My Mathematical Contributions

My initial work was concerned with the rigidity of lattices in real and p-adic semi-simple groups. I recall that a lattice in a locally compact group G is a discrete subgroup Γ such that the homogeneous space G/Γ carries a finite G-invariant Borel measure. A lattice is said to be *strongly rigid* if it "determines" the ambient topological group G. In my first paper [1], written with M.S. Raghunathan, it was shown that the $\mathbb{R}$-rank of a real semi-simple Lie group can be determined from the group theoretic structure of any lattice in it. Later, I proved that every irreducible non-cocompact lattice in a semi-simple Lie group which has a factor of $\mathbb{R}$-rank 1 is strongly rigid ([2]). This, combined with the beautiful results of G.D. Mostow and G.A. Margulis on strong rigidity, implies that any irreducible lattice in a real semi-simple Lie group G is strongly rigid provided G does not contain any nontrivial compact normal subgroups and it is not locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$. Mostow's proof of strong rigidity requires an equivariant *pseudo-isometry* between the associated symmetric spaces. The proof of existence of such a pseudo-isometry is fairly simple if the lattice Γ is cocompact. However, if G/Γ is noncompact, the existence of an equivariant pseudo-isometry is not completely obvious. But if G has a factor of $\mathbb{R}$-rank 1, using the geometric properties of a nice fundamental domain, and the fact proven in [3] that an isomorphism $\Gamma \to \Gamma'$, of an irreducible lattice Γ in a connected semi-simple Lie group G which does not contain a nontrivial connected compact normal subgroup and which is not locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$ onto a lattice Γ' in a connected semi-simple Lie group G' which also does not contain a nontrivial connected compact normal subgroup, takes unipotent elements to unipotent elements, I was able to show that a desired pseudo-isometry exists.

In [4], it is proved that if G, G' are connected semi-simple Lie groups, G is not locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$, Γ is an irreducible lattice and Γ' is a discrete subgroup of G' isomorphic to Γ, then Γ' is a lattice in G' if and only if the dimension of the symmetric space associated with G' equals the dimension of the symmetric space associated with G. This implies, in particular, that Γ is co-Hopfian, i.e., it is not isomorphic to a proper subgroup.

After the above work was done, I began to investigate lattices in semi-simple groups over nonarchimedean local fields and also began considering arithmetic questions on semi-simple groups defined over global fields. In 1973, I proved that any irreducible cocompact lattice in a product of semi-simple groups over local fields is strongly rigid if the sum of the local ranks is at least two ([5]), but in case the local fields are of positive characteristic, a lattice need not be infinitesimally rigid ([6]). My proof of strong rigidity makes use of the geometry of the Bruhat-Tits buildings and the fundamental ideas of Mostow from his proof of strong rigidity of lattices in real semi-simple Lie groups.

In 1976, I, and independently, Margulis, proved the strong approximation property for all simply connected semi-simple groups over global fields of arbitrary characteristics ([7]). Over number fields, this result was proved by M. Kneser and V.P. Platonov; however, their methods did not work if the global field is of positive characteristic, i.e., it is the function field of a curve over a finite field. My proof, as well Margulis', for groups over a global function field required geometric and ergodic theoretic considerations, and also the Kneser-Tits property of simple simply connected isotropic groups over local fields, i.e. the projective simplicity of the abstract group $G(k)$, if G is a simple simply connected group defined and *isotropic* over a local field k. It has been known for quite sometime that most of the classical groups have the Kneser-Tits property. Platonov proved in 1969 that in fact any simple simply connected isotropic group over a local field has this property. Using a Galois-cohomological argument, in 1981 I gave a simple, and conceptually more satisfactory, proof of this result; this inspired the work contained in the joint paper [10]. In [10] it has been proved that over an arbitrary field k, all simple simply connected k-isotropic groups have the Kneser-Tits property if all such groups of k-rank 1 have this property. I have recently shown that over a global field, the triality forms (of type D_4) of rank 1 have the Kneser-Tits property. So, now, except for the two outer forms ${}^2E_{6,1}^{29}$ and ${}^2E_{6,1}^{35}$ of rank 1, all simple simply connected isotropic groups over a global field are known to possess the Kneser-Tits property. I am currently looking at the form ${}^2E_{6,1}^{29}$.

A good part of my work during the late seventies and early eighties was in collaboration with M.S. Raghunathan and it was devoted to determining topological central extensions of semi-simple groups over local fields and to the computation of the *metaplectic kernel*

$$M(S,G) := \mathrm{Ker}(H_m^2(G(A(S)),\mathbb{R}/\mathbb{Z}) \to H^2(G(K),\mathbb{R}/\mathbb{Z})),$$

for an absolutely simple simply connected group G defined and isotropic over a global field K, where S is a finite set of places of K and $A(S)$ is the K-algebra of S-adeles of K. Our approach to settle these problems was different from all the earlier work in the area. It used the structure and geometry of the groups, provided by the Borel-Tits and the Bruhat-Tits theories, intimately. Details of this work are given in the papers [8], [9] and [11], where topological central extensions of $G(K_v)$ have been determined (by computing the continuous cohomology group $H_c^2(G(K_v),\mathbb{R}/\mathbb{Z})$) for every place v of K, and the metaplectic kernel $M(S,G)$ has been computed for all *isotropic* G. A survey of the known results and open problems in the area can be found in my address at the International Congress of Mathematicians held in Kyoto in 1990. It is known that whenever the congruence subgroup kernel is central, it is isomorphic to the dual of the metaplectic kernel. Thus the computation of the metaplectic kernel leads to a precise solution of the congruence subgroup problem once the centrality of the congruence subgroup kernel is established. A precise computation of the metaplectic kernel is also reqired in the theory of automorphic forms of fractional weights, and this computation has been used to construct examples of smooth complex projective varieties whose fundamental group is not residually finite.

The centrality of the congruence subgroup kernel is known for all isotropic and several anisotropic groups. The goal of [25] is to present a short and simple proof of the centrality.

In 1993, A.S. Rapinchuk and I computed the metaplectic kernel for all absolutely simple simply connected groups (including the anisotropic groups); see [18]. In [18] we have also shown that the topological central extensions of the adele group, and of the group of rational points of an absolutely simple simply connected isotropic group over a nonarchimedean local field, constructed by Pierre Deligne in his paper in Publ. Math. IHES **84**(1996), are universal. This, together with the main result of [9], implies that the second continuous cohomology group $H_c^2(G(k), \mathbb{R}/\mathbb{Z})$ of $G(k)$, where G is an absolutely simple simply connected group defined and isotropic over a nonarchimedean local field k, is isomorphic to the dual $\hat{\mu}(k)$ of the finite cyclic group $\mu(k)$ of roots of unity in k. In [22] a purely local proof of the fact that Deligne's topological central extension of $G(k)$ is universal has been given. It has also been shown there that $\mathrm{Aut}(G)(k)$ acts trivially on $H_c^2(G(k), \mathbb{R}/\mathbb{Z})$.

In 1987 I gave a formula for the volume of the quotient of a semi-simple group by a principal S-arithmetic subgroup; see [12]. Using this formula and some number theoretic estimates, in a subsequent paper [13], written in collaboration with Armand Borel, finiteness of the number of S-arithmetic subgroups with covolume bounded by a given number, and the finiteness of the set of semi-simple groups of compact type defined over some global field and with a given class number were established. A result of [12] inspired Benedict Gross to construct a "motive" for reductive groups. More recently, the volume formula of [12], and the results and techniques of [13], have been used to count maximal arithmetic subgroups.

In another work [14] with Borel, I investigated the density of the set of values of an irrational indefinite quadratic form at S-integral points. Some of the techniques introduced in this work have turned out to be quite useful.

I have been currently also interested in the representation theory of reductive p-adic groups. In the papers [15, 16] written in collaboration with Allen Moy, natural filtrations of parahoric subgroups and their Lie algebras have been given, and a uniform proof of existence and associativity of unrefined minimal K-types in an arbitrary admissible representation of a connected reductive p-adic group has been given using the Bruhat-Tits Theory and a result of G. Kempf and G. Rousseau in Invariant Theory. We have defined a new invariant of an admissible representation, called its *depth*, which is a nonnegative rational number associated to the representation. In [16] we proved that the depth does not change under parabolic induction and Jacquet restrictions. In that paper we also studied irreducible admissible representations of depth zero in detail and completely determined the supercuspidal representations of depth zero. These results include, as a particular case, the well-known results of Borel on irreducible representations which contain a nonzero vector fixed under an Iwahori subgroup. A result of [16] on the equivalence of the category of depth zero representations with the category of representations of certain Hecke algebras has been used by George Lusztig to classify the unipotent representations of depth zero.

The description of supercuspidal representations of depth zero given in [16] was crucial for J.K. Yu's construction of new supercuspidal representations. The filtration of parahoric subgroups and Lie algebras introduced in [15-17], and the geometric techniques developed in these papers have been used by Stephen DeBacker to classify maximal unramified tori in a reductive p-adic group and the nilpotent orbits in its Lie algebra, and also in his proof of the "Hales-Moy-Prasad conjecture" on the range of validity of Harish Chandra-Howe local character expansion. Several other mathematicians have used the tecniques and the results of [15-17] to settle important questions in the representation theory of, and harmonic analysis on, reductive p-adic groups. For example, Ju-Lee Kim's recent work in which she has shown that the supercuspidal representations constructed by J.K. Yu exhaust all the "tame" ones, and J.-L. Waldspurger's proof of the fact that the "Fundamental Lemma" in positive characteristics implies it in characteristic zero, use the formalism and the results of [15] and [16].

In [20], J.K. Yu and I have proved that if F is a finite subgroup of the group $\mathrm{Aut}(G)(k)$ of automorphisms of a connected reductive group G defined over a nonarchimedean local field k, then the set of points fixed under F in the Bruhat-Tits building of G/k is the Bruhat-Tits building of the connected reductive subgroup $(G^F)^o/k$ provided the order of F is prime to p.

Given a finitely generated subfield K of $\mathbb{R}$, a connected semi-simple K-group G, and a Zariski-dense subgroup Γ of G, it is shown in [19] and [21] that Γ contains a regular $\mathbb{R}$-regular element γ such that the centralizer T_γ of γ in G is a maximal torus the Galois group of whose splitting field contains the Weyl group of G. Given such an element γ, the cyclic group generated by it is dense in T_γ in the Zariski-topology. These elements have been used by A. Katok and R. Spatzier and Margulis and N. Qian in their study of Anosov and hyperbolic actions of lattices and abelian groups on manifolds, and by H. Abels, Margulis and G.A. Soifer in their investigation of the Auslander problem.

A consequence of the main result of [21] is that given a positive integer n, there exist regular $\mathbb{R}$-regular elements $\gamma_1, \ldots, \gamma_n \in \Gamma$, whose eigenvalues under any faithful representation of G are "multiplicatively independent". With this result at hand, in [23] we were able to use some known results in transcendental number theory, and a well-known conjecture due to Schanuel, to settle a question of Y. Benoist, and also study lengths of closed geodesics in locally symmetric spaces. The existence of such elements implies that if G does not split over $\mathbb{R}$, and $G(\mathbb{R})$ is noncompact, then the eigenvalues of regular $\mathbb{R}$-regular elements in Γ generate a dense subgroup of $\mathbb{C}^\times$. This last result is being used by Y. Guivarc'h and A. Raugi to determine minimal Γ-invariant closed subsets of some quotients of $G(\mathbb{R})$.

In [24] J.K. Yu and I have shown that a flat affine group scheme $\mathcal{G}$ over a local ring R, whose generic fiber is connected and smooth, and whose reduced special fiber is a reductive group of same dimension as the generic fiber, is of finite type, and except when the generic fiber contains SO_{2n+1}, for some n, as a normal subgroup, and the residue field of R is of characteristic 2, $\mathcal{G}$ is reductive (in particular, smooth). (This result implies, for example, that a flat affine group scheme over a Dedekind scheme S, all whose fibers are connected reductive algebraic groups of the same dimension, is a reductive group over S.) In [24] we have classified all "good quasi-reductive" models of SO_{2n+1} over a local ring R. If the residue field of R is of characteristic 2, then there are good quasi-reductive models which are not smooth. The results of [24] have been used by Mirkovic and Vilonen to give a geometric interpretation of the Langlands dual through a Tannakian formalism.

Selected publications

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23. G. Prasad and A. S. Rapinchuk, *Zariski-dense subgroups and transcendental number theory*, Math. Res. Letters **12**(2005), 239-249.

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