contradiction! □

In the rest of this section, we will use the notation introduced in 1.6. The centralizer $Z(S)$ of S in G is an almost direct product of its central torus and its derived subgroup $Z(S)'$. The enlarged Bruhat-Tits building $\mathcal{B}(Z(S)/K)$ will be viewed as the union of apartments in the building $\mathcal{B}(G/K)$ corresponding to maximal K-split tori of G_K which contain S_K. Now we note that $Z(S)$ contains a special k-torus T (see 1.12), the maximal k-split subtorus of T is S, and the special k-apartment A corresponding to T is contained in $\mathcal{B}(Z(S)/K)$. As $Z(S)'$ is anisotropic over k, according to the previous proposition $\mathcal{B}(Z(S)'/K)^\Gamma$ consists of a single point. This immediately implies the following:

Proposition 2.6. $\mathcal{B}(Z(S)/K)^\Gamma = A^\Gamma$. *(Note that A^Γ is an affine space under $\mathbb{R} \otimes_{\mathbb{Z}} \mathrm{X}_*(S) =: V(S)$.)*

2.7. Let $N(S)$ be the normalizer of S in G. As the subgroup $N(S)$ normalizes $Z(S)$, there is a natural action of $N(S)(K)$ on $\mathcal{B}(Z(S)/K)$ and $N(S)(k)$ stabilizes $\mathcal{B}(Z(S)/K)^\Gamma$ under this action. For $n \in N(S)(K)$, the action of n carries an apartment A of $\mathcal{B}(Z(S)/K)$ to the apartment $n \cdot A$ by an affine transformation.

Now let T be a special k-torus of G containing S. Let $A := A_T$ be the special k-apartment of $\mathcal{B}(G/K)$ corresponding to the torus T. As $T \supset S$, this apartment is contained in $\mathcal{B}(Z(S)/K)$ and it follows from the previous proposition that $\mathcal{B}(Z(S)/K)^\Gamma = A^\Gamma$. So we can view $\mathcal{B}(Z(S)/K)^\Gamma$ as an affine space under $V(S) = \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{X}_*(S)$. We will now show, using the proof of the lemma in 1.6 of [PY1], that $\mathcal{B}(Z(S)/K)^\Gamma$ has the properties required of an apartment corresponding to the maximal k-split torus S in the Bruhat-Tits building of G/k if such a building exists. We need to check the following three conditions.

A1: *The action of $N(S)(k)$ on $\mathcal{B}(Z(S)/K)^\Gamma = A^\Gamma$ is by affine transformations and the maximal bounded subgroup $Z(S)(k)_b$ of $Z(S)(k)$ acts trivially.*

Let $\mathrm{Aff}(A^\Gamma)$ be the group of affine automorphisms of A^Γ and $f : N(S)(k) \to \mathrm{Aff}(A^\Gamma)$ be the action map.

A2: *The group $Z(S)(k)$ acts by translations, and the action is characterized by the following formula: for $z \in Z(S)(k)$,*

$$\chi(f(z)) = -\omega(\chi(z)) \text{ for all } \chi \in \mathrm{X}_k^*(Z(S)) \, (\hookrightarrow \mathrm{X}_k^*(S)),$$

here we regard the translation $f(z)$ as an element of $V(S) = \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{X}_*(S)$.

A3: *For $g \in \mathrm{Aff}(A^\Gamma)$, denote by $dg \in \mathrm{GL}(V(S))$ the derivative of g. Then the map $N(S)(k) \to \mathrm{GL}(V(S))$, $n \mapsto df(n)$, is induced from the action of $N(S)(k)$ on $\mathrm{X}_*(S)$ (i.e., it is the Weyl group action).*

Moreover, as G has been assumed to be semi-simple, these three conditions determine the affine structure on $\mathcal{B}(Z(S)/K)^\Gamma$, see [T2, 1.2].

Proposition 2.8. *Conditions* A1, A2 *and* A3 *hold.*

Proof. The action of $n \in N(S)(k)$ on $\mathcal{B}(G/K)$ carries the special k-apartment $A = A_T$ via an affine isomorphism $\varphi(n) : A \to A_{nTn^{-1}}$ to the special k-apartment $A_{nTn^{-1}}$ corresponding to the special k-torus nTn^{-1} containing S. As $(A_{nTn^{-1}})^\Gamma = \mathcal{B}(Z(S)/K)^\Gamma = A^\Gamma$, we see that $\varphi(n)$ keeps A^Γ stable and so $\varphi(n)|_{A^\Gamma}$ is an affine automorphism of A^Γ.

Let $V(T_K) = \mathbb{R}\otimes_{\mathbb{Z}}\mathrm{Hom}_K(\mathrm{GL}_1, T_K)$ and $V(nT_Kn^{-1}) = \mathbb{R}\otimes_{\mathbb{Z}}\mathrm{Hom}_K(\mathrm{GL}_1, nT_Kn^{-1})$. The derivative $d\varphi(n) : V(T_K) \to V(nT_Kn^{-1})$ is induced from the map

$$\mathrm{Hom}_K(\mathrm{GL}_1, T_K) = \mathrm{X}_*(T_K) \to \mathrm{X}_*(nT_Kn^{-1}) = \mathrm{Hom}_K(\mathrm{GL}_1, nT_Kn^{-1}),$$

$\lambda \mapsto \mathrm{Int}\, n \cdot \lambda$, where $\mathrm{Int}\, n$ is the inner automorphism of G determined by n. So, the restriction $df(n) : V(S) \to V(S)$ is induced from the homomorphism $\mathrm{X}_*(S) \to \mathrm{X}_*(S)$, $\lambda \mapsto \mathrm{Int}\, n \cdot \lambda$. This proves A3.

Condition A3 implies that df is trivial on $Z(S)(k)$. Therefore, $Z(S)(k)$ acts by translations. The action of the bounded subgroup $Z(S)(k)_b$ on A^Γ admits a fixed point by the fixed point theorem of Bruhat-Tits. Therefore, $Z(S)(k)_b$ acts by the trivial translation. This proves A1.

Since the image of $S(k)$ in $Z(S)(k)/Z(S)(k)_b \simeq \mathbb{Z}^r$ is a subgroup of finite index, to prove the formula in A2, it suffices to prove it for $z \in S(k)$. But for $z \in S(k)$, $zTz^{-1} = T$, and $f(z)$ is a translation of the apartment A ($f(z)$ is regarded as an element of $V(T_K)$) which satisfies (see 1.8):

$$\chi(f(z)) = -\omega(\chi(z)) \quad \text{for all } \chi \in \mathrm{X}^*_K(T_K).$$

This implies the formula in A2, since the restriction map $\mathrm{X}^*_K(T_K) \to \mathrm{X}^*_K(S_K)$ ($= \mathrm{X}^*_k(S)$) is surjective and the image of the restriction map $\mathrm{X}^*_k(Z(S)) \to \mathrm{X}^*_k(S)$ is of finite index in $\mathrm{X}^*_k(S)$. □

2.9. By definition, the *apartments of $\mathcal{B}$* are A^Γ, for special k-apartments A of $\mathcal{B}(G/K)$. Let A be a special k-apartment of $\mathcal{B}(G/K)$, T the corresponding k-torus and S the maximal k-split torus of G contained in T. Then (Proposition 2.6) $A^\Gamma = \mathcal{B}(Z(S)/K)^\Gamma$ and it is an affine space under $V(S) = \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{X}_*(S)$. Now as maximal k-split tori of G are conjugate to each other under $G(k)$, we conclude that $G(k)$ acts transitively on the set of apartments of $\mathcal{B}$.

3. Main results.

Theorem 3.1. *Let A and A' be special k-apartments in $\mathcal{B}(G/K)$; T, T' be the corresponding special k-tori. Let Ω be a Γ-stable nonempty bounded convex subset of $A \cap A'$ and $\mathscr{G}_\Omega^\circ$ be the smooth affine $\mathfrak{o}$-group scheme associated to Ω in 1.9. Then there is an element $g \in \mathscr{G}_\Omega^\circ(\mathfrak{o}) \subset G(k)$ that carries A^Γ onto ${A'}^\Gamma$.*

If the residue field κ is of dimension $\leqslant 1$, then there exists an element $g \in \mathscr{G}_\Omega^\circ(\mathfrak{o}) \subset G(k)$ that conjugates T onto T', hence it carries the apartment A onto the apartment A'.

As g belongs to $\mathscr{G}_\Omega^\circ(\mathfrak{o})$, it fixes Ω pointwise.

Proof. Let S and S' be the maximal k-split tori of G contained in T and T' respectively. Let $\mathscr{G} := \mathscr{G}_\Omega^\circ$ and $\mathscr{T}$ and $\mathscr{T}'$ be the maximal $\mathfrak{o}$-tori in $\mathscr{G}$ with generic fibers T and T' respectively (see 1.9). Let $\mathscr{S}$ and $\mathscr{S}'$ be the maximal $\mathfrak{o}$-split subtori of $\mathscr{T}$ and $\mathscr{T}'$ respectively. Then the special fibers $\overline{\mathscr{S}}$ and $\overline{\mathscr{S}}'$ of $\mathscr{S}$ and $\mathscr{S}'$ respectively are maximal κ-split tori in the special fiber $\overline{\mathscr{G}}$ of $\mathscr{G}$. Hence there exists an element $\overline{g} \in \overline{\mathscr{G}}(\kappa)$ that conjugates $\overline{\mathscr{S}}$ onto $\overline{\mathscr{S}}'$. The transporter scheme $\mathfrak{T} := \mathrm{Transp}_{\mathscr{G}}(\mathscr{S}, \mathscr{S}')$ consisting of points of the scheme $\mathscr{G}$ that conjugate $\mathscr{S}$ onto $\mathscr{S}'$ is a closed smooth $\mathfrak{o}$-subscheme of $\mathscr{G}$ (see [C, Prop. 2.1.2] or [SGA3$_{\mathrm{II}}$, Exp. XI, 2.4bis]). Let $\overline{\mathfrak{T}}$ be the special fiber of $\mathfrak{T}$. Then $\overline{g}$ belongs to $\overline{\mathfrak{T}}(\kappa)$. Now as $\mathfrak{o}$ is Henselian, the natural homomorphism $\mathfrak{T}(\mathfrak{o}) \to \overline{\mathfrak{T}}(\kappa)$ is surjective. Therefore, there exists an element $g \in \mathfrak{T}(\mathfrak{o})\,(\subset \mathscr{G}(\mathfrak{o}) \subset G(k))$ that maps onto $\overline{g}$. As $g\mathscr{S}g^{-1} = \mathscr{S}'$, we infer that $gSg^{-1} = S'$, so

$$g \cdot A^\Gamma = g \cdot \mathcal{B}(Z(S)/K)^\Gamma = \mathcal{B}(Z(S')/K)^\Gamma = {A'}^\Gamma,$$

and g fixes Ω pointwise.

Now let us assume that κ is of dimension $\leqslant 1$. Then the smooth connected κ-group $\overline{\mathscr{G}}$ is quasi-split (1.7) and hence any maximal κ-split torus of $\overline{\mathscr{G}}/\mathscr{R}_u(\overline{\mathscr{G}})$ is contained in a unique maximal torus. Since the element $\overline{g}$ chosen in the previous paragraph conjugates $\overline{\mathscr{S}}$ onto $\overline{\mathscr{S}}'$, it conjugates $\overline{\mathscr{T}}$ onto a maximal κ-torus of the solvable subgroup $\overline{\mathscr{H}} := \overline{\mathscr{T}}' \cdot \mathscr{R}_u(\overline{\mathscr{G}})$. As any two maximal κ-tori of the solvable κ-group $\overline{\mathscr{H}}$ are conjugate under an element of $\overline{\mathscr{H}}(\kappa)$ [Bo, Thm. 19.2], we conclude that $\overline{\mathscr{T}}'$ is conjugate to $\overline{\mathscr{T}}$ under an element of $\overline{\mathscr{G}}(\kappa)$. Now considering the closed smooth $\mathfrak{o}$-subscheme $\mathrm{Transp}_{\mathscr{G}}(\mathscr{T}, \mathscr{T}')$ of $\mathscr{G}\ (= \mathscr{G}_\Omega^\circ)$, we see that there exists an element $g \in \mathrm{Transp}_{\mathscr{G}}(\mathscr{T}, \mathscr{T}')(\mathfrak{o})\,(\subset \mathscr{G}(\mathfrak{o}) \subset G(k))$ that conjugates $\mathscr{T}$ onto $\mathscr{T}'$, so $gTg^{-1} = T'$, and hence g carries A onto A' fixing Ω pointwise. □

3.2. Polysimplicial structure on $\mathcal{B}$. The *facets* (resp. *chambers*) of $\mathcal{B}$ are by definition the subsets $\mathcal{F} := F \cap \mathcal{B}$ (resp. $\mathcal{C} := C \cap \mathcal{B}$) for k-facets F (resp. k-chambers C) of $\mathcal{B}(G/K)$.

Let F be a minimal k-facet in $\mathcal{B}(G/K)$ and A be a special k-apartment containing F. We will show that F contains a unique point fixed under Γ (i.e., F meets $\mathcal{B}$ in a single point). Every special k-apartment is stable under the action of the Galois group Γ which acts on it by affine automorphisms. Now if x and y are two distinct points in $F \cap \mathcal{B}$, then the whole straight line in the apartment A passing through x and y is pointwise fixed under Γ. This line must meet the boundary of F, this contradicts the minimality of F. By definition, a *vertex* of $\mathcal{B}$ is the unique point of $F \cap \mathcal{B}$ for any minimal k-facet F in $\mathcal{B}(G/K)$.

Let F be a k-facet in $\mathcal{B}(G/K)$ (F is not assumed to be minimal) and $\mathscr{V}_F$ be the set of vertices of $\mathcal{B}$ contained in $\overline{F}$. For $v \in \mathscr{V}_F$, let F_v be the face of F which contains v. Since v is vertex of $\mathcal{B}$, F_v is a minimal k-facet. Now if x and y are two distinct k-vertices in $\mathscr{V}_F$, then $\overline{F}_x \cap \overline{F}_y$ is empty. To see this note that this intersection is convex and stable under Γ and hence if it is nonempty, it will contain a Γ-fixed point (i.e., a point of $\mathcal{B}$). This will contradict the minimality of k-facets F_x and F_y. Thus the sets of vertices (we call them K-vertices) of the facets F_x and F_y are disjoint, and each one of these sets is Γ-stable. The union of the sets of K-vertices of F_x, for $x \in \mathscr{V}_F$, is the set of K-vertices of F. To see this, we observe that any K-vertex of F is a K-vertex of a face of F which is a minimal k-facet and so it contains a (unique) point of $\mathscr{V}_x$. Arguing by induction on dimension of F, we easily see that $\overline{F} \cap \mathcal{B}$ is the convex hull of the set $\mathscr{V}_F$ of vertices of $\mathcal{B}$ contained in $\overline{F}$. The points of $\mathscr{V}_F$ are by definition the vertices of the facet $\mathcal{F} := F \cap \mathcal{B}$ of $\mathcal{B}$.

Given a k-facet F of $\mathcal{B}(G/K)$, using the description of parabolic κ-subgroups of $\overline{G}_F^{\mathrm{red}}$ up to conjugacy, we see that κ-rank of the derived subgroup of $\overline{G}_F^{\mathrm{red}}$ is equal to the codimension of $\mathcal{F} := F \cap \mathcal{B}$ in $\mathcal{B}$.

We assume in this paragraph that G is absolutely almost simple. Then the Bruhat-Tits building $\mathcal{B}(G/K)$ is a simplicial complex, and in this case $\mathcal{B}$ is also a simplicial complex with simplices $\mathcal{F} := F \cap \mathcal{B}$, for k-facets F of $\mathcal{B}(G/K)$ (F is a simplex!). To see this, note that given a nonempty subset $\mathscr{V}'$ of $\mathscr{V}_F$, the k-facet F' whose set of K-vertices is the union of the set of K-vertices of F_x for $x \in \mathscr{V}'$ is a face of F, so $\mathcal{F}' := F' \cap \mathcal{B}$ is a face of $\mathcal{F}$ and its set of vertices is $\mathscr{V}'$.

3.3. If G is simply connected, then for any k-facet F, the stabilizer of the facet $\mathcal{F} = F \cap \mathcal{B}$ of $\mathcal{B}$ in $G(k)$ (resp. $G(K)$) is $\mathscr{G}_F^\circ(\mathfrak{o})$ (resp. $\mathscr{G}_F^\circ(\mathcal{O}) = \mathscr{G}_F(\mathcal{O})$), hence the stabilizer of $\mathcal{F}$ fixes both F and $\mathcal{F}$ pointwise. This follows from the fact that the stabilizer of $\mathcal{F}$ also stabilizes F. But, in case G is simply connected, the stabilizer of F in $G(K)$ is the subgroup $\mathscr{G}_F^\circ(\mathcal{O}) = \mathscr{G}_F(\mathcal{O})$ ($\subset G(K)$) and this subgroup fixes F pointwise.

3.4. Proposition *Let $\mathcal{A}$ be an apartment in $\mathcal{B}$. Then there is a unique maximal k-split torus S of G such that $\mathcal{A} = \mathcal{B}(Z(S)/K)^\Gamma$. So the stabilizer of $\mathcal{A}$ in $G(k)$ is $N(S)(k)$.*

Proof. After replacing G with its simply connected central cover, we may (and do) assume that G is simply connected. Let S be a maximal k-split torus of G such that $\mathcal{A} = \mathcal{B}(Z(S)/K)^\Gamma$. We will show that S is uniquely determined by $\mathcal{A}$. For this purpose, we observe that the subgroup $\mathcal{Z}$ of $G(k)$ consisting of elements that fix $\mathcal{A}$ pointwise is the intersection of the subgroups $\mathscr{G}_\mathcal{C}^\circ(\mathfrak{o})$, for $\mathcal{C}$ varying over the set of chambers of $\mathcal{A}$. As $N(S)(k)$ acts on $\mathcal{A}$, we conclude that $\mathscr{Z}$ is a bounded subgroup of $G(k)$ that is normalized by $N(S)(k)$. Now, using the Bruhat decomposition of $G(k)$ with respect to S, we see that every bounded subgroup of $G(k)$ that is normalized by $N(S)(k)$ is a normal subgroup of the latter. Since $\mathscr{G}_{\mathcal{C}}^{\circ}(\mathfrak{o})$ contains the maximal bounded subgroup $Z(S)(k)_b$ of $Z(S)(k)$ for every chamber $\mathcal{C}$ of $\mathcal{A}$, $Z(S)(k)_b$ is contained in $\mathscr{Z}$. So the identity component of the Zariski-closure of $\mathscr{Z}$ is $Z(S)$. As S is the unique maximal k-split torus of G contained in $Z(S)$, both the assertions follow. □

Proposition 3.5. *Let $\mathcal{A}$ be an apartment in $\mathcal{B}$, and $\mathcal{C}$, $\mathcal{C}'$ two chambers in $\mathcal{A}$. Then there is a gallery joining $\mathcal{C}$ and $\mathcal{C}'$ in $\mathcal{A}$, i.e., there is a finite sequence*

$$\mathcal{C} = \mathcal{C}_0,\, \mathcal{C}_1,\, \ldots,\, \mathcal{C}_m = \mathcal{C}'$$

of chambers in $\mathcal{A}$ such that for i with $1 \leqslant i \leqslant m$, $\mathcal{C}_{i-1}$ and $\mathcal{C}_i$ share a face of codimension 1.

Proof. Let $\mathcal{A}_2$ be the codimension 2-skelton of $\mathcal{A}$, i.e., the union of all simplices in $\mathcal{A}$ of codimension at least 2. Then $\mathcal{A}_2$ is a closed subset of $\mathcal{A}$ of codimension 2, so $\mathcal{A} - \mathcal{A}_2$ is a connected open subset of the affine space $\mathcal{A}$. Hence $\mathcal{A} - \mathcal{A}_2$ is arcwise connected. This implies that for any points $x \in \mathcal{C}$ and $x' \in \mathcal{C}'$, there is piecewise linear curve in $\mathcal{A} - \mathcal{A}_2$ joining x and x'. Now the chambers in $\mathcal{A}$ which meet this curve make a gallery joining $\mathcal{C}$ to $\mathcal{C}'$. □

As G has been assumed to be semi-simple, the dimension of any apartment, or any chamber, in $\mathcal{B}$ is equal to the k-rank of G. A *panel* in $\mathcal{B}$ is by definition a facet of codimension 1.

Proposition 3.6. *Let $\mathcal{A}$ be an apartment in $\mathcal{B}$ and S be the maximal k-split torus of G corresponding to this apartment.* (*Then $\mathcal{A} = (\mathcal{B}(Z(S)/K)^{\Gamma}$.*) *The group $N(S)(k)$ acts transitively on the set of chambers of $\mathcal{A}$.*

Proof. According to the previous proposition, given any two chambers in $\mathcal{A}$, there exists a minimal gallery in $\mathcal{A}$ joining these two chambers. So to prove the proposition by induction on the length of a minimal gallery joining two chambers, it suffices to prove that given two different chambers $\mathcal{C}$ and $\mathcal{C}'$ in $\mathcal{A}$ which share a panel $\mathcal{F}$, there is an element $n \in N(S)(k)$ such that $n \cdot \mathcal{C} = \mathcal{C}'$. Let $\mathscr{G} := \mathscr{G}_{\mathcal{F}}^{\circ}$ be the smooth $\mathfrak{o}$-group scheme associated with the facet $\mathcal{F}$ and $\mathscr{S} \subset \mathscr{G}$ be the maximal $\mathfrak{o}$-split torus with generic fiber S. Let $\overline{\mathscr{G}}$ be the special fiber of $\mathscr{G}$, $\overline{\mathscr{S}}$ the special fiber of $\mathscr{S}$, and $\overline{G}_{\mathcal{F}}^{\mathrm{red}}$ be the maximal reductive quotient of $\overline{\mathscr{G}}$. The image $\overline{S}$ of $\overline{\mathscr{S}}$ in $\overline{G}_{\mathcal{F}}^{\mathrm{red}}$ is a maximal κ-split torus of $\overline{G}_{\mathcal{F}}^{\mathrm{red}}$. The chambers $\mathcal{C}$ and $\mathcal{C}'$ correspond to minimal parabolic κ-subgroups $\overline{P}$ and $\overline{P}'$ of $\overline{G}_{\mathcal{F}}^{\mathrm{red}}$. Both of these minimal parabolic κ-subgroups contain $\overline{S}$ since the chambers $\mathcal{C}$ and $\mathcal{C}'$ lie on $\mathcal{A}$. But then there is an element $\overline{n} \in \overline{G}_{\mathcal{F}}^{\mathrm{red}}(\kappa)$ which normalizes $\overline{S}$ and conjugates $\overline{P}$ onto $\overline{P}'$.

We consider the normalizer $\mathfrak{o}$-subgroup scheme $N_{\mathscr{G}}(\mathscr{S})$ of $\mathscr{G}$. This subgroup scheme is a closed smooth $\mathfrak{o}$-subscheme (see [C, Prop. 2.1.2]). So, since $\mathfrak{o}$ is Henselian, the natural homomorphism $N_{\mathscr{G}}(\mathscr{S})(\mathfrak{o}) \to N_{\overline{\mathscr{G}}}(\overline{\mathscr{S}})(\kappa)$ is surjective. As κ is perfect, the natural homomorphism $N_{\overline{\mathscr{G}}}(\overline{\mathscr{S}})(\kappa) \to N_{\overline{G}_{\mathcal{F}}^{\mathrm{red}}}(\overline{S})(\kappa)$ is also surjective. Therefore, there is an element $n \in N_{\mathscr{G}}(\mathscr{S})(\mathfrak{o})$ that maps onto $\overline{n}$. It is clear that n normalizes S and hence it lies in $N(S)(k)$; it fixes $\mathcal{F}$ pointwise and $n \cdot \mathcal{C} = \mathcal{C}'$. □

Proposition 3.7. *$\mathcal{B}$ is thick, that is any panel is a face of at least three chambers, and every apartment of $\mathcal{B}$ is thin, that is any panel lying in an apartment is a face of exactly two chambers of the apartment.*

Proof. Let F be a k-facet of $\mathcal{B}(G/K)$ that is not a chamber, and C be a k-chamber of which F is a face. Then there is an $\mathfrak{o}$-group scheme homomorphism $\mathscr{G}_C^\circ \to \mathscr{G}_F^\circ$. The image of $\mathscr{G}_C^\circ(\mathcal{O})$ in $\mathscr{G}_F^\circ(\mathcal{O})$ maps onto the group of $\overline{\kappa}$-rational points of a minimal parabolic κ-subgroup of $\overline{G}_F^{\mathrm{red}}$, and conversely, any minimal parabolic κ-subgroup of the latter determines a k-chamber with F as a face. Now as the projective variety of minimal parabolic κ-subgroups of the κ-group $\overline{G}_F^{\mathrm{red}}$ has at least three κ-rational points, we see that F is a face of at least three distinct k-chambers.

To prove the second assertion, let $\mathcal{F} := F^\Gamma$ be a panel in an apartment $\mathcal{A}$ of $\mathcal{B}$, where F is a k-facet in $\mathcal{B}(G/K)$. Let S be the maximal k-split torus of G corresponding to $\mathcal{A}$. Let $\mathscr{G}_F^\circ$ be the smooth affine $\mathfrak{o}$-group scheme associated with F in 1.9 and $\overline{G}_F^{\mathrm{red}}$ be the maximal reductive quotient of the special fiber of this group scheme. Let $\mathscr{S}$ be the maximal $\mathfrak{o}$-split torus of $\mathscr{G}_F^\circ$ with generic fiber S. Then the chambers of $\mathcal{B}$ lying in $\mathcal{A}$ are in bijective correspondence with minimal parabolic κ-subgroups of $\overline{G}_F^{\mathrm{red}}$ which contain the image $\overline{S}$ of the special fiber of $\mathscr{S}$. The κ-rank of the derived subgroup of $\overline{G}_F^{\mathrm{red}}$ is 1 since $\mathcal{F}$ is of codimension 1 in $\mathcal{B}$ (3.2). This implies that $\overline{G}_F^{\mathrm{red}}$ has exactly two minimal parabolic κ-subgroups containing $\overline{S}$.

The second assertion also follows at once from the following well-known result in algebraic topology: In any simplicial complex whose geometric realization is a topological manifold without boundary (such as an apartment $\mathcal{A}$ in $\mathcal{B}$), any simplex of codimension 1 is a face of exactly two chambers (i.e., maximal dimensional simplices). □

We now assert that $\mathcal{B} = \mathcal{B}(G/K)^\Gamma$ is an affine building. As explained in 1.11, to establish this assertion it suffices to prove it for absolutely almost simple groups. Now in case G is an absolutely almost simple k-group, $\mathcal{B}$ is a simplicial complex (3.2). Propositions 2.4, 3.5, 3.7 and Theorem 3.1 show that all the four conditions, recalled in 1.11, in the definition of buildings are satisfied for $\mathcal{B}$, if $\mathcal{B}(Z(S)/K)^\Gamma = \mathcal{B}(Z(S)/K) \cap \mathcal{B}$, for maximal k-split tori S of G, are taken to be its apartments, and $\mathcal{F} := F \cap \mathcal{B}$, for k-facets F of $\mathcal{B}(G/K)$, are taken to be its facets. Thus we obtain the following:

Theorem 3.8. *$\mathcal{B} = \mathcal{B}(G/K)^\Gamma$ is a building. Its apartments are the affine spaces $\mathcal{B}(Z(S)/K)^\Gamma$ under $V(S) := \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{X}_*(S)$, for maximal k-split tori S of G. Its chambers are $\mathcal{C} := C \cap \mathcal{B}$ for k-chambers C of $\mathcal{B}(G/K)$, and its facets are $\mathcal{F} :=$*

$F \cap \mathcal{B}$ for k-facets F of $\mathcal{B}(G/K)$. The group $G(k)$ acts on $\mathcal{B}$ by polysimplicial automorphisms.

Definition 3.9. $\mathcal{B}$ is called the *Bruhat-Tits building of* $G(k)$.

Since $G(k)$ acts transitively on the set of maximal k-split tori of G, it acts transitively on the set of apartments of $\mathcal{B}$. Now Proposition 3.6 implies the following:

Proposition 3.10. *G(k) acts transitively on the set of pairs $(\mathcal{A},\mathcal{C})$ consisting of an apartment $\mathcal{A}$ of $\mathcal{B}$ and a chamber $\mathcal{C}$ lying in the apartment $\mathcal{A}$.*

3.11. Parahoric subgroups of $G(k)$. Let, as before, G be a connected semi-simple k-group. If F is a k-facet of $\mathcal{B}(G/K)$, then the Bruhat-Tits smooth affine $\mathfrak{o}$-group scheme with connected fibers associated with the facet $\mathcal{F} := F \cap \mathcal{B}$ of $\mathcal{B}$ is the group scheme $\mathscr{G}^{\circ}_{\mathcal{F}}\,(= \mathscr{G}^{\circ}_{F})$ defined in 1.9. The generic fiber of $\mathscr{G}^{\circ}_{\mathcal{F}}$ is G, and the subgroup $\mathscr{G}^{\circ}_{\mathcal{F}}(\mathfrak{o}) = \mathscr{G}^{\circ}_{F}(\mathcal{O})^{\Gamma}\,(\subset G(k))$ fixes F pointwise. Since F is the unique facet of $\mathcal{B}(G/K)$ containing $\mathcal{F}$, the stabilizer of $\mathcal{F}$ also stabilizes F. But $\mathscr{G}^{\circ}_{F}(\mathcal{O})$ is of finite index in the stabilizer of F in $G(K)$. Therefore, $\mathscr{G}^{\circ}_{\mathcal{F}}(\mathfrak{o})$ is of finite index in the stabilizer of $\mathcal{F}$ in $G(k)$. The subgroups $\mathscr{G}^{\circ}_{\mathcal{F}}(\mathfrak{o})$, for k-facets F of $\mathcal{B}(G/K)$, are by definition the *parahoric subgroups* of $G(k)$. For a k-chamber C of $\mathcal{B}(G/K)$, let $\mathcal{C} = C \cap \mathcal{B}$ denote the corresponding chamber of $\mathcal{B}$. The subgroup $\mathscr{G}^{\circ}_{\mathcal{C}}(\mathfrak{o})$ is then a minimal parahoric subgroup of $G(k)$, and all minimal parahoric subgroups of $G(k)$ arise this way. Now let P be a parahoric subgroup of $G(K)$ which is stable under the action of Γ on $G(K)$, then the facet F in $\mathcal{B}(G/K)$ corresponding to P is Γ-stable, i.e., it is a k-facet. Let $\mathcal{F} = F \cap \mathcal{B}$ be the corresponding facet of $\mathcal{B}$, and $\mathscr{G}^{\circ}_{\mathcal{F}}$ be the associated $\mathfrak{o}$-group scheme with generic fiber G and with connected special fiber. Then $\mathscr{G}^{\circ}_{\mathcal{F}}(\mathfrak{o}) = \mathscr{G}^{\circ}_{F}(\mathcal{O})^{\Gamma} = P^{\Gamma}$ is a parahoric subgroup of $G(k)$. Thus the parahoric subgroups of $G(k)$ are the subgroups of the form P^{Γ}, for Γ-stable parahoric subgroups P of $G(K)$.

We will say that a semi-simple k-group G is *residually quasi-split* if any k-chamber is actually a chamber, or, equivalently, if for any k-chamber C, the special fiber of the $\mathfrak{o}$-group scheme $\mathscr{G}^{\circ}_{C}$ is solvable. If the residue field κ of k is of dimension $\leqslant 1$, then by Proposition 2.1, every semi-simple k-group is residually quasi-split. For residually quasi-split G, the minimal parahoric subgroups of $G(k)$ are called the *Iwahori subgroups* of $G(k)$. They are of the form I^{Γ} for Γ-stable Iwahori subgroups I of $G(K)$.

Proposition 3.12. *The minimal parahoric subgroups of $G(k)$ are conjugate to each other under $G(k)$.*

Proof. The minimal parahoric subgroups of $G(k)$ are the subgroups $\mathscr{G}^{\circ}_{\mathcal{C}}(\mathfrak{o})$ for chambers $\mathcal{C}$ in the building $\mathcal{B}$. Proposition 3.10 implies that $G(k)$ acts transitively on the set of chambers of $\mathcal{B}$. □

3.13. Tits systems in suitable subgroups of $G(k)$ given by the building $\mathcal{B}$. Let $\pi : \widehat{G} \to G$ be the simply connected central cover of G. Then $\mathcal{B}$ is also the Bruhat-Tits building of $\widehat{G}(k)$, this group acts on the building via π, and the action is type-preserving (or "special"). Now let $\mathcal{G}$ be a subgroup of $G(k)$ which contains $\pi(\widehat{G}(k))$ and acts on $\mathcal{B}$ by type-preserving automorphisms. Let C be a k-chamber and A be a special k-apartment of $\mathcal{B}(G/K)$ containing C. Let $\mathcal{C} = C \cap \mathcal{B}$ and $\mathcal{A} = A \cap \mathcal{B}$. Then $\mathcal{C}$ is a chamber and $\mathcal{A}$ is an apartment in the Bruhat-Tits building $\mathcal{B}$ of $G(k)$. Let T be the special k-torus of G corresponding to A and S be the maximal k-split torus of G contained in T. Let $N(S)$ be the normalizer of S in G. Let B (resp. $\widehat{B}$) be the subgroup consisting of elements in $\mathcal{G}$ (resp. $\widehat{G}(k)$) which stabilize $\mathcal{C}$, and N (resp. $\widehat{N}$) be the group of elements in $\mathcal{G}$ (resp. $\widehat{G}(k)$) which stabilize $\mathcal{A}$. Then in view of Proposition 3.8, according to [T1, Prop. 3.11], (B, N) is a saturated Tits system in $\mathcal{G}$ and $(\widehat{B}, \widehat{N})$ is a saturated Tits system in $\widehat{G}(k)$, and $\mathcal{B}$ is the Tits building determined by either of these two Tits systems. Note that $\mathcal{G} \cap \mathscr{G}^{\circ}_{\mathcal{C}}(\mathfrak{o})$ is a subgroup of B of finite index, and $N = \mathcal{G} \cap N(S)(k)$ since the stabilizer of $\mathcal{A}$ in $G(k)$ is $N(S)(k)$ by Proposition 3.4.

As $\widehat{G}(k)$ acts transitively on the set of pairs consisting of an apartment of $\mathcal{B}$ and a chamber lying in the apartment (Proposition 3.10 for $\widehat{G}$ in place of G), and the stabilizer of the pair $(\mathcal{A}, \mathcal{C})$ in $\mathcal{G}$ is $B \cap N$, we conclude that $\mathcal{G} = (B \cap N) \cdot \pi(\widehat{G}(k))$. Hence, the Weyl group $N/(B \cap N)$ of the above Tits system is isomorphic to the Weyl group $\widehat{N}/(\widehat{B} \cap \widehat{N})$ of the Tits system $(\widehat{B}, \widehat{N})$ in $\widehat{G}(k)$.

3.14. If G is simply connected, then $\mathcal{G} = G(k)$ and $B = \mathscr{G}^{\circ}_{\mathcal{C}}(\mathfrak{o})$ is a minimal parahoric subgroup of $G(k)$, $N = N(S)(k)$, and the Weyl group of the above Tits system is the "affine Weyl group" $N/(B \cap N) = N(S)(k)/Z(S)(k)_b$. The parahoric subgroups of $G(k)$ are simply the stabilizers (in $G(k)$) of facets in the building $\mathcal{B}$, and so they are subgroups of $G(k)$ which contain a conjugate of B under $G(k)$. The normalizer of any parahoric subgroup P of $G(k)$ is P itself, for if P is the stabilizer of the facet $\mathcal{F}$ of $\mathcal{B}$, then the normalizer of P also stabilizes $\mathcal{F}$, and hence it coincides with P.

4. Filtration of the root groups and valuation of root datum.

4.1. In this section, we will assume that G is simply connected. Fix a maximal k-split torus S of G, and let $\Phi(G, S)$ be the root system of G with respect to S. Let $\mathcal{B}$ be the Bruhat-Tits building of G/k and $\mathcal{A}$ be the apartment corresponding to S. For a nondivisible root a, let U_a be the root group corresponding to a. If $2a$ is also a root, the root group U_{2a} is a subgroup of U_a. Let S_a be the identity component of the kernel of a. Let H_a be the centralizer of S_a and G_a be the derived subgroup of H_a. Then H_a is a Levi-subgroup of G and G_a is a simply connected semi-simple group of k-rank 1. Let C_a be the central torus of H_a. Then S_a is the maximal k-split subtorus of C_a. The root groups of G_a and H_a with respect to S are $U_{\pm a}$, and also $U_{\pm 2a}$ in case $\pm 2a$ are roots too.

There is a $G_a(K)$-equivariant embedding of the Bruhat-Tits building $\mathcal{B}(G_a/K)$ of $G_a(K)$ into the Bruhat-Tits building $\mathcal{B}(G/K)$ of $G(K)$, [BrT1, §7.6]; such an embedding is unique up to translation by an element of $V((C_a)_K) := \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{X}_*((C_a)_K)$. Thus the set of $G_a(K)$-equivariant embeddings of $\mathcal{B}(G_a/K)$ into $\mathcal{B}(G/K)$ is an affine space under $V((C_a)_K)$ on which the Galois group Γ of K/k acts through a finite quotient. Therefore, there is a $\Gamma \ltimes G_a(K)$-equivariant embedding of $\mathcal{B}(G_a/K)$ into $\mathcal{B}(G/K)$. This implies that there is a $G_a(k)$-equivariant embedding ι of the Bruhat-Tits building $\mathcal{B}(G_a/K)^\Gamma$ of $G_a(k)$ into the Bruhat-Tits building $\mathcal{B}(G/K)^\Gamma$ of $G(k)$. (In fact, such embeddings form an affine space under $V(S_a) := \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{X}_*(S_a) = V((C_a)_K)^\Gamma$.) We shall consider the Bruhat-Tits building of $G_a(k)$, which is a Bruhat-Tits tree since G_a is of k-rank 1, embedded in the Bruhat-Tits building of $G(k)$ in terms of ι.

For a real valued affine function ψ on $\mathcal{A}$ with gradient a, let z be the point on the apartment $\mathcal{A}_a$ ($\subset \mathcal{A}$), corresponding to the maximal k-split torus of G_a contained in S, in the Bruhat-Tits tree of $G_a(k)$ such that $\psi(z) = 0$. Let $\mathscr{G}$ be the Bruhat-Tits $\mathfrak{o}$-group scheme with generic fiber G_a corresponding to the point z. We will view $\mathscr{G}(\mathfrak{o})$ as a subgroup of $G(k)$. Denote by U_ψ the subgroup $\mathscr{G}(\mathfrak{o}) \cap U_a(k)$. Using the last assertion of Proposition 2.1.8(3) of [CGP] (with k, which is an an arbitrary commutative ring in that assertion, replaced by $\mathfrak{o}$, and G replaced by $\mathscr{G}$), one can see that U_ψ is a smooth $\mathfrak{o}$-subgroup scheme of $\mathscr{G}$, but we will not use this fact here.

4.2. We will now work with a given $u \in U_a(k) - \{1\}$. Let ψ_u be the largest real valued affine function on $\mathcal{A}$ with gradient a such that u lies in U_{ψ_u} and let $z = z(u)$ be the unique point on the apartment $\mathcal{A}_a$ where ψ_u vanishes. We observe that z is a vertex in $\mathcal{A}_a$. For otherwise, it would be a point of a chamber $\mathcal{C}$ (i.e., a 1-dimensional simplex) of $\mathcal{A}_a$ and then as u fixes z it would fix the chamber $\mathcal{C}$ pointwise, and hence it would fix both the vertices of $\mathcal{C}$. Now let $\psi > \psi_u$ be the affine function with gradient a which vanishes at the vertex of $\mathcal{C}$ where ψ_u takes a negative value. Then u belongs to U_ψ, contradicting the choice of ψ_u to be the largest of such affine functions. As in the previous paragraph, let $\mathscr{G}$ be the Bruhat-Tits group scheme with generic fiber G_a corresponding to the vertex $z = z(u)$. Then u lies in $\mathscr{G}(\mathfrak{o})$. Let $\overline{\mathscr{G}}$ be the special fiber of $\mathscr{G}$. We assert that the image $\overline{u}$ of u in $\overline{\mathscr{G}}(\kappa)$ does not lie in $\mathscr{R}_u(\overline{\mathscr{G}})(\kappa)$, for if it did, then u would fix the unique chamber of $\mathcal{A}_a$ which has z as a vertex and ψ_u takes negative values on it. Then, as before, we would be able to find an affine function $\psi > \psi_u$ with gradient a such that $u \in U_\psi$, contradicting the choice of ψ_u.

4.3. Let $\mathscr{S}$ be the 1-dimensional split torus of $\mathscr{G}$ whose generic fiber is the maximal k-split torus of G_a contained in S. Let $\overline{\mathscr{S}}$ be the special fiber of $\mathscr{S}$. As $\overline{\mathscr{G}}(\kappa)$ contains an element which normalizes $\overline{\mathscr{S}}$ and whose conjugation action on $\overline{\mathscr{S}}$ is by inversion, as in the proof of Proposition 3.6, by considering the smooth normalizer subgroup scheme $N_{\mathscr{G}}(\mathscr{S})$, we conclude that $\mathscr{G}(\mathfrak{o})$ contains an element n which normalizes $\mathscr{S}$ and whose conjugation action on this torus is by inversion.

Let $\lambda : \mathrm{GL}_1 \to \mathscr{S}$ be the $\mathfrak{o}$-isomorphism such that $\langle a, \lambda \rangle > 0$. We shall now use the notation introduced in §2.1 of [CGP]. According to Remark 2.1.11, and the last assertion of Proposition 2.1.8(3) of [CGP] (with k, which is an arbitrary commutative ring in that assertion, replaced by $\mathfrak{o}$, and G replaced by $\mathscr{G}$), the multiplication map

$$U_{\mathscr{G}}(-\lambda) \times Z_{\mathscr{G}}(\lambda) \times U_{\mathscr{G}}(\lambda) \to \mathscr{G}$$

is an open immersion of $\mathfrak{o}$-schemes. We shall denote $U_{\mathscr{G}}(\lambda)$, $Z_{\mathscr{G}}(\lambda)(= Z_{\mathscr{G}}(\mathscr{S}))$ and $U_{\mathscr{G}}(-\lambda)$ by $\mathscr{U}_a$, $\mathscr{Z}$ and $\mathscr{U}_{-a}$ respectively, and the special fibers of these $\mathfrak{o}$-subgroup schemes by $\overline{\mathscr{U}}_a$, $\overline{\mathscr{Z}}$ and $\overline{\mathscr{U}}_{-a}$ respectively. Note that $\mathscr{U}_{\pm a}$ are the $\pm a$-root groups of $\mathscr{G}$ and $\overline{\mathscr{U}}_{\pm a}$ are the $\pm a$-root groups of $\overline{\mathscr{G}}$. Now since $n^{-1}\mathscr{U}_a n = \mathscr{U}_{-a}$, we see that $\Omega := \mathscr{U}_{-a}\mathscr{Z}n\mathscr{U}_{-a}$ is an open subscheme of $\mathscr{G}$. Let $\overline{\Omega} = \overline{\mathscr{U}}_{-a}\overline{\mathscr{Z}}n\overline{\mathscr{U}}_{-a}(\subset \overline{\mathscr{G}})$ be the special fiber of Ω.

According to [CGP, Prop. 2.1.12(1)], the open immersion

$$(\mathscr{R}_u(\overline{\mathscr{G}}) \cap \overline{\mathscr{U}}_a) \times (\mathscr{R}_u(\overline{\mathscr{G}}) \cap \overline{\mathscr{Z}}) \times (\mathscr{R}_u(\overline{\mathscr{G}}) \cap \overline{\mathscr{U}}_{-a}) \to \mathscr{R}_u(\overline{\mathscr{G}}),$$

defined by multiplication, is an isomorphism of schemes. Using this we see that $\overline{\Omega} \cdot \mathscr{R}_u(\overline{\mathscr{G}}) = \overline{\Omega}$. Now let $\pi : \overline{\mathscr{G}} \to \overline{\mathscr{G}}^{\rm red} := \overline{\mathscr{G}}/\mathscr{R}_u(\overline{\mathscr{G}})$ be the natural projection. Then $\pi(\overline{\Omega}) = \pi(\overline{\mathscr{U}}_{-a})\pi(\overline{\mathscr{Z}})\pi(n)\pi(\overline{\mathscr{U}}_{-a})$ and $\pi(\overline{u})$ is a nontrivial element of $\pi(\overline{\mathscr{U}}_a)(\kappa)$. Note that $\pi(\overline{\mathscr{U}}_{\pm a})$ are the $\pm a$-root groups of the reductive group $\overline{\mathscr{G}}^{\rm red}$ with respect to the maximal κ-split torus $\pi(\overline{\mathscr{S}})$ [CGP, Cor. 2.1.9]. Hence, $\pi(\overline{u})$ lies in $\pi(\overline{\Omega})(\kappa)$ (see [BoT, §5] or [CGP, Prop. C.2.24(i)]). So $\overline{u} \in \overline{\Omega}(\kappa)$. Now the following well-known lemma implies at once that u is contained in $\Omega(\mathfrak{o})$. Therefore, there exist $u', u'' \in \mathscr{U}_{-a}(\mathfrak{o})$, such that $m(u) := u'uu'' \in \mathscr{Z}(\mathfrak{o})n\,(\subset N_{\mathscr{G}}(\mathscr{S})(\mathfrak{o}) \subset \mathscr{G}(\mathfrak{o}))$.

Lemma 4.4. *Let X be a scheme, and $\Omega \subset X$ an open subscheme. If for a local ring R, $f : \mathrm{Spec}(R) \to X$ is a map carrying the closed point into Ω, then f factors through Ω.*

Proof. Since Ω is an open subscheme of X, the property of f factoring through Ω is purely topological; i.e., it is equivalent to show that the open subset $f^{-1}(\Omega) \subset \mathrm{Spec}(R)$ is the entire space. Our hypothesis says that this latter open subset contains the closed point, so our task reduces to showing that the only open subset of a local scheme that contains the unique closed point is the entire space. Said equivalently in terms of its closed complement, we want to show that the only closed subset Z of $\mathrm{Spec}(R)$ not containing the closed point is the empty set. For an ideal $J \subset R$ defining Z, this is the obvious assertion that if J is not contained in the unique maximal ideal of R then $J = (1)$. □

4.5. We recall that there exist unique $u', u'' \in U_{-a}(k)$ such that $u'uu''$ normalizes S [CGP, Prop. C.2.24]. Thus the above $m(u)$ is uniquely determined by u. It acts on the apartment $\mathcal{A}$ by an affine reflection $r(u)$ whose derivative (or, vector part) is the reflection associated with a. As $m(u) \in \mathscr{G}(\mathfrak{o})$, $r(u)$ fixes the point $z = z(u)$ defined above. Hence, the fixed point set of the affine reflection $r(u)$ is the hyperplane spanned by $S_a(k) \cdot z$ in $\mathcal{A}$. As $\psi_u(z) = 0$, this hyperplane is the vanishing hyperplane of the affine function ψ_u. This observation implies at once that the filtration subgroups of $U_a(k)$ as defined in [T2, §1.4] are same as the subgroups U_ψ described above. We also note that the largest half-apartment in $\mathcal{A}$ that is fixed pointwise by the element u is $\psi_u^{-1}([0, \infty))$.

4.6. As above, let $u', u'' \in U_{-a}(k)$ be such that $m(u) = u'uu''$ normalizes S. Then $m(u) = (m(u)u''m(u)^{-1})u'u = uu''(m(u)^{-1}u'm(u))$. Since $m(u)^{-1}u'm(u)$ and

$m(u)u''m(u)^{-1}$ belong to $U_a(k)$, we conclude that $m(u') = m(u) = m(u'')$. Hence, $\psi_{u'} = -\psi_u = \psi_{u''}$. Also, $m(u^{-1}) = m(u)^{-1}$, and hence $r(u) = r(u^{-1})$, and so $\psi_u = \psi_{u^{-1}}$.

4.7. Now assume that $2a$ is also a root of G with respect to S, and $u \in U_{2a}(k)$–$\{1\}(\subset U_a(k)$–$\{1\})$. Let u', u'' be as in 4.5. Considering the semi-simple subgroup generated by the root groups $U_{\pm 2a}$, we see that u', $u'' \in U_{-2a}(k)$. Let ψ_u be the affine function as in 4.2. Then $2\psi_u$ is the affine function with gradient $2a$ whose vanishing hyperplane is the fixed point set of the reflection $r(u)$. Thus if we consider u to be an element of $U_{2a}(k)$–$\{1\}$, then the associated affine function with gradient $2a$ is $2\psi_u$.

4.8. The valuation φ_a on the root group $U_a(k)$, corresponding to a given point $s \in \mathcal{A}$ is defined as follows: For $u \in U_a(k)$–$\{1\}$, let $\varphi_a(u) = \psi_u(s)$. According to a result of Tits (Theorem 10.11 of [R]), (φ_a) are a valuation of the root groups $(U_a(k))$. From the results in 4.6, 4.7 we see that for $u \in U_a(k)$–$\{1\}$, if $m(u) = u'uu''$ is as above, then $\varphi_{-a}(u') = -\varphi_a(u) = \varphi_{-a}(u'')$, and $\varphi_a(u) = \varphi_a(u^{-1})$. Moreover, if $2a$ is also a root, then $\varphi_{2a} = 2\varphi_a$ on $U_{2a}(k)$–$\{1\}$.

5. Residue field κ of dimension $\leqslant 1$.

In this section we will assume that the residue field κ is of dimension $\leqslant 1$. Then according to Proposition 2.1, every k-chamber is a chamber in $\mathcal{B}(G/K)$, in other words, every semi-simple k-group is residually quasi-split.

Theorem 5.1. (i) *Any two special k-tori of G are conjugate to each other under an element of $G(k)$.*

(ii) *Let S be a maximal k-split torus of G, then any two special k-tori contained in $Z(S)$ are conjugate to each other under an element of the bounded subgroup $Z(S)'(k)$ of $Z(S)(k)$, where $Z(S)' = (Z(S), Z(S))$ is the derived subgroup of $Z(S)$.*

Proof. (i) For $i = 1, 2$, let T_i be a special k-torus of G and A_i the corresponding special k-apartment in $\mathcal{B}(G/K)$. If $A_1 \cap A_2$ is nonempty, the first assertion follows immediately from the second assertion of Theorem 3.1. So let us assume that $A_1 \cap A_2$ is empty. We fix a k-chamber C_i in A_i, for $i = 1, 2$ (Proposition 2.1). According to Proposition 2.4, there is a special k-apartment A containing C_1 and C_2. Let T be the special k-torus of G corresponding to this apartment. Then using the second assertion of Theorem 3.1 twice, first for the pair $\{A, A_1\}$, and then for the pair $\{A, A_2\}$ we see that T is conjugate to both T_1 and T_2 under $G(k)$. So T_1 and T_2 are conjugate to each other under an element of $G(k)$.

(ii) Let $\mathcal{S}$ be the maximal central k-torus of $Z(S)$ which splits over K. Then any special k-torus of $Z(S)$ is of the form $\mathcal{S} \cdot \mathcal{T}$, where $\mathcal{T}$ is a special k-torus of the semi-simple k-group $Z(S)'$. Now the second assertion follows from the first assertion applied to $Z(S)'$ in place of G. ☐

Theorem 5.2. *Let T be a special k-torus of G and S be the maximal k-split torus of G contained in T. Then $N(T)(k) \subset N(S)(k) = Z(S)'(k) \cdot N(T)(k)$. Therefore, the natural homomorphism $N(T)(k) \to N(S)(k)/Z(S)(k)_b$, induced by the inclusion of $N(T)(k)$ in $N(S)(k)$, is surjective.*

Proof. Any k-automorphism of T carries the unique maximal k-split subtorus S to itself. So $N(T)(k) \subset N(S)(k)$. Now let $n \in N(S)(k)$, then nTn^{-1} is a special k-torus that contains S. So T and nTn^{-1} are special k-tori contained in $Z(S)$. Now Theorem 5.1(ii) implies that there is a $g \in Z(S)'(k)$ such that $g^{-1}Tg = nTn^{-1}$. Hence, gn belongs to $N(T)(k)$, and $n = g^{-1} \cdot gn$. □

The following result is in [BrT3, 4.4-4.5] for complete k.

Theorem 5.3. *Assume that G is absolutely almost simple and anisotropic over k. Then it splits over the maximal unramified extension K of k and is of type A_n for some n.*

Proof. We know from Proposition 2.5 that $\mathcal{B} = \mathcal{B}(G/K)^\Gamma$ consists of a single point, say x. Let A be a special k-apartment of $\mathcal{B}(G/K)$, and C be a k-chamber in A (Proposition 2.1). Then $C^\Gamma = C \cap \mathcal{B}$ is nonempty, and hence it equals $\{x\}$. Let I be the Iwahori subgroup of $G(K)$ determined by the chamber C and T be the k-torus of G corresponding to the apartment A. Then I is stable under Γ, and T_K is a maximal K-split torus of G_K. We consider the affine root system of G_K with respect to T_K and let Δ denote its basis determined by the Iwahori subgroup I. Then Δ is stable under the natural action of Γ on the affine root system and there is a natural Γ-equivariant bijective correspondence between the set of vertices of C and Δ. As $\mathcal{B}$ does not contain any facets of positive dimension, we see from the discussion in 3.2 that Γ acts transitively on the set of vertices of C, and hence it acts transitively on Δ. Now from the description of irreducible affine root systems, we see that G_K is K-split and its root system with respect to the split maximal K-torus T_K is of type A_n for some n, for otherwise, the action of the automorphism group of the Dynkin diagram of Δ is not transitive on Δ. □

Remark 5.4. If k is a locally compact nonarchimedean field (that is, k is complete and its residue field κ is finite), then any absolutely almost simple k-anisotropic group G is of *inner* type A_n for some n. This assertion was proved by Martin Kneser for fields of characteristic zero, and Bruhat and Tits in general. In view of the previous theorem, to prove it, we just need to show that any simply connected absolutely almost simple k-group G of *outer* type A_n for $n \geqslant 2$ is k-isotropic. Since there does not exist a noncommutative finite dimensional division algebra with center a quadratic Galois extension of k which admits an involution of the second kind with fixed field k, see [Sch, Ch. 10, Thm. 2.2(ii)], if G is of outer type, there is a quadratic Galois extension ℓ of k and a nondegenerate hermitian form h on ℓ^{n+1} such that $G = \mathrm{SU}(h)$. But any hermitian form over a nonarchimedean locally compact field in at least 3 variables represents zero nontrivially, and hence $\mathrm{SU}(h)$ is isotropic for $n \geqslant 2$.

The following example of an absolutely almost simple k-anisotropic group of outer type A_{r-1} (over a suitable k) was communicated to me by Philippe Gille. As usual, $\mathbb{C}$ will denote the field of complex numbers; for a positive integer r, let μ_r denote the group of r-th roots of unity; $F = \mathbb{C}(x)$ and $F' = \mathbb{C}(x')$ with $x' = \sqrt{x}$. We take $k = F((t))$ and $k' = F'((t))$. The residue maps induce isomorphisms:

$$\ker({}_r\mathrm{Br}(k') \xrightarrow{N_{k'/k}} {}_r\mathrm{Br}(k)) \xrightarrow{\simeq} \ker(\mathrm{H}^1(k', \mu_r) \xrightarrow{N_{k'/k}} \mathrm{H}^1(k, \mu_r))$$
$$\xrightarrow{\simeq} \ker({k'}^{\times}/{k'}^{\times r} \xrightarrow{N_{k'/k}} k^{\times}/k^{\times r}).$$

The element $u := (1 + x')/(1 - x')$ is a nontrivial element of the last group, it has trivial norm and has a pole of order 1 at $x' = 1$, so it cannot be an r-th power. It defines a central simple k'-algebra $\mathscr{D}$ which is division and cyclic of degree r. By Albert's theorem, $\mathscr{D}$ carries a k'/k-involution τ of the second kind. The k-group $\mathrm{SU}(\mathscr{D}, \tau)$ is of outer type A_{r-1} and is anisotropic over k.

In the following theorem, and in its proof, we will use the notation introduced earlier in the paper, and assume, as before, that k is a discretely valued field with Henselian local ring and perfect residue field κ of dimension $\leqslant 1$.

Theorem 5.5. *Let G be a simply connected semi-simple k-group. Then the Galois cohomology set $\mathrm{H}^1(k, G)$ is trivial.*

Proof. By Steinberg's theorem (1.7), $\mathrm{H}^1(K, G)$ is trivial, so $\mathrm{H}^1(k, G) \simeq \mathrm{H}^1(K/k, G(K))$. Let $c : \gamma \mapsto c(\gamma)$ be a 1-cocycle on the Galois group Γ of K/k with values in $G(K)$ and ${}_cG$ be the Galois-twist of G with the cocycle c. The k-groups G and ${}_cG$ are isomorphic over K and we will identify ${}_cG(K)$ with $G(K)$. (Recall that with identification of ${}_cG(K)$ with $G(K)$ as an abstract group, the "twisted" action of Γ on ${}_cG(K)$ is described as follows: For $x \in {}_cG(K)$, and $\gamma \in \Gamma$, $\gamma \circ x = c(\gamma)\gamma(x)c(\gamma)^{-1}$, where $\gamma(x)$ denotes the γ-transform of x considered as a K-rational element of the given k-group G.)

Now let I be a Γ-stable Iwahori subgroup of $G(K)$, say $I = \mathscr{G}_C^{\circ}(\mathcal{O})$ for a k-chamber C of $\mathcal{B}(G/K)$. The subgroup I is also an Iwahori subgroup of ${}_cG(K)$ (as ${}_cG(K)$ has been identified with $G(K)$ in terms of a K-isomorphism ${}_cG_K \to G_K$). However, under the twisted action of Γ on ${}_cG(K)$, I may not be Γ-stable. But as ${}_cG$ is a residually quasi-split semi-simple k-group, ${}_cG(K)$ certainly contains an Iwahori subgroup which is stable under the twisted action of Γ. Since any two Iwahori subgroups of ${}_cG(K)$ are conjugate under ${}_cG(K) = G(K)$ (Proposition 3.10 for K in place of k), there exists a $g \in G(K)$ such that gIg^{-1} is stable under the twisted action of Γ. Then $c(\gamma)\gamma(g)I\gamma(g)^{-1}c(\gamma)^{-1} = gIg^{-1}$ for all $\gamma \in \Gamma$. Hence, for $\gamma \in \Gamma$, $c'(\gamma) := g^{-1}c(\gamma)\gamma(g) \in G(K)$ normalizes the Iwahori subgroup I. As the normalizer of I is I itself (3.12 for K in place of k), we conclude that c', which is a 1-cocycle on Γ cohomologous to c, takes values in $I = \mathscr{G}_C^{\circ}(\mathcal{O})$. So to prove the theorem, it suffices to prove the triviality of $\mathrm{H}^1(\Gamma, \mathscr{G}_C^{\circ}(\mathcal{O}))$.

By unramified Galois descent over discrete valuation rings [BLR, §6.2, Ex. B], this cohomology set classifies $\mathscr{G}_C^{\circ}$-torsors $\mathscr{X}$ over $\mathfrak{o}$ which admit an $\mathcal{O}$-point. (As $\mathscr{X}$

inherits $\mathfrak{o}$-smoothness from $\mathscr{G}_C^\circ$, and $\mathcal{O}$ is Henselian with algebraically closed residue field $\overline{\kappa}$, $\mathscr{X}$ does automatically admit $\mathcal{O}$-points.) Thus, it suffices to prove that every such torsor admits an $\mathfrak{o}$-point. By $\mathfrak{o}$-smoothness of $\mathscr{X}$, and the henselian property of $\mathfrak{o}$, it suffices to prove that the special fiber of $\mathscr{X}$ admits a rational point. But the isomorphism class of the special fiber as a torsor is classified by the set $\mathrm{H}^1(\Gamma, \mathscr{G}_C^\circ(\overline{\kappa}))$ which is trivial by Steinberg's Theorem (1.7) since κ has cohomological dimension $\leqslant 1$. □

Remark 5.6. The above theorem was first proved by a case-by-case analysis by Martin Kneser for k a nonarchimedean local field of characteristic zero with finite residue field. It was proved for all discretely valued *complete* fields k with perfect residue field of dimension $\leqslant 1$ by Bruhat and Tits [BrT3, Thm. in §4.7]. If $\widehat{k}$ denotes the completion of k, then the natural map $\mathrm{H}^1(k, G) \to \mathrm{H}^1(\widehat{k}, G)$ is bijective [GGM, Prop. 3.5.3(ii)]. So the vanishing theorem of Bruhat and Tits over the completion $\widehat{k}$ also implies the above theorem.

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