Typos/improvements in "Pseudo-reductive groups" (1st printing), 10/8/11

Inside flap: for the description of the jobs of Conrad and Gabber, replace "a Professor of Mathematics" with "Professor of Mathematics" for Conrad, and replace "a Directeur de Recherches" with "Directeur de Recherche" for Gabber. Also remove "(IHÉS)".

p. xi: On line 10 of the text, replace "component $\mathrm{Aut}^0_{X/k}$." with "component $\mathrm{Aut}^0_{X/k}$ apart from that it is a k-group scheme of finite type."

p. xvi: On line 21 replace"(assuming $[k : k^2] \leqslant 2$ when $\mathrm{char}(k) = 2$)" with "(apart from some exceptions when $\mathrm{char}(k) = 2$)", and on line 27 replace "would be" by "is".

p. xix: On line -2, replace "(such as" with "(e.g., we can have".

p. xx: At the end of the text on this page, insert a new paragraph:

For a smooth affine group G over a field k and a k-split k-torus $T \subset G$, $\Phi(G, T)$ denotes the set of *non-trivial* weights of T under the adjoint action of G on $\mathrm{Lie}(G)$ (i.e., the set of nontrivial k-homomorphisms $a : T \to \mathbf{G}_{\mathrm{m}}$ such that the a-weight space in $\mathrm{Lie}(G)$ is nonzero). This definition also makes sense for any group scheme G locally of finite type over k, but we only require the smooth affine case. (By using Lemma A.8.8, this definition carries over to smooth group schemes and split tori over any connected non-empty base scheme, but we will not need this generalization.)

p. 3: On line -8, replace "In particular" with "As a special case".

p. 7: On line 8 of Remark 1.1.7, insert "bijectively" immediately after "projecting". On the last line of Remark 1.1.7, replace "B" with "B mod W".

p. 12: On line 5, replace [Bo2, §2.1] with [Bo2, I, 2.1, 2.2]

p. 13: On line 2, replace "between maximal k-tori" with "to a maximal k-torus in G".

p. 14: On line -18, replace "the" with "an".

p. 15: On line -5, omit the second N.

p. 16: At the end of line -16 (end of the first full paragraph of 1.3), insert the sentence "For later purposes, we now consider any smooth connected affine group G over an arbitrary field k, but the reader may wish to keep in mind the above special case."

On line -15, replace $G^{\mathrm{ss}}_{\overline{k}}$ with $G^{\mathrm{red}}_{\overline{k}}$. On line -14, replace "but this" with "but if G is pseudo-reductive and non-reductive then this". On line -10, remove "$= \mathscr{R}(G_{\overline{k}})$". On line -7, replace "absolutely simple and semisimple" with "reductive".

On lines -4 and -3, remove the sentence "Note that the definition ... affine k-group G."

On line -1 replace "for which" with "for which $G^{\mathrm{ss}}_{\overline{k}}$ is simple and".

p. 17: On line 4, replace "The" with "Consider a non-reductive pseudo-reductive k-group G such that $G^{\mathrm{red}}_{\overline{k}}$ is a simple semisimple group." On line 5, remove "therefore". On lines 5–6, remove "pseudo-reductive".

On line -4, replace "To see this, choose" with "To see this, first note that for any finite extension of fields L/K and integer $m > 0$, applying the snake lemma for fppf abelian sheaves to the m-power endomorphism of the short exact sequence of K-groups

$$1 \to \mathrm{GL}_1 \to \mathrm{R}_{L/K}(\mathrm{GL}_1) \to \mathrm{R}_{L/K}(\mathrm{GL}_1)/\mathrm{GL}_1 \to 0$$

proves that the natural map $\mathrm{R}_{L/K}(\mu_m)/\mu_m = \mathrm{R}_{L/K}(\mathrm{GL}_1)[m]/\mu_m \to (\mathrm{R}_{L/K}(\mathrm{GL}_1)/\mathrm{GL}_1)[m]$ is an isomorphism. Now choose"

p. 21: On line 1, replace "surjectivity" with "non-surjectivity".

p. 22: At the end of line 4, replace "$p-1$." with "$p-1$ (see (1.3.2))."

p. 37: On line -14, replace "is a" by "is then a".

p. 39: On line -14, just before the right parenthesis insert ", and the snake lemma argument in Example 1.3.2 describes $U[p]$".

p. 41: Replace the text from line 12 (beginning of second full paragraph) until the end of the page with:

Define $G'' = G' \times G'$ and $T'' = T' \times T'$. By Proposition 1.3.4, the k-groups $G = \mathrm{R}_{k'/k}(G'')/Z$ and $C = \mathrm{R}_{k'/k}(T'')/Z$ are pseudo-reductive and G is perfect. Since C is commutative and acts on $\mathrm{R}_{k'/k}(G'')$ in the evident manner, we get a standard presentation of G via the natural isomorphism

$$G \simeq (\mathrm{R}_{k'/k}(G'') \rtimes C)/\mathrm{R}_{k'/k}(T'').$$

Let $N \subseteq G$ be the smooth connected normal k-subgroup that is the image in G of the first factor $\mathrm{R}_{k'/k}(G')$ of $\mathrm{R}_{k'/k}(G'')$, so N is perfect. We shall prove that the quotient G/N is not pseudo-reductive over k.

p. 42: On lines 8–9, remove "as well as the exact sequence (1.6.3)," and on line 10 change the equation label to (1.6.3). On line 14, replace (1.6.4) with (1.6.3).

p. 43: In paragraph 2, replace the text beginning at "but it will suffice . . . " and ending at the end of the paragraph with "but it will suffice for our needs in the main body of this book to have this theory for pseudo-split groups. In Appendix C we give a detailed exposition of k-root systems, relative Weyl groups, and k-root groups, as well as construct a Tits system in $G(k)$ for any smooth connected affine group G over any field k; see C.2.11–C.2.23."

In paragraph 3, replace the parenthetical in the last two lines with "if $\mathrm{char}(k) \neq 2$, and we prove a related result when $\mathrm{char}(k) = 2$." (Note that no parentheses are used here.)

p. 44: On line 2 replace "smooth" with "smooth and closed and the quotient", and replace lines -10 through -3 with the following:

where $\{\chi_1, \dots, \chi_m\}$ is the finite set of pairwise distinct weights for the GL_1-action on V via λ (and $m \geqslant 0$). In particular, the V_{χ_i} are nonzero finitely generated projective k-modules and $\chi_i(t) = t^{c_i}$ for a unique $c_i \in \mathbf{Z}$. By working Zariski-locally on $\mathrm{Spec}\, k$ for now, we may assume that $V \neq 0$ and the V_{χ_i} are free k-modules. We also may and do arrange the χ_i so that c_i is strictly decreasing in i.

p. 45: On line 4, replace $\langle \chi_{i_r}, \lambda \rangle$ with c_{i_r}. On the right side of the displayed expression (2.1.1), replace $t^{\langle \chi_{i_r}, \lambda \rangle - \langle \chi_{i_s}, \lambda \rangle}$ with $t^{c_{i_r} - c_{i_s}}$. On line 11, remove "or constancy of the $\langle \chi_i, \lambda \rangle$ above". On lines -4, -5, and -8, replace $\mathrm{gr}^\bullet(V)$ with $\mathrm{gr}_\bullet(V)$.

p. 51: On line -2, replace "adjoint representation" with "derivative".

p. 57: In the *first* line of the statement of Proposition 2.1.10, remove the word "connected". Replace the entire proof with the following:

Proof. By Theorem B.3.4, any smooth connected unipotent k-subgroup U contains a unique k-split smooth connected normal k-subgroup U_{split} such that U/U_{split} is k-wound in the sense of Definition B.2.1 (i.e., admits no non-constant k-morphism from $\mathbf{A}^1_k$), and moreover the formation of U_{split} commutes with separable extension on k. To prove that $U := U_G(\lambda)$ is k-split, we will show that $U = U_{\mathrm{split}}$. Now we can replace k with k_s and assume that $k = k_s$. For any nontrivial element g in $U(k)$, the map $t \mapsto \lambda(t)g\lambda(t)^{-1}$ extends to a morphism $\mathbf{A}^1_k \to U$ carrying 0 to 1. The induced morphism $\phi : \mathbf{A}^1_k \to U/U_{\mathrm{split}}$ is constant since U/U_{split} is k-wound. This implies that $g \in U_{\mathrm{split}}(k)$. Thus we have shown that $U(k) \subseteq U_{\mathrm{split}}(k)$. From the Zariski-density of $U(k)$ in U we conclude that $U = U_{\mathrm{split}}$. □

pp. 67–68: Replace Definition 2.3.1 and the two paragraphs following it by the following two paragraphs:

Let k be a field, and G a smooth connected affine k-group. In Theorem C.2.3 we shall prove that all maximal k-split k-tori in G are $G(k)$-conjugate. The proof of this result rests on the structure theory of root groups that is developed below.

Definition 2.3.1 The k-group G is *pseudo-split* (over k) if it contains a maximal k-torus T that is k-split. (If G is connected reductive then this coincides with the usual notion of being k-split.) We call any such (G, T) a *pseudo-split pair.*

Now let T be an arbitrary k-split k-torus in G. Since T is k-split, $\mathrm{Lie}(G)$ equipped with its adjoint action by T decomposes as a direct sum of weight spaces for k-rational characters of T (Lemma A.8.8). Define $\Phi(G, T) \subset \mathrm{X}(T)$ to be the set of nontrivial characters of T that occur in $\mathrm{Lie}(G)$.

p. 68: Replace line -10 on p. 68 by "Returning to the general setup, we fix a k-split k-torus T in a smooth connected affine k-group G, and for each nonzero $a \in \mathrm{X}(T)_{\mathbf{Q}}$ let T_a be the". Delete the word "maximal" on line -2 of this page.

p. 70: On line 3, replace "$n \geq 1$" with "$n \geqslant 1$". On line -4, replace "f" with "π". Replace the entirety of Remark 2.3.6 (the third, fourth and the fifth paragraph on this page) with the following two paragraphs:

Remark 2.3.6 Let G be a smooth connected affine k-group and T a k-split k-torus of G. For any smooth connected k-subgroup $H \subset G$ that contains T, and any nonzero $a \in \mathrm{X}(T)_{\mathbf{Q}}$, since $Z_H(T_a) = H \cap Z_G(T_a)$ for $T_a = (\ker a)^0_{\mathrm{red}}$ we see (via Lemma 2.1.5) that $U^H_{(a)} = H \cap U^G_{(a)}$

(where $U_{(a)}^{G}$ is the group $U_{(a)}$ in Lemma 2.3.3 associated to (G,T,a) and similarly with $U_{(a)}^{H}$ and (H,T,a)). In particular, $H\cap U_{(a)}^{G}$ is smooth and connected.

Consider a quotient map $\pi : G \twoheadrightarrow G'$ over k, and let $T' = \pi(T)$, so $\pi|_T$ induces an embedding of $\mathrm{X}(T')_{\mathbf{Q}}$ into $\mathrm{X}(T)_{\mathbf{Q}}$ that we use to identify the former with a subspace of the latter. From Corollary 2.1.9 we see that for any nonzero $a\in \mathrm{X}(T')_{\mathbf{Q}}$, π carries $U_{(a)}^{G}$ onto $U_{(a)}^{G'}$ (note that Proposition A.2.8 implies that $\pi(Z_G(T_a)) = Z_{G'}(T'_a))$. In particular, if $a\in\Phi(G',T')$ then $U_{(a)}^{G'}\neq 1$, so $U_{(a)}^{G}\neq 1$ and hence some positive rational multiple of a lies in $\Phi(G,T)$. If G is pseudo-reductive and $(\ker\pi_{\overline{k}})_{\mathrm{red}}^{0}$ is unipotent then for any smooth connected k-subgroup $H\subset G$ containing T and $a\in\Phi(H,T)$, the map π cannot kill $U_{(a)}^{H}$ (by Lemma 1.2.1). Thus, for $H'=\pi(H)$ the k-group $U_{(a)}^{H'}$ is nontrivial, which is to say that the element $a\in\Phi(H,T)$ cannot completely "disappear" in $\Phi(H',T')$. That is, some positive rational multiple of a lies in $\Phi(H',T')$ when G is pseudo-reductive and $(\ker\pi_{\overline{k}})_{\mathrm{red}}^{0}$ is unipotent. The same assertion clearly holds if instead of assuming that $(\ker\pi_{\overline{k}})_{\mathrm{red}}^{0}$ is unipotent, we assume that $\ker\pi_{\overline{k}}\subseteq Z_G(T)$.

p. 71: On line 5 replace "U, and" with "U when A is local, and". On lines 6–7, replace "G is pseudo-reductive" with "(G,T) is a pseudo-split pair with G pseudo-reductive".

p. 73: On line 12, in item (iii) replace "normal" with "normal subgroup".

p. 75: On line 13, replace "We call" with "For a pseudo-split pair (G,T) over k, we call". On lines -4, -5, as well as on lines -2, -1, replace "settles the existence … structure claims" with "can be taken to be U_a, and it admits the required unique linear structure".

On lines -2, -1, replace "settles the existence … structure claims" with "can be taken to be U_{2a} and it admits the required unique linear structure".

p. 78: At the end of the statement of Proposition 2.3.15 add "$:= U_a^{G'}$", and replace the last sentence of the proof of this proposition with "This is a special case of an assertion in Remark 2.3.6.".

p. 82: On line 14 remove "if k is infinite then", and on lines 15 and 16 remove the final sentence "For completeness … of $(G_{\overline{k}})_t$."

p. 84: In Remark 2.4.7, replace "the same result" with "the equivalence among conditions (i), (ii), and (iii)".

p. 85: On line 6 replace "so" with "and for such a an argument similar to the proof of uniqueness implies that", and replace "endomorphism and thus" with "endomorphism. Thus, for such a the kernel". On line -5 replace "so" with "and similarly" and replace "and hence" with "so it". On line -4 replace "has smooth" with "has a smooth".

p. 90: On lines 6–7, replace "groups … that" with "the automorphism scheme of a group of multiplicative type". On line 9, replace "the" with "an". On line 2 of paragraph 2, replace "Proposition" with "Corollary". On line 4 of paragraph 2 replace "containing Z then" with “then $Z \subseteq T$ and”, and on line 5 of paragraph 2 replace “$Z_{\overline{k}}$ were not maximal” with “the image of $Z_{\overline{k}}$ in G' is not the maximal central torus”

p. 92: On line 10, replace “(take N to be the final nontrivial term of the derived series of H)” with “(take N to be the smallest nontrivial term in the derived series of a smooth connected nontrivial proper normal k-subgroup of H)”.

p. 94: In the displayed expression on the third line, remove the final comma. At the start of the next line, insert “(to verify the first inclusion, work modulo G_J and use that C_N commutes with both C and G_{I-J}),”

On the second and third lines of Lemma 3.2.1, replace F with k (three times, including replacing $f_{\overline{F}}$ with $f_{\overline{k}}$). Likewise, replace F with k on each of lines -7, -8, -12, -15, -16, -17, -18 (once per line).

p. 95: In the displayed expression on line 9, replace “$=$” with “$\mapsto$”.

p. 99: On lines 3–4 of Proposition 3.2.10, replace “collections $\{T_i\}$ for a maximal k-torus” with “choices $\{T_i\}$ of maximal k-tori”.

p. 100; On line 19, delete the word “maximal”.

p. 102: Delete the paragraph appearing just before Proposition 3.3.5, and add the following to the statement of Proposition 3.3.5: *If H'' normalizes H', then any S-weight of* $\mathrm{Lie}(H'H'')$ *admits a positive rational multiple that is an S-weight of* $\mathrm{Lie}(H')$ *or* $\mathrm{Lie}(H'')$.

Then insert the following immediately below Proposition 3.3.5:

Proof. The first two assertions of the proposition are obtained by repeated application of Lemma 3.3.4. To prove the last assertion, assume that H'' normalizes H' and consider the S-equivariant surjective homomorphism $H' \rtimes H'' \to H'H''$ defined by multiplication. By Proposition B.4.5, any S-weight a that occurs on $\mathrm{Lie}(H'H'')$ is a positive integral multiple of an S-weight that occurs on $\mathrm{Lie}(H' \rtimes H'') = \mathrm{Lie}(H') + \mathrm{Lie}(H'')$. Thus, some positive rational multiple of a is an S-weight of either $\mathrm{Lie}(H')$ or $\mathrm{Lie}(H'')$. ☐

p. 108: On line 14, replace “Corollary B.2.6” with “Lemma B.1.10”.

p. 116: From the last line on this page delete “*, and moreover* $u', u'' \neq 1$”

p. 117: On line -4, replace $R_{\overline{k}}$ with R.

p. 118: On line 5, replace “normalizing” with “compatible with”.

p. 120: On lines 8–9, replace “Corollary 3.3.13(2) is precisely (DR2),” with “Axiom (DR2) (in which it should be assumed that b is not a positive multiple of $-a$) is an immediate consequence of Corollary 3.3.13(2),”

p. 122: On line -16, insert an extra half-space in front of “k'-split”.

p. 125: On lines 1 and 2 of paragraph 2, replace “Choose . . . and” with “We” and after “of” insert “the given positive system of roots”.

pp. 125–126: Replace lines -8 of page 125 through line 2 of page 126 with "Applying the same uniqueness consideration to the product map $i_G \times i_G$, we deduce from the fact that j_0 and i_G are k-homomorphisms that f is a k-rational homomorphism. Hence, f uniquely extends to a k-homomorphism $F : G_0 \to G$ (as we see via Galois descent by using the covering of $(G_0)_{k_s}$ by $\Omega(k_s)$-translates of Ω_{k_s}). This must be a k-subgroup inclusion since $i_G \circ F = j_0$ is a k-subgroup inclusion. By construction $F(G_0)$ contains T and every E_a $(a \in \Delta)$. It follows that F identifies G_0 with a connected reductive k-subgroup L of G having T as a maximal k-torus and Δ as a basis for a positive system of roots (with associated root groups E_a for $a \in \Delta$). Since $\pi \circ F_{k'} : (G_0)_{k'} \to G'$ induces $(T_0)_{k'} \simeq T'$ respecting the common basis Δ for the root systems, $\pi : L_{k'} \to G'$ induces an isomorphism between root data and hence is an isomorphism. Thus, L is the desired Levi k-subgroup of G." Please keep the "$\square$" at the end of this paragraph.

p. 132: In the displayed expression in the middle of the page, replace $\gamma^*(\lambda')$ with $\gamma(\lambda')$.

p. 138: On lines 5, 7,8,9, 10, 11, and 17, replace H with $\overline{P}$ (twice on lines 9 and 11, including replacing $N_H(\overline{T})$ with $N_{\overline{P}}(\overline{T})$ on line 11).

p. 139: On line -3, at the end of the line replace Φ with Φ'.

p. 143: On line -10, near the end of the line replace "in" with "of".

p. 146: Please insert the following after the text presently on this page.

One consequence of Proposition 3.5.10 is that in a smooth connected affine group G over a field k, any pseudo-parabolic k-subgroup is uniquely determined by its k-unipotent radical. The following result (which is also a significant refinement of Proposition 3.5.1(3)) shows something much stronger: containment relations among pseudo-parabolic k-subgroups are equivalent to containment relations (in the opposite direction) among their k-unipotent radicals. (For connected reductive G this is a well-known fact.)

Proposition 3.5.14 *Let G be a smooth connected affine group over a field k, and let P and Q be pseudo-parabolic k-subgroups of G. The following conditions are equivalent: (i) $P \subseteq Q$, (ii) $\mathscr{R}_{u,k}(Q) \subseteq \mathscr{R}_{u,k}(P)$, (iii) $\mathscr{R}_{us,k}(Q) \subseteq \mathscr{R}_{us,k}(P)$.*

Proof. We first prove the equivalence of (i) and (ii). We may and do assume $k = k_s$, and (by Proposition 2.2.10) that G is pseudo-reductive over k, as well as non-commutative (so the root system associated to G is non-empty). Assume (i), and let T be a maximal k-torus in P (so also $T \subseteq Q$). Let Ψ_P denote the set of non-divisible roots $a \in \Phi(G,T)$ such that $-a$ is not a T-weight on Lie(P), and define Ψ_Q similarly. The set $\Phi(P,T)$ of nontrivial T-weights on Lie(P) is contained in $\Phi(Q,T)$, so $\Psi_Q \subseteq \Psi_P$. This latter containment implies (ii), due to Proposition 3.5.1(2).

Conversely, assume (ii) holds. By Proposition 3.5.12(1), there is a maximal k-torus T contained in both P and Q. By Proposition 3.5.1(3), to prove (i) it is equivalent to prove that $\Phi(P,T) \subseteq \Phi(Q,T)$. Suppose to the contrary that some $a \in \Phi(P,T)$ does not lie in $\Phi(Q,T)$. Since $\Phi(P,T)$ is a parabolic subset of $\Phi(G,T)$, we may arrange that a is non-divisible in

$\Phi(G,T)$. It follows from Proposition 3.5.1(2) that $-a$ is a T-weight on $\mathrm{Lie}(\mathscr{R}_{u,k}(Q))$, so $-a$ is a T-weight on $\mathrm{Lie}(\mathscr{R}_{u,k}(P))$ due to (ii). Once again applying Proposition 3.5.1(2), we conclude that $a \notin \Phi(P,T)$, a contradiction. We have proved that (i) and (ii) are equivalent.

Next, assume (ii). Consider the natural k-homomorphism $\mathscr{R}_{us,k}(Q) \to \mathscr{R}_{u,k}(P)/\mathscr{R}_{us,k}(P)$. By Corollary B.3.5 the target of this map is k-wound, so the k-homomorphism is trivial. Thus, (iii) holds. Conversely, if we assume (iii) then Corollary 2.2.5 implies that $\mathscr{R}_{u,k}(Q) = \mathscr{R}_{us,k}(Q)\mathscr{R}_{u,k}(G) \subseteq \mathscr{R}_{us,k}(P)\mathscr{R}_{u,k}(G) = \mathscr{R}_{u,k}(P)$. $\square$

p. 155: On the fourth line below diagram (4.2.1), replace “object in” with “object (up to isomorphism) in”.

p. 158: On line 13, insert comma before “and” (i.e., replace “$G_{\overline{k}}$ and” with “$G_{\overline{k}}$, and”).

p. 160: On line -14, remove “therefore”.

p. 164: On line 12, insert “(in view of Proposition A.5.9)” at the start of the line and replace “Weil restriction” with “Weil restriction, as any non-empty open subset of a smooth connected scheme over a field is Zariski-dense.” On line -14, replace “rational maps” with “rational maps (that are even morphisms)”.

p. 170: On line -1, replace $Z_{G'}(T')$ with T'.

p. 171: On line 2, replace $Z_{G'}(T')$ with T'. In the displayed expression on line 9, replace $Z_{G'}(T')$ with T'. In the displayed expression on line 11, replace C_1 with $\mathscr{C}$ and replace $Z_{G'}(S')$ with T'.

p. 186: On line 1, just before “has” insert “(see the snake lemma argument in Example 1.3.2, taking $m = p^n$)”

p. 191: On line 4, replace “$G_K^{\mathrm{ss}} =$” with “$G_K^{\mathrm{ss}} :=$”. On line 5, replace G_k^{ss} with G_K^{ss}.

p. 196: Remove “of interest” on the first line of the first full paragraph (beginning with “Returning”).

p. 198: On line 6, replace $-\sum_{j\geqslant 1} c_j x^j$ with $-x^n - \sum_{j=1}^{n-1} c_j x^j$.

p. 213: On line 2, replace “$\mathrm{C}_n (n \geqslant 1)$” with “$\mathrm{B}_n, \mathrm{C}_n (n \geqslant 1)$, or F_4”.

p. 214: In Remark 6.3.7 there is too much empty space between words. Can this be improved?

p. 218: On lines -10 through -9, replace “*has preimage* … *are long.*” with “*goes over to the intersection in* $(T'/Z_{G'})[p]$ *of the kernels of the long roots in* Φ *when these roots are viewed as characters on* $T'/Z_{G'}$*, and it is not annihilated by any short root.*” On line -7 replace (2) with (3), and on line -6 replace $T[p]$ with $(T/Z_G)[p]$.

p. 219: In the displayed expression on line 15, at the right side the last subscript i should be j (i.e., $b_j^{\vee}$ rather than $b_i^{\vee}$). Replace lines -13 up through “$a(\mu_<) = 1$.” on line -3 with:

The assertion in (2) is a well-known property of root systems that are reduced and irreducible (in view of our hypotheses on the edge multiplicities in the Dynkin diagram).

To prove (3), by construction it is clear that $\mu_<$ contains the intersection of the kernels in $(T'/Z_{G'})[p]$ for the long roots. To prove that this containment is an equality, we just have to show that $\mu_<$ is killed by all long roots. But we saw above that every $a \in \Phi_>$ has $\Delta_<$-coefficients in $p\mathbf{Z}$, so $a(\mu_<) = 1$ by definition of $\mu_<$.

Make a new paragraph beginning with "It remains" on line -3.

p. 220: On line 2, replace $T[p]$ with $(T'/Z_{G'})[p]$ and on line 4 replace $\prod_{c\in(g(\Delta))_>}\mu_p$ by $\prod_{c\in g(\Delta_<)}\mu_p$.

In the statement of Lemma 7.1.2, replace the final sentence with: "*Every nonzero G-submodule of $\mathfrak{g}$ distinct from $\mathfrak{z} := \mathrm{Lie}(Z_G)$ contains $\mathfrak{n}$. In particular, except for types* B_n $(n \geqslant 2)$ *and* C_{2m} $(m \geqslant 2)$*, $\mathfrak{n}$ is the unique irreducible G-submodule of $\mathfrak{g}$.*" On lines -13 through -12, replace "the center ... in such cases." with "and type C with even rank at least 4 (so $\mathrm{char}(k) = 2$), the center $Z_G = \mu_2$ is contained in $\prod_{c\in\Delta_<} c^\vee(\mathrm{GL}_1)$, so $\mathfrak{z} \subseteq \mathfrak{t}_< \subseteq \mathfrak{n}$ in these cases."

Delete "Since $\mathfrak{sl}_2$ is generated as" from line -1 of page 220, also delete the first line on page 221 and insert "We conclude that" at the beginning of the second line on page 221.

p. 221: On line 10, replace "system and" with "system for $(Z_G(S), T)$ and", and before "coroot" insert "corresponding". Also replace "give that the root spaces $\mathfrak{g}_a$ and $\mathfrak{g}_b$ have *non-vanishing* pairing under the Lie bracket, valued in the root space $\mathfrak{g}_{a+b}$ for the short root $a + b$.", occurring on lines 12–14, with "$[\mathfrak{g}_a, \mathfrak{g}_b] = \mathfrak{g}_{a+b}$, and $a + b$ is a short root."

pp. 221–222: Delete "It therefore" from the last line on p. 221 and also delete the first 16 lines on page 222, and insert the following in their place together with an empty box at the end (after the period).

Fix a k-isomorphism $x_b : \mathbf{G}_\mathrm{a} \simeq U_b$ for all $b \in \Phi^+$. Letting (c, c') denote the set of roots in Φ of the form $ic + jc'$ with $i, j \geqslant 1$, equipped with a fixed ordering, there is an identity

$$x_c(s)x_{c'}(t)x_c(s)^{-1} = \prod_{ic+jc'\in(c,c')} x_{ic+jc'}(r_{i,j}s^it^j) \cdot x_{c'}(t)$$

with $r_{i,j} \in k$. Passing to the t-derivative at $t = 0$ reduces us to the problem of proving that $r_{i,1} = 0$ if $ic + c'$ is a long root (with c' short). Hence, we may and do assume (c, c') is non-empty, so for $T_{c,c'} := (\ker c \cap \ker c')^0_{\mathrm{red}}$ the rank-2 connected semisimple k-subgroup $G_{c,c'} := \mathscr{D}(Z_G(T_{c,c'}))$ is simple (i.e., not of type $\mathrm{A}_1 \times \mathrm{A}_1$). The k-torus $S := T \cap G_{c,c'}$ in $G_{c,c'}$ is maximal, and the identification $\mathrm{X}(T)_\mathbf{Q} = \mathrm{X}(S)_\mathbf{Q} \oplus \mathrm{X}(T_{c,c'})_\mathbf{Q}$ is equivariant with respect to the inclusion $W(G_{c,c'}, S) \to W(G, T)$ when using the trivial $W(G_{c,c'}, S)$-action on $\mathrm{X}(T_{c,c'})_\mathbf{Q}$. Thus, since the root lengths in an irreducible root system can be defined using any Weyl-invariant inner product, the injection of $\Phi_{c,c'} = \Phi(G_{c,c'}, S)$ into $\Phi(G, T)$ induced by the inclusion $\mathrm{X}(S)_\mathbf{Q} \hookrightarrow \mathrm{X}(T)_\mathbf{Q}$ preserves the distinction between long and short roots (with c' short). Our problem is therefore intrinsic to the central isogeny class of $G_{c,c'}$, and we may assume that the irreducible rank-2 root system $\Phi_{c,c'}$ has two root lengths (as it contains $c'|_S$). Hence, it is of type B_2 or G_2, with the latter occurring if and only if $p = 3$. The

Chevalley commutation relations in [Hum2, §33.4(b), §33.5(b)] imply that when c' is short and $ic + c'$ is long, $r_{i,1} = 0$ for type B_2 with $p = 2$ and for type G_2 with $p = 3$.

On lines -18 through -16 (on p. 222), replace "non-central … G-action" with "minimal non-central normal k-subgroup scheme of G with vanishing Frobenius."

p. 226 On lines 2–4 of Remark 7.1.6, replace "construction related … for $n \geq 2$:" with "construction. Let (V, q) be a quadratic space of odd dimension $2n + 1$, $n \geqslant 2$, that is non-degenerate (in the sense of [EKM, 7.A]):"

p. 227: On line -17, replace (2) with (3). On line 19, replace "Proposition A.8.10(2) applied over $\overline{k}$" with "Proposition A.8.12".

p. 229: On line -16, please replace "k-split" with "pseudo-split".

p. 230: On line 13, replace "function field" with "function field of characteristic 2". On line 15, after "Witt index $n-1$." insert "(See [EKM, 7.A, 8.A] for the notions of non-degeneracy and Witt index of finite-dimensional quadratic spaces over any field F. In the non-degenerate case, the Witt index equals the F-rank of the associated special orthogonal group.)"

p. 239: The 3rd and 4th paragraphs on this page should be made into a single paragraph, and the entire text "Upon choosing … proof of Proposition A.7.8." on lines -12 through -8 should be replaced with "This holds by Lemma A.7.7."

p. 243: On line 2 of the second paragraph, remove "the non-commutative aspect of" and replace "rests on" with "can be given in terms of".

p. 244: On line 6 of the second paragraph, replace "with $n \geqslant 2$," with "(with $n \geqslant 2$) or".

p. 245: On line -9, remove "preceding", replace "result for" with "of", and just before "implies that" insert "proved above".

p. 248: At the end of line -11, insert the sentence "By our hypotheses, $k' = k^{1/p}$."

p. 249: On line 15, remove the extra space before the second right parenthesis (i.e., replace "7.2.7(2))" with "7.2.7(2))"). On lines 4 through 10, replace the text "But the very … over k." with the following text:

This is a factorization of the very special isogeny as a composition of surjections with $\phi_{k'}$ not an isomorphism, so by Lemma 7.1.2 either π is an isomorphism (i.e., $\Pi_{k'} = q$) or else $\mathrm{char}(k) = 2$ with G' of type B or C and $\ker \pi = Z_{G'} = \mu_2$. Hence, the k-subgroup $\ker \Pi \subset \mathscr{G}$ is a k-descent of either $\ker q$ or $q^{-1}(Z_{G'})$ respectively, so to complete the proof that ϕ is an isomorphism it suffices to prove that $q^{-1}(Z_{G'})$ does not descend to a k-subgroup $\mathscr{H} \subset \mathscr{G}$ (as this implies the same for $\ker q$). Necessarily $\mathscr{H} \subseteq \mathscr{G} \cap \mathrm{R}_{k'/k}(Z_{G'}) = Z_{\mathscr{G}}$ (Corollary 7.2.5(1)) and $\ker q \subseteq \mathscr{H}_{k'}$. The explicit description of $Z_{\mathscr{G}}$ inside of $\mathrm{R}_{k'/k}(G')$ in Corollary 7.2.5(2) shows that $\dim \mathscr{H} \leqslant \dim Z_{\mathscr{G}} \leqslant [k' : k]$, but Theorem 7.2.3 gives that

$$\dim \ker q = \dim \mathscr{G} - \dim G' = (\#\Delta_< + \#\Phi_<)([k' : k] - 1) \geqslant 3([k' : k] - 1) > [k' : k],$$

so no such $\mathscr{H}$ can exist.

p. 251: On line 3 replace ", so" with "then".

p. 266: In item (d), remove the word "suitable", and inside of the parentheses insert the following sentence at the end: "Proposition 7.3.1 gives necessary and sufficient conditions on $\overline{G}$ so that its preimage under $\mathrm{R}_{K'/K}(\pi')$ is K-smooth."

p. 279: On line 9, switch "C_n" and "B_n".

p. 283: On line 6, replace "to whether" with "as".

p. 284: On line 3, immediately after "extension" insert "of an arbitrary field k". On line 7, insert "define" before "the smooth", and on line 9 replace "is" with "to be". On line 14, replace "in the SL_2-case or the PGL_2-case" with "inside SL_2 or PGL_2". On line 16, replace "result." with "result concerning the data k $(= k_s)$, G, K, T, $H = i_G(G)$, V, and $G_0 := G_K^{\mathrm{ss}}$ $(= \mathrm{SL}_2$ or $\mathrm{PGL}_2)$ in the discussion preceding Lemma 9.1.1."

On the first line of the proof of Proposition 9.1.5, remove the first sentence "Without loss of generality we can assume $k = k_s$." On the third line of the proof, replace ", so we just apply" with " and $k = k_s$ by hypothesis, so it suffices to apply".

p. 285: Replace line 5 with "so we see using Lemma 3.1.5 that $\mathscr{H}$ is Zariski-dense".

p. 286: On line 15, "$U_V^{\pm} \subseteq U^{\pm}$" should be "$U_V^{\pm} \subseteq \mathrm{R}_{K/k}(U^{\pm})$". At the end of line 17, $\underline{V}$ should be replaced with $U_V^{\pm}$; the same correction should be made in the middle of line -8.

p. 291: On line 4, near the end of the displayed expression, replace "$T')$" with "$T'_{K_s})$. On line -9, move "over k" to line -8 just before the comma.

p. 293: On line 2 of paragraph 3, remove the period after "subgroup".

p. 295: On line -9, remove "pullback property".

p. 296: Please replace the first paragraph of the proof of Proposition 9.2.4 with the following:

If the rank of G is 1 then the surjectivity of ξ_G follows from Remark 9.1.3, as $[K : k] = 2$. We assume now that the rank of G is greater than 1 and identify $\Phi(G', T')$ with the set of non-multipliable roots in $\Phi(G, T)$. By Lemma 9.2.1, $\xi_G(Z_G(T)) = \mathrm{R}_{K/k}(T')$ where $T' := T_K$ is a maximal K-torus of G'. We may and do view T as a maximal k-torus of $\mathrm{R}_{K/k}(G')$ via ξ_G. For any short root $b \in \Phi(G', T')\,(\subset \Phi(G, T))$ there exists $c \in \Phi(G', T')$ such that $\langle b, c^\vee \rangle = -1$, so the equality $\xi_G(Z_G(T)) = \mathrm{R}_{K/k}(T')$ implies that the containment of $\xi_G(U_b)$ in the b-root group of $\mathrm{R}_{K/k}(G')$ is an equality. Any long root $a \in \Phi(G', T')$ is divisible when viewed in $\Phi(G, T)$, so the rank-1 pseudo-simple derived group of the centralizer of $T_a := (\ker a)^0_{\mathrm{red}}$ in G has a non-reduced root system and hence the settled rank-1 case implies that $\xi_G(U_a)$ is the full a-root group of $\mathrm{R}_{K/k}(G')$ (with respect to T). We have shown that $\xi_G(G)$ contains all root groups of $\mathrm{R}_{K/k}(G')$ with respect to T. As these root groups generate $\mathrm{R}_{K/k}(G')$, the surjectivity of $\xi_G : G \to \mathrm{R}_{K/k}(G')$ follows when G has rank greater than 1. This settles (1).

p. 299: On line 1, replace ρ with λ and replace λ with ρ.

p. 306: In the statement of Lemma 9.3.4, on the third line, replace "algebraic" with "regular".

p. 310: In the statement of Lemma 9.3.7, on the second line of the second paragraph, replace "an algebraic" with "a regular".

p. 311: Replace lines 3–8 with the following text:

square if and only if the corresponding partial derivative vanishes. To express this criterion in a convenient form for our problem on $\mathrm{R}_{K/k}(\mathrm{Sym}_n)$, we shall use the decomposition $\mathrm{R}_{K/k}(\mathrm{Sym}_n) = \underline{V} \times \underline{V'}$ with the affine spaces associated to the respective k-vector spaces $V = \mathrm{Sym}_n(k)$ and $V' = \mathrm{Sym}_n(k)\alpha$ of "real" and "imaginary" symmetric matrices over K.

Let Ω denote the dense open domain of the dominant rational endomorphism $f : m_0 \mapsto -m_0^{-1}$ of $\mathrm{R}_{K/k}(\mathrm{Sym}_n)$, so for any k-algebra R and $m_0 \in \Omega(R)$ the derivative $\mathrm{d}f(m_0) \in \mathrm{End}_R(\mathrm{Sym}_n(K \otimes_k R))$ has $\mathrm{Sym}_n(K \otimes_k R)$-valued components

$$(L_V(m_0), L_{V'}(m_0)) = \mathrm{Hom}_R(V_R, \mathrm{Sym}_n(K \otimes_k R)) \oplus \mathrm{Hom}_R(V'_R, \mathrm{Sym}_n(K \otimes_k R)).$$

Concretely, $L_V(m_0)$ is the derivative in the "real symmetric" direction and $L_{V'}(m_0)$ is the derivative in the "imaginary symmetric" direction. By identifying V' with V via multiplication by α, we view L_V and $L_{V'}$ as k-morphisms from Ω to $\underline{\mathrm{Hom}}_k(V, \mathrm{Sym}_n(K))$, where we write $\underline{\mathrm{Hom}}_k(W, W')$ to denote the affine space over k associated to $\mathrm{Hom}_k(W, W')$ for finite-dimensional k-vector spaces W and W'.

Consider the natural K-linear decomposition $\mathrm{Sym}_n(K) = K^n \oplus \mathrm{Sym}_n^0(K)$, where K^n is the space of diagonal matrices and $\mathrm{Sym}_n^0(K)$ is the space of off-diagonal symmetric matrices (i.e., those with vanishing diagonal). Composition with the resulting projection $\mathrm{R}_{K/k}(\mathrm{Sym}_n) \to \mathrm{R}_{K/k}(\mathbf{G}_\mathrm{a})^n$ defines "diagonal components" of L_V and $L_{V'}$ as k-morphisms $\Omega \rightrightarrows \underline{\mathrm{Hom}}_k(V, K^n)$. Each of these has a "real part" valued $\underline{\mathrm{Hom}}_k(V, k^n)$. Our problem is exactly to show that the real part of the diagonal component of $L_{V'}$ vanishes (i.e., the derivative in the imaginary symmetric directions has "pure imaginary" diagonal component) and the real part of the diagonal component of L_V takes values in $\underline{\mathrm{Hom}}_k(V/V^0, k^n)$ (i.e., the derivative in the real symmetric off-diagonal directions has "pure imaginary" diagonal component).

On line 10, replace "total differential" with "the derivative $L_{V'}(m_0)$". Replace line 13 with "so the linear map $L_{V'}(m_0)$ is identified with the". Replace line -9 with "the restriction of the derivative $L_V(m_0)$ to the off-diagonal subspace Sym_n^0 of Sym_n is", and on line -8 replace "the space" with "the space Sym_n^0".

p. 314: In Theorem 9.3.10(4), replace $(\Phi_n)_>$ in the subscript on the second line with $\Phi_>^+ \cup \Phi_>^-$ and after "where" insert "$\Phi_>^- := -\Phi_>^+$ and". On the third line of this part (4), replace "$(\Phi_n^+)_>$ before $(\Phi_n^-)_>$" with "$\Phi_>^+$ before $\Phi_>^-$".

p. 317: On line 10, replace "maps" with "is mapped". On line 19, replace $(\Phi_n^+)_>$ in the subscript with $\Phi_>^+$. On line -10, replace $(\Phi_n^+)_>$ with $\Phi_>^+$.

On line 15, replace "of ξ_{G_α}" with "of ξ_{G_α} $(= \xi_\alpha)$".

p. 321: On line 3, replace $(1/yc)$ with $(yc)^{-1}$, and on line 9 replace $1/yc$ with $(yc)^{-1}$.

pp. 322–323: Replace the last line of p. 322 and the top two lines of p. 323 with:

Theorem 9.3.5 that: (i) the k-group U^+ coincides with the k-group $\mathscr{U}$ in Proposition 9.3.3, and (ii) when the product variety $U^+\mathbf{w}\mathrm{R}_{K/k}(D)U^+$ is equipped with the strict birational group law arising from Theorem 9.3.5 then the natural k-variety map $\Omega := U^+\mathbf{w}Z_G(T)U^+ \to U^+\mathbf{w}\mathrm{R}_{K/k}(D)U^+$ (using the identity on the U^+-factors and the quotient map $Z_G(T) \twoheadrightarrow \mathrm{R}_{K/k}(D)$) is a map of strict birational group laws. Thus, this extends to a surjective k-group homomorphism $\pi : G \to G_{\alpha,n}$ through which ξ_G factors (as such properties can be checked using strict birational group laws on dense open subvarieties). Since π restricts to a map between $\mathbf{w}$-translated open Bruhat cells respecting the decompositions as product varieties, we see by inspection that the intersection of $\ker\pi$ with the open Bruhat cell $\mathbf{w}\Omega$ is contained in $Z_G(T)$. But $Z_G(T)$ is closed in G, so the topological closure of the open subscheme $\ker\pi \cap \mathbf{w}\Omega$ of $\ker\pi$ is contained in $Z_G(T)$. It follows that $(\ker\pi)^0 \subseteq Z_G(T)$ scheme-theoretically. Since G is generated by its maximal k-tori (Proposition A.2.11), all of which are $G(k)$-conjugate to T (Proposition A.2.10), normality of $(\ker\pi)^0$ in G implies centrality. Hence, $\ker\pi$ is central in G (Lemma 5.3.2), so G is a central extension of $G_{\alpha,n}$ by $\ker\pi$. But $\ker\pi$ is contained in the unipotent k-group scheme $\ker\xi_G$, so this central extension satisfies the hypotheses of Proposition 9.4.2. It follows that G is the split central extension $G_{\alpha,n} \times \ker\pi$, so smoothness and connectedness of G force $\ker\pi = 1$. Thus, π is an isomorphism!

p. 324: The last sentence of the proof of Theorem 9.4.3 should be replaced with:

But $\mathrm{H}^1(K_s/K, \mathrm{Sp}_{2n}) = 1$ since up to K-isomorphism there is a unique symplectic space of dimension $2n$ over K.

p. 333: The last paragraph before the statement of Theorem 9.4.10 should be replaced with:

We can now classify all absolutely pseudo-simple groups with a non-rduced root system, provided that the imperfect ground field k of characteristic 2 is “almost perfect”, i.e., $[k : k^2] = 2$.

pp. 345-346: Replace the text from line 11 of page 345 beginning at “The rank …” up through line 5 of page 346 with the following:

More generally, for an arbitrary field k we claim that this map of Isom-sets is bijective for any finite purely inseparable extension of fields k'/k and any pseudo-reductive k'-groups G' and H'. By Proposition A.5.11 (with $B = k$, $B' = k'$, and $X' = G'$), the natural k'-homomorphism $q_{G'} : \mathrm{R}_{k'/k}(G')_{k'} \to G'$ is smooth and surjective with connected unipotent kernel, so it is the maximal pseudo-reductive quotient over k'. The same applies to $q_{H'}$, so any k-isomorphism $f : \mathrm{R}_{k'/k}(G') \simeq \mathrm{R}_{k'/k}(H')$ induces a k'-isomorphism $f' : G' \simeq H'$ between the maximal pseudo-reductive quotients over k'. Functoriality of Weil restriction implies that $\mathrm{R}_{k'/k}(\varphi)' = \varphi$ for any k'-isomorphism $\varphi : G' \simeq H'$, and by Proposition 1.2.2 any k-isomorphism f is determined by f'. Thus, $f \mapsto f'$ is an inverse to $\varphi \mapsto \mathrm{R}_{k'/k}(\varphi)$.

p. 350: On line 8 make the following two changes: replace “such” with “the” and after “pseudo-reductivity” insert “of $\mathrm{R}_{k'/k}(G')/Z$”.

p. 356: On line -5, G' should be G_c.

p. 357: On the line immediately below the displayed diagram (10.2.1), replace "The top line of (10.2.1)" with "of $\mathrm{R}_{\mathsf{k}/k}(\widetilde{\mathscr{G}})$ which".

pp. 364–5: In Example 11.2.1, remove the text from the third paragraph ("To make such G for which ...") up to the end including the top two lines on page 365, and replace it with the following paragraph:

To make such G for which $G \neq \mathscr{D}(G)$, let k be an imperfect field of characteristic $p > 0$ and let k'/k be a degree-p purely inseparable extension. Let $G = \mathrm{R}_{k'/k}(\mathrm{PGL}_p)$. This is pseudo-reductive and has trivial (scheme-theoretic) center since PGL_p has trivial center and the formation of the scheme-theoretic center commutes with Weil restriction through a finite extension of fields (Proposition A.5.15(1)). To see that $G \neq \mathscr{D}(G)$ it suffices to exhibit a nontrivial commutative quotient of the group $G(\overline{k}) = \mathrm{PGL}_p(A)$, where $A = \overline{k} \otimes_k k' \simeq \overline{k}[x]/(x^p)$. Since the pth powers in A are the elements of $\overline{k}$, the determinant on GL_p induces a surjective homomorphism from $\mathrm{PGL}_p(A)$ onto $A^\times/\overline{k}^\times \neq 1$. (For a variant on this explanation for the non-perfectness of G, see Example A.7.9 applied to $\mathrm{SL}_p \to \mathrm{PGL}_p$.)

p. 376: In the middle of the page, the equation label (11.4.1) for the displayed expression should be moved down a half-line (so midway between the two lines containing the text of the displayed expression).

p. 393: On line 12, just before "The k-split property" insert the following: "(By [Bo2, I, 4.8], when U is a smooth affine k-group such an upper-triangular form is equivalent to all elements of $U(\overline{k})$ being unipotent. This latter condition is how unipotence for U is defined in [Bo2] and other classical references on linear algebraic groups, so for smooth affine k-groups the classical notion of unipotence is equivalent to Definition A.1.3.)"

On line -3, before "The *kernel*" insert "Let k be a ring."

p. 394: On line 3, at the beginning insert "Now assume k is a field." and before "is the unique" insert "in Definition A.1.8". On line 11, replace "G be a scheme" with "G be a group scheme", and on line 18 replace "is finite type" with "is of finite type" twice.

p. 395: On line 4, replace the *second* "of" with "on". Immediately after the end of the sentence preceding Example A.1.12, insert "An *isogeny* $f : G \to G'$ between locally finite type k-group schemes is a k-homomorphism that is flat and surjective with $H = \ker f$ a finite k-group. Note that the k-finiteness of H is equivalent to the finiteness of $H(\overline{k})$, and by descent theory an isogeny is necessarily a finite morphism. Thus, the following example implies that a k-homomorphism $G \to G'$ between smooth k-groups of finite type is an isogeny if and only if $G(\overline{k}) \to G'(\overline{k})$ is surjective with finite kernel (thereby recovering the classical notion of isogeny for smooth connected groups)."

Please link the above definition of *isogeny* to a new index entry called "isogeny".

At the end of the final line of page 395, insert the sentence "In general, G/H is affine when G is affine and H is normal in G: for smooth H and G this follows from the construction of

G/H in [Bo2, II, §5–§6], and the general case is deduced from the smooth case in the proof of [SGA3, $\mathrm{VI_B}$, 11.17]."

p. 398: On line -11, insert "an" before "affine". On line -4, replace "must be" with "is".

p. 399: On lines 3–7, replace the text "The construction ... to construct" with "The existence of the directed union is proved in [SGA3, $\mathrm{VI_B}$, Lemme 11.8]. Next, we construct".

p. 400: In the statement of Proposition A.2.8, interchange G and G', and also S and S' everywhere. Replace its proof with the following:

Proof. That S' is a maximal torus of G' is [Bo2, Prop. 11.14], and that $f(Z_G(S)) = Z_{G'}(S')$ is [Bo2, Cor. 2, 11.14]. □

p. 401: In the proof of Proposition A.2.10, on line 11 of the proof, replace "n-torsion subgroups in a k-torus for n" with "k-subgroups $S[n]$ for a k-torus S as n varies through all positive integers".

In the statement of Proposition A.2.11, replace "an infinite" with "a" on the first line, replace "of $G_{\overline{k}}$." with "of $G_{\overline{k}}$, and $G(k)$ is Zariski-dense in G if k is infinite and G is perfect.", and remove the 4 lines at the bottom of the page (beneath the statement of Proposition A.2.11).

p. 402: On line 8 replace "A.2.8)" with "A.2.8 applied to the smooth connected preimage in G of a maximal k-torus of G/G_t)", on line 9 replace "unipotent" with "solvable", on line -15 replace "since" with "when", on line -13 replace "G_t" with "G_t when k is infinite" and remove "indeed", on line -12 replace "field" with "field when k is infinite", on line -11 insert "when k is infinite" just before "it suffices".

Replace lines -10 through -4 with: "to prove that the open subset $V := \mathscr{T} - Z$ in $\mathscr{T}$ is empty. Assume $V \neq \emptyset$, so V contains the unique generic point of $\mathscr{T}$, which the map $\pi : \mathscr{T} \to \mathrm{Tor}_G$ sends to the generic point of Tor_G. Thus, the constructible set $\pi(V)$ in Tor_G contains the generic point and so contains a dense open subset. But that must meet $\mathrm{Tor}_G(k)$ since k is infinite, which is a contradiction since V has empty fibers over $\mathrm{Tor}_G(k)$ (by definition of Z)."

At the very end of the proof of Proposition A.2.11, insert the following text and then place the "end of proof" white box at the end of this new material:

Now we address the case where k is finite. As a convenient notation, for any smooth connected k-group G we will denote by G_τ the Galois descent of the normal k_s-subgroup $(G_{k_s})_t$ in G_{k_s}, so G_τ is a smooth connected normal k-subgroup of G that contains all k-tori of G. We have to prove that G_τ is generated by the k-tori of G (i.e., $G_\tau = G_t$). We will treat the solvable case by an argument that works over any perfect field with nonzero characteristic, and then deduce the general case from the solvable case (using that any connected semisimple k-group contains a Borel k-subgroup, since k is finite).

So we now prove that if G is a smooth connected solvable group over a perfect field k of characteristic $p > 0$ then $G_\tau = G_t$. Writing $G = T \ltimes U$ for a maximal k-torus T in G

and $U := \mathscr{R}_{u,k}(G)$, we first consider the case where U is commutative and p-torsion. Let U' be the maximal smooth connected T-stable k-subgroup of U such that U'^T is trivial; see Theorem B.4.3 for the existence of U'. We will use the following lemma.

Lemma A.2.12 *The $U'(k)$-conjugates of T generate $T \ltimes U'$.*

Proof. We may replace U with U', and may assume $U' \neq 1$. By Theorem B.4.3 there is a T-equivariant isomorphism of U' onto the vector group $\underline{V}$ associated to a finite-dimensional k-linear representation V of T with $V^T = 0$. It is well-known that all k-linear representations of a k-torus are completely reducible over k. (A deduction from the case of split k-tori in Lemma A.8.8 is that if K/k is an extension of fields and $r : K \to k$ is a k-linear retraction then for any exact sequence $0 \to W' \to W \to W'' \to 0$ of k-linear representations of a k-group scheme G and any G_K-splitting $W''_K \to W_K$ of the associated K-linear sequence, a G-splitting of the original sequence is provided by $W'' \to W''_K \to W_K \to W$, using r at the final step.) By expressing V as a direct sum of T-irreducibles we may reduce to the case where V is irreducible over k.

Now choose $x \in V - \{0\} = U'(k) - \{1\}$, so x does not commute with the T-action (as $V^T = 0$) and hence $xTx^{-1} \neq T$. Thus, T and xTx^{-1} generate a smooth connected k-subgroup H in $T \ltimes U'$ that strictly contains T, so $H = T \ltimes (H \cap U')$ with $H \cap U'$ a nontrivial smooth connected k-group (as H is smooth and connected). Then $\mathrm{Lie}(H \cap U') \subseteq \mathrm{Lie}(U') = V$ is a nonzero T-stable subspace, so the irreducibility of the T-action on V implies $H \cap U' = U'$. $\square$

By Lemma A.2.12 we have $(T \ltimes U')_t = T \ltimes U'$. Thus, to settle the case where U is commutative and p-torsion it suffices to prove that the inclusion $T \ltimes U' \subseteq (T \ltimes U)_\tau$ is an equality. This follows from the normality of $T \ltimes U'$ in $T \ltimes U$ and the unipotence of $(T \ltimes U)/(T \ltimes U')$.

Consider now the general solvable case. To prove that the containment $G_t \subseteq G_\tau$ is an equality we induct on $\dim G$ (the case $\dim G = 0$ is clear) and may replace G with G_τ so that $G_\tau = G$. Let $U = \mathscr{R}_{u,k}(G)$, which we may assume is nontrivial. Let $C(U)$ be the last nontrivial term in the descending central series for U (so $C(U)$ is central in U and normal in G), and let $n \geqslant 0$ be maximal such that $V := p^n C(U) \neq 1$. Hence, V is normal in G, central in U, and p-torsion. Clearly $(G/V)_\tau = G/V$ since $G_\tau = G$, so by induction on dimension it follows that $G/V = (G/V)_t = G_tV/V$. Thus $G_tV = G$. From this we see that if V is a central subgroup of G then G_t is a normal subgroup, and as G/G_t is a unipotent group, it follows that $G_\tau = G_t$.

Let us now assume that V is not a central subgroup of G, so (by centrality of V in U) the natural action of the torus $T := G/U$ on V is nontrivial. Let V' be the maximal smooth connected k-subgroup of V such that V'^T is trivial (cf. Theorem B.4.3), so V' is nontrivial and normal in G. Let $\pi : G \to G/V'$ be the quotient map. By induction on dimension, $(G/V')_t = G/V'$, so to prove that $G_t = G$ it suffices to show that for any maximal k-torus S in G/V', $\pi^{-1}(S)_t = \pi^{-1}(S)$. Since V' is commutative and p-torsion and V'^T is trivial, we can apply Lemma A.2.12 to conclude that $\pi^{-1}(S)_t = \pi^{-1}(S)$, and thus $G_t = G$. This completes our treatment of the case of solvable G.

Turning to the general case, consider an arbitrary k-split torus S in G, and choose a cocharacter $\lambda \in \mathrm{X}_*(S)$ such that $Z_G(\lambda) = Z_G(S)$. (The construction of such a λ is explained in a self-contained manner near the end of the proof of Proposition 3.3.10, for part (b).) The multiplication map

$$U_G(\lambda) \times Z_G(S) \times U_G(-\lambda) \to G$$

is an open immersion (Proposition 2.1.8(3)), so G is generated by the k-subgroups $U_G(\pm\lambda)$ and the k-group $Z_G(S) = Z_G(\lambda)$ that normalizes each $U_G(\pm\lambda)$. Hence, the smooth connected k-subgroup H_S of G generated by S, $U_G(\lambda)$, and $U_G(-\lambda)$ is normal in G.

The k-subgroup of G generated by S and the image of the commutator map

$$c_\pm : S \times U_G(\pm\lambda) \to U_G(\pm\lambda)$$

is clearly equal to $(S \ltimes U_G(\pm\lambda))_\tau$. Thus, provided that $c_\pm$ has dense image, $(S \ltimes U_G(\pm\lambda))_\tau = S \ltimes U_G(\pm\lambda)$ and hence $S \ltimes U_G(\pm\lambda)$ is generated by k-tori (by the solvable case). To prove that the image of $c_\pm$ is dense, it suffices to show that $\mathrm{Lie}(U_G(\pm\lambda))_{k_s}$ is spanned by the images of the derivative of $c_\pm$ at k_s-points in $c_\pm^{-1}(1)$. The image of this derivative at $(\lambda(t), 1)$ is the image of the endomorphism $v \mapsto \mathrm{Ad}_G(\lambda(t))(v) - v$ of the underlying k_s-vector space of $\mathrm{Lie}(U_G(\pm\lambda))_{k_s} = (\mathfrak{g}_{k_s})_{\pm\lambda>0}$ (where $\mathfrak{g} = \mathrm{Lie}(G)$), so the desired spanning property holds.

We conclude that H_S is generated by k-tori. The k-subgroups H_S (with S varying) generate a smooth connected normal k-subgroup H of G that contains all k-split tori and is contained in G_t. The quotient G/H contains no nontrivial k-split k-tori (apply Proposition A.2.8 to the pullback of $G \to G/H$ along the inclusion into G/H of a maximal k-split k-torus), and hence likewise the same holds for its maximal semisimple quotient $(G/H)^{\mathrm{ss}}$. But $(G/H)^{\mathrm{ss}}$ contains a Borel k-subgroup (as k is finite) and hence contains a nontrivial split k-torus if $(G/H)^{\mathrm{ss}} \neq 1$. This forces $(G/H)^{\mathrm{ss}} = 1$, so G/H is solvable. By the settled solvable case, $(G/H)_\tau = (G/H)_t$. But $H \subseteq G_t$, so $(G/H)_t = G_t/H$ and $(G/H)_\tau = G_\tau/H$. Hence, $G_\tau = G_t$ as desired.

p. 404: On lines -7, -8, and -10, replace “radiciel” with “radicial”.

p. 408: On line 4, replace “only normal” with “only finite normal”.

p. 420: On line -6, replace “The definition” with “In the other direction, the definition of a pinning”, and on lines -3, -4, replace “and the action … $\mathrm{Aut}_k(U^+)$.” with “is an identification of $k^\times$-torsors and φ_a is unique up to precisely the natural faithful action on SL_2 by $\overline{D}(k) = k^\times$, where $\overline{D} = \mathrm{GL}_1$ is the diagonal torus in PGL_2. We do *not* claim to canonically determine φ_c for all $c \in \Phi(G,T)$ in terms of the choice of the maps φ_a for $a \in \Delta$, as there are $k^\times$-scaling ambiguities on the root groups U_c; we will return to this at the end of Remark A.4.14.”

p. 423: On line 14, replace “this hypothesis” with “this quasi-projectivity hypothesis”. On lines 16–17, replace the sentence “In particular … construction.” with “To be precise, the gluing is initially done (as a locally finite type B-scheme) using all open subschemes of all open affines in X'. To find finitely many open affines in X' whose Weil restrictions cover $\mathrm{R}_{B'/B}(X')$ (so $\mathrm{R}_{B'/B}(X')$ is finite type over B) we may assume $B \to B'$ has constant fiber-degree $n > 0$. Consider the quasi-compact *topological* n-fold power $P := X'^n$ (with the product topology). By quasi-projectivity of X' over B', any n-tuple in X' (allowing repetitions) is contained in an open affine U', so the corresponding open subsets U'^n cover P. By quasi-compactness of P, there are finitely many $\{U'_i\}$ such that the powers $U_i'^n$ cover P. We claim that the open affines $\mathrm{R}_{B'/B}(U'_i)$ in $\mathrm{R}_{B'/B}(X')$ are a covering. For any field k and map $t : \mathrm{Spec}(k) \to \mathrm{Spec}(B)$, the pullback $\mathrm{Spec}(k \otimes_B B')$ has at most n points, so any B'-map $\mathrm{Spec}(k \otimes_B B') \to X'$ has image consisting of at most n points. Thus, such a B'-map lands in some U'_i, so any point in $\mathrm{R}_{B'/B}(X')(k)$ over t lies in one of the sets $\mathrm{R}_{B'/B}(U'_i)(k)$. This proves that $\mathrm{R}_{B'/B}(X')$ is covered by the finitely many open affine $\mathrm{R}_{B'/B}(U'_i)$."

p. 425: In Example A.5.3, on lines 4–5 replace "the decomposition" with "a choice of k_s-isomorphism" and remove "canonically". On the line preceding Corollary A.5.4, replace "are" with "can be".

p. 426: On lines 1 and -18, replace "radiciel" with "radicial".

p. 427: On line 6, replace B with B'.

p. 428: On line 15, replace "$k[t]/(t^2)$)" with "$k[t]/(t^2)$ with non-zero k)".

p. 432: On the 10th line above Proposition A.5.8, replace "surjectivity claim" with "closed immersion claim under a surjectivity hypothesis". On the 8th line above Proposition A.5.8, replace $\mathrm{R}_{B'/B}(Z)$ with $\mathrm{R}_{B'/B}(Z_{B'})$ and replace "surjective)" with "faithfully flat)".

On the 7th and 8th lines above Proposition A.5.8, replace "Hence, we can work Zariski-locally on Y to reduce to the case when Y is affine and then even the case when" with "As U varies through the set of all affine open subschemes of Y, $\mathrm{R}_{B'/B}(U_{B'})$ varies through an affine open cover of $\mathrm{R}_{B'/B}(Y_{B'})$ (by construction of the latter). Hence, we may assume that Y is affine, and then even that".

On 2nd and 3rd lines above Proposition A.5.8, replace "The same calculation using $n = 0$ and any $d \geqslant 0$" with "The functor F represented by $\mathrm{Spec}(B)$ assigns a one-point set to every B-scheme, so the identification of $j_{\mathrm{Spec}(B)}$ with j_F".

p. 434: On lines 14–15, replace "regular sequence of parameters in the k-smooth B at y" with "regular system of parameters of the local ring B_y". On lines 15–16, replace "action ... on the parameters" with "map $(u_1, \dots, u_n) \mapsto (tu_1, \dots, tu_n)$".

On line -5, append the following to the end of the paragraph: "The following refinement in the smooth case is a generalization of [SGA3, XVII, App. III, Prop. 5.1(i)]."

p. 435: On line 5, replace "dimension" with "pure dimension". Replace the last full paragraph on the page (the one beginning with "The assertion on fiber dimensions") with the following:

For the assertion on fiber dimensions in (1), we may continue to assume that B is an algebraically closed field k as above. Now B' is just a finite k-algebra (not necessarily local). Let $y'_1, \dots, y'_m$ be the maximal ideals of B', $d_i = \dim X'_{y'_i} \geqslant 0$, and $e_i = \dim_k(B'_{y'_i}) \geqslant 1$. The hypothesis on fibral degrees implies that $\delta_i := (\sum_j e_j d_j) - d_i > 0$ for each i (as one sees by separately considering the cases that $d_i = 0$ or $d_i > 0$). Each fiber of $q_{y'_i}$ is already known to be smooth and non-empty; we will show that the tangent space at any of its k-points has dimension δ_i. The target of the smooth surjective map $q_{y'_i}$ is non-empty of pure dimension d_i, so we just have to check that the smooth y'_i-fiber of $X_{B'}$ has pure dimension $\sum e_j d_j$. More specifically, it is equivalent to show that all tangent spaces at k-points of the smooth X have dimension $\sum e_j d_j$. For $x \in X(k)$ and the corresponding section $x' \in X'(B') = \prod X'_{y'_j}(B'_{y'_j})$ with jth component $x'_j \in X'_{y'_j}(B'_{y'_j})$, the tangent space $\mathrm{Tan}_x(X)$ is identified with $\prod_j \mathrm{Tan}_{x'_j}(X'_j)$ (by consideration with dual numbers). But $\mathrm{Tan}_{x'_j}(X'_j)$ is a free $B'_{y'_j}$-module of rank d_j (by the smoothness of X'_j at x'_j), so its k-dimension is $e_j d_j$.

p. 436: Immediately after the proof of Proposition A.5.12, please insert the following new paragraph: "The preceding considerations with Weil restrictions lead to an interesting phenomenon, as follows. Let k'/k be a nontrivial purely inseparable finite extension of fields, and G' a pseudo-reductive k'-group (e.g., a connected reductive k'-group). By Proposition A.5.11 (applied to k'/k), the kernel U' of the natural map $q : \mathrm{R}_{k'/k}(G')_{k'} \to G'$ is the k'-unipotent radical. By [SGA3, XVII, App. III, Prop. 5.1(ii)], if a smooth closed k-subgroup H of $\mathrm{R}_{k'/k}(G')$ has the property that $U' \subseteq H_{k'}$ then $H = \mathrm{R}_{k'/k}(G')_{k'}$. In particular, $\mathrm{R}_{k'/k}(G')$ does not contain a proper parabolic k-subgroup, even if $k = k_s$ (even though it does contain a split maximal k-torus when $k = k_s$, as does any smooth connected affine k-group). This is a striking contrast with the case of connected reductive groups."

On line -2 of the proof of Proposition A.5.12, replace ", so even the quasi-projective hypothesis on H" with ", and likewise with each B_i in place of B, so the hypothesis that H is quasi-projective".

p. 441: On line 5 of the proof of Corollary A.5.16, replace "whose composition with $\iota_{k'}$ is the identity" with "for which $\iota_{k'}$ is a section".

p. 449: On line -11 replace $k[\epsilon'][\epsilon]$ with $k[\epsilon][\epsilon']$, on line -10 at the beginning replace $k[\epsilon]$ with $k[\epsilon']$, and on line -7 replace "a $k[\epsilon]$-point of G. As such, since $1 + \epsilon v$ is trivial modulo ϵ" with "an element of $\mathrm{Tan}_{e_{k[\epsilon]}}(G_{k[\epsilon]})$. By definition of ad_G," On line -3, at the end of the sentence insert "by computing the effect of both sides on the structure sheaf of $G_{k[\epsilon,\epsilon']}$ via right translation".

p. 451: On line 14, replace "5.1]" with "(i) in Cor. to Prop. 5.1]", and in the two-line displayed expression in the middle of the page (above Lemma A.7.7) please move the top line to the left and the bottom line to the right so that the symbol "$\subseteq$" in the top line is directly over the leftmost symbol "=" in the bottom line.

p. 457: On line 8, insert the following additional text at the end of this paragraph: "In the special case that G also has height $\leqslant 1$, N is normal in G if and only if $\mathfrak{n}$ is an ideal in $\mathfrak{g}$. This amounts to verifying that $\mathfrak{n}$ is an ideal in $\mathfrak{g}$ if and only if $\mathfrak{n}$ is stable under the adjoint action of G on $\mathfrak{g}$. Consider the closed k-subgroup scheme $H \subseteq G$ defined as the preimage under $\mathrm{Ad}_G : G \to \mathrm{GL}(\mathfrak{g})$ of the subgroup that stabilizes $\mathfrak{n}$. Since G has height $\leqslant 1$, By Proposition A.7.14 we have $H = G$ if and only if the inclusion of Lie algebras $\mathfrak{h} \hookrightarrow \mathfrak{g}$ is an equality. By Proposition A.7.5, $\mathfrak{h}$ is the Lie-theoretic normalizer of $\mathfrak{n}$ in $\mathfrak{g}$. Hence, $\mathfrak{h} = \mathfrak{g}$ if and only if $\mathfrak{n}$ is an ideal in $\mathfrak{g}$. This proves that for G of height $\leqslant 1$, normality of N in G is equivalent to $\mathfrak{n}$ being an ideal in $\mathfrak{g}$."

Insert the following new text immediately above section A.8, as part of Example A.7.16:

Suppose G has height $\leqslant 1$ and that N is normal in G, or equivalently (as we have seen above) that $\mathfrak{n}$ is an ideal in $\mathfrak{g}$. In such cases $\mathfrak{g}' := \mathfrak{g}/\mathfrak{n}$ is a p-Lie algebra, and we claim that the natural map $\mathfrak{g}' \to \mathrm{Lie}_p(G/N)$ of p-Lie algebras is an isomorphism. This follows from Proposition A.7.14 because these p-Lie algebras (compatibly) represent the same covariant functor. (More specifically, a k-homomorphism $G \to H$ to a finite k-group scheme of height $\leqslant 1$ factors through G/N if and only if the induced map of p-Lie algebras $\mathfrak{g} \to \mathfrak{h}$ factors through $\mathfrak{g}'$.) In particular, the equivalence $\mathscr{G} \rightsquigarrow \mathrm{Lie}_p(\mathscr{G})$ from the category of finite k-group schemes of height $\leqslant 1$ to the category of p-Lie algebras carries short exact sequences of such k-group schemes to short exact sequences of p-Lie algebras.

p. 459: On line 11 replace "$\mathscr{F}$ by $\mathbf{Z}/p\mathbf{Z}$" with "$\mathbf{Z}/p\mathbf{Z}$ by $\mathscr{F}$", on line 12 replace "a $\mathbf{Z}/p\mathbf{Z}$-torsor over $\mathscr{F}$" with "an $\mathscr{F}$-torsor over $\mathbf{Z}/p\mathbf{Z}$", and on line 16 replace "that is is" with "which is".

On line -5, append the following sentence to the end of the paragraph: "A more elementary example is $G = \{x^p + ay^p = 0\}$ with $a \in k - k^p$."

p. 460: In the displayed expression on line 13 replace $\Lambda \otimes \mathrm{GL}_1$ with $\mathbf{Hom}(\Lambda, \mathrm{GL}_1)$, and make the same change on line 14. On lines 14–15, replace "sheafification of ... topology" with "functor on k-algebras defined by $B \rightsquigarrow \mathrm{Hom}_{\mathbf{Z}}(\Lambda, B^{\times})$". In the displayed expression on line -14 replace $\Lambda \otimes \mathrm{GL}_1$ with $\mathbf{Hom}(\Lambda, \mathrm{GL}_1)$.

p. 463: On line 5, immediately before "then" insert "and $k \neq 0$". On line 25, replace "Lemma A.8.8 linear reductivity" with "Lemma A.8.8 and an averaging argument, linear reductivity".
"

p. 467: On line 12, replace "identity map" with "canonical inclusion".

p. 468: On line -8, insert the following additional sentence: "Over any field k of nonzero characteristic there are counterexamples with connected semisimple H of adjoint type A."

p. 470: In the statement of Proposition A.8.14(1), insert "smooth closed" immediately before "normal".

p. 471: On line 6, insert the sentence "Also, in (1) the smoothness of N cannot be removed when allowing non-smooth H, as one sees by considering the Frobenius isogeny $G \to G^{(p)}$ of a split connected semisimple group G in characteristic $p > 0$ containing a non-central μ_p; let $H = \mu_p$ act by conjugation on G and act trivially on $G^{(p)}$."

p. 473: On line 2, before "concerning" insert "(building on earlier work of M. Rosenlicht)", and on line 13 replace "many" by "some".

pp. 474–475: Replace the last paragraph on page 474 and the first paragraph on page 475 with the following single paragraph:

For any $\alpha \in G(k)$, $f(x+\alpha)$ and $f(x)$ have the same zero scheme (namely, G) inside of $\mathbf{G}_{\mathrm{a}}^n$. Thus, $f(x+\alpha) = c(\alpha)f(x)$ for a unique $c(\alpha) \in k^\times$. Consideration of a highest-degree monomial term appearing in f implies that $c = 1$. Pick $\beta \in k^n$, so $f(\beta+\alpha) - f(\beta) = 0$ for all $\alpha \in G(k)$. Thus $f(\beta + x) - f(\beta)$ vanishes on G, so $f(\beta + x) - f(\beta) = g(\beta)f(x)$ for a unique $g(\beta) \in k$. Consideration of a highest-degree monomial term in f forces $g(\beta) = 1$. Hence, $f(\beta + x) = f(\beta) + f(x)$ for all $\beta \in k^n$. This says that f is additive. ☐

p. 475: In the statement of Lemma B.1.7(2), at the end of the sentence replace "in k" with "in k^n". On line -2, replace $\mathbf{G}_{\mathrm{a}}^n$ with $\mathbf{G}_{\mathrm{a}}$.

p. 476: In the displayed equation on line 4, replace p_i with p in the superscript (so the full superscript reads as $s_i p^{m_i}$ rather than as $s_i p_i^{m_i}$).

p. 477: On line -8, replace"Since $k = \overline{k}$," with "By [Bo2, 10.6(2), 10.9],".

p. 478: Replace lines 1–6 beginning at "The p-torsion ..." with:
"In other words, there is a unique pullback diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{G}_{\mathrm{a}} \arrow[r] \arrow[d, equal] & U \arrow[r] \arrow[d, "f'"] & \mathbf{G}_{\mathrm{a}} \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & \mathbf{G}_{\mathrm{a}} \arrow[r] & W_2 \arrow[r] & \mathbf{G}_{\mathrm{a}} \arrow[r] & 0
\end{tikzcd}$$

and we claim that if U is p-torsion then $f = 0$ (so the top row is a split sequence). Clearly $f'(U) \subset W_2[p]$, but the maximal smooth k-subgroup of $W_2[p]$ is the kernel term $\mathbf{G}_{\mathrm{a}}$ along the bottom row. Hence, $f'(U)$ is killed by the quotient map along the bottom row, so $f = 0$."

p. 482: Just after line 2, insert the following three paragraphs:

Example B.2.8 Let k be a field and let G be a commutative pseudo-reductive k-group (e.g., $\mathrm{R}_{k'/k}(T')$ for a purely inseparable finite extension k'/k and a k'-torus T'). For the maximal k-torus T in G, consider the smooth connected commutative unipotent quotient $U = G/T$. We claim that U is k-wound. Since G_{k_s} is pseudo-reductive we may assume $k = k_s$, so T is k-split. By definition, we need to prove that any map of k-schemes $f : \mathbf{A}_k^1 \to U$ is constant.

Consider the pullback $G \times_U \mathbf{A}_k^1$. This is a T-torsor over $\mathbf{A}_k^1$, so it is trivial since T is split and $\mathrm{Pic}(\mathbf{A}_k^1) = 1$. A choice of splitting defines a k-scheme morphism $\widetilde{f} : \mathbf{A}_k^1 \to G$ over f, so it suffices to prove that $\widetilde{f}$ is constant. Using a translation, we may assume $\widetilde{f}(0) = 1$. We claim that for any smooth connected commutative k-group C and any k-scheme morphism $h : \mathbf{A}_k^1 \to C$ satisfying $h(0) = 1$, the smooth connected k-subgroup of C generated by the image of h is unipotent. Applying this to the commutative pseudo-reductive G would then force $\widetilde{f} = 1$, as desired.

To prove our claim concerning C we may assume that k is algebraically closed, so C is a direct product of a torus and a unipotent group. Using projections to factors, it suffices to treat the case $C = \mathrm{GL}_1$. In this case h is a nowhere-vanishing polynomial in one variable with value 1 at the origin, so $h = 1$.

In the statement of Proposition B.3.2, part (3), replace "of F" with "F" (i.e., remove the first "of" on that line).

p. 483: On line 7, remove "(whose first term is $\mathscr{D}(U)$)". On line -19, replace "$xF \to$" with "$xF \mapsto$". On line -8, replace "In particular" with "Moreover".

p. 485: On line 9 of section B.4, replace $t^p x^p$ with $(t^p - t)x^p$, and on lines 11–13 of section B.4 replace "compose the T-action ... is linear" with "define the additive non-linear automorphism $f : U \simeq U$ by $f(x, y) = (x, y + x^p)$ then the f-twisted T-action $(t, (x, y)) \mapsto f^{-1}(t.f(x, y))$ is the usual linear action $(t, (x, y)) \mapsto (tx, ty)$."

pp. 485-487: In the second line of the statement of Proposition B.4.1, replace "*a k-torus T*" with "*an affine k-group scheme T of finite type*". In the last line of the statement of this proposition, replace V with $\underline{V}$. Replace the proof of Proposition B.4.1 (from line -2 on page 485 through line -9 on page 487) with the following:

Proof. Let $\mathbf{Hom}(U, \mathbf{G}_{\mathrm{a}})$ be the covariant functor assigning to any k-algebra R the R-module $\mathrm{Hom}_R(U_R, \mathbf{G}_{\mathrm{a}})$ of R-group morphisms $\phi : U_R \to \mathbf{G}_{\mathrm{a}}$ (with R-module structure defined via the R-linear structure on the R-group $\mathbf{G}_{\mathrm{a}}$). There is a natural R-linear injection $\mathbf{Hom}(U, \mathbf{G}_{\mathrm{a}})(R) \hookrightarrow R[U_R] = R \otimes_k k[U]$ defined by $\phi \mapsto \phi^*(x)$ (where x is the standard coordinate on $\mathbf{G}_{\mathrm{a}}$), and its image is the R-submodule of "group-like" elements: those f satisfying $m_R^*(f) = f \otimes 1 + 1 \otimes f$ (where $m : U \times U \to U$ is the group law). This is an R-linear condition on f and is functorial in R, so by k-flatness the R-module of group-like elements over R is J_R where $J \subset k[U]$ is the k-subspace of group-like elements over k. In particular, the natural map $R \otimes_k \mathrm{Hom}(U, \mathbf{G}_{\mathrm{a}}) \to \mathrm{Hom}_R(U_R, \mathbf{G}_{\mathrm{a}})$ is an isomorphism.

The (left) T-action on U defines a left T-action on $\mathbf{Hom}(U, \mathbf{G}_{\mathrm{a}})$ (via $(t.\phi)(u) = \phi(t^{-1}.u)$) making the k-linear inclusion $\mathrm{Hom}(U, \mathbf{G}_{\mathrm{a}}) \hookrightarrow k[U]$ a T-equivariant map. Thus, $\mathrm{Hom}(U, \mathbf{G}_{\mathrm{a}})$ is the directed union of T-stable finite-dimensional k-subspaces, due to the same property for $k[U]$ (Proposition A.2.3). By Lemma B.1.10 there is a k-subgroup inclusion $j : U \hookrightarrow \mathbf{G}_{\mathrm{a}}^n$ for some $n \geqslant 1$. Let $W \subset \mathrm{Hom}(U, \mathbf{G}_{\mathrm{a}})$ be a T-stable finite-dimensional k-subspace containing $j^*(x_1), \ldots, j^*(x_n)$. The canonical map $U \to W^* = \mathrm{Spec}(\mathrm{Sym}(W))$ is a T-equivariant closed immersion that is a k-homomorphism (since W consists of group-like elements in $k[U]$ that generate $k[U]$ as a k-algebra). $\square$

pp. 487-491: Replace Proposition B.4.2 and its proof with the following:

Proposition B.4.2 *Let T, U, and V be as in Proposition* B.4.1*, with T a k-torus, and let $V = V_0 \times V'$ be the unique T-equivariant k-linear decomposition of V with $V_0 = V^T$ (so V' is the span of the isotypic k-subspaces for the nontrivial irreducible representations of T over k that occur in V). The product map*

$$\iota : (U \cap \underline{V}_0) \times (U \cap \underline{V}') \to U$$

is an isomorphism and there is a T-equivariant k-linear decomposition $V' = V'_1 \times V'_2$ of V' and a T-equivariant k-automorphism α of the additive k-group $\underline{V}$ such that

$$\alpha(U) = (\alpha(U) \cap \underline{V}_0) \times \underline{V}'_1.$$

In particular, if $V^T = 0$ then the k-group U is a vector group admitting a T-equivariant linear structure.

Proof. Clearly $\underline{V}_0 = Z_{\underline{V}}(T)$ as k-subgroups of $\underline{V}$, so $U_0 := U \cap \underline{V}_0$ is $Z_U(T)$. This is smooth since U is smooth. We will first prove that ι is an isomprphism, so $U \cap \underline{V}'$ is smooth.

Since the formation of V' clearly commutes with scalar extension on k, to establish that ι is an isomorphism we may assume k is algebraically closed. Choose $s \in T(k)$ such that for every weight χ of T in V', $\chi(s) \neq 1$. Consider the k-linear map $f : V \to V$ defined by $f(v) = s \cdot v - v$. It is obvious that f maps V onto V' with $\ker f = Z_V(s) = V_0$ and that the restriction of f to V' is a linear automorphism. The image $f(U)$ is a smooth k-subgroup of $\underline{V}'$, and it lies in U due to the T-stability of U inside of $\underline{V}$. By definition, $\underline{V}'$ has a T-equivariant composition series whose successive quotients are 1-dimensional vector groups with a nontrivial T-action. Hence, all T-stable k-subgroup schemes of $\underline{V}'$ are connected. In particular, $f(U)$ is connected.

Since $U_0 \cap f(U) = 0$ (as $\underline{V}_0 \cap \underline{V}' = 0$), under addition $U_0 \times f(U)$ is a k-subgroup of U. Thus, $f : U \to f(U)$ is a map onto a k-subgroup of U and the restriction of this map to $f(U)$ is therefore an endomorphsm $f(U) \to f(U)$ with trivial kernel. But $f(U)$ is smooth and connected, so this endomorphism is an automorphism. In other words, $f : U \twoheadrightarrow f(U)$ is a projector up to an automorphism of $f(U)$. Since $U \cap \ker f = U \cap \underline{V}_0 = U_0$, this shows that the k-subgroup inclusion $U_0 \times f(U) \hookrightarrow U$ is an isomorphism, so $f(U) = U \cap \underline{V}'$. This completes the proof that ι is an isomorphism.

Let $U' = U \cap \underline{V}'$ and define $V'_1 = \mathrm{Lie}(U')$. Then V'_1 is a T-stable k-linear subspace of V'. Complete reducibility of k-linear representations of T provides a T-stable k-linear complement V'_2 of V'_1 in V'. Using the decomposition $\underline{V}' = \underline{V}'_1 \times \underline{V}'_2$, the projection $U' \to \underline{V}'_1$ is an isomorphism on Lie algebras, so it is étale. By T-equivariance, the finite étale kernel is T-stable and therefore centralized by the connected T. But $Z_{\underline{V}'}(T) = 0$, so this kernel vanishes. In other words, $U' \to \underline{V}'_1$ is an isomorphism. It follows that the k-subgroup $U' \subseteq \underline{V}' = \underline{V}'_1 \times \underline{V}'_2$ is the graph of a T-equivariant k-homomorphism $g : \underline{V}'_1 \to \underline{V}'_2$. The T-equivariant k-automorphism α of $\underline{V}$ may be taken to be the automorphism that is the identity on $\underline{V}_0$ and is the inverse of the map $(v_1, v_2) \mapsto (v_1, g(v_1) + v_2)$ on $\underline{V}'_1 \times \underline{V}'_2$.

pp. 491–493: In the statement of Theorem B.4.3, append the following sentence at the end: "Moreover, U' is uniquely determined and is functorial in U."

Replace the proof of Theorem B.4.3 with the following:

Proof. By Propositions B.4.1 and B.4.2 we get the existence of U'. To prove the uniqueness and functoriality of U', we may assume $k = k_s$. Under the decomposition of U' into weight spaces relative to a T-equivariant linear structure on U', all T-weights must be nontrivial due to the definition of U_0. Hence, the canonical map $T \times U \to U$ defined by $(t, u) \mapsto t.u - u$ has image U'. This proves the uniqueness and functoriality of U'. $\square$

p. 493: Immediately preceding the statement of Corollary B.4.4, insert the following paragraph:

"If U in Theorem B.4.3 is k-wound, then it must coincide with U_0 and so have trivial T-action. This is a special case of the following general consequence of invariance of the wound property with respect to separable extension of the ground field (Proposition B.3.2):"

Replace the second paragraph of the proof of Corollary B.4.4 with the following paragraph:

"By Proposition B.3.2, we may extend scalars to k_s, so T is k-split. Assume that the T-action on $\mathrm{Lie}(U)$ is nontrivial, so the action on $\mathrm{Lie}(U)$ by some GL_1 in T is nontrivial. We may replace T by this GL_1, so there is an action $\lambda : \mathrm{GL}_1 \times U \to U$ that induces a nontrivial action on $\mathrm{Lie}(U)$. Composing the action with inversion on GL_1 allows us to arrange that there is a nonzero weight space in $\mathrm{Lie}(U)$ with a positive weight. We now apply the theory of §2.1 to the semidirect product $G = U \rtimes \mathrm{GL}_1$ (see Remark 2.1.11). By Lemma 2.1.5 and Proposition 2.1.8(1), there is a smooth connected k-subgroup $U_G(\lambda) \subseteq U$ whose Lie algebra is the span in $\mathrm{Lie}(U)$ of the positive weight spaces. Hence, $U_G(\lambda) \neq 0$. But $U_G(\lambda)$ is k-split (Proposition 2.1.10), contradicting that U is k-wound."

Also add the following text at the bottom of the page.

Proposition B.4.5 *Let S be a split torus over a field k of characteristic exponent $p \geqslant 1$, and let $f : G \to G'$ be a surjective S-equivariant k-homomorphism between smooth affine k-groups equipped with S-actions. Every S-weight on* $\mathrm{Lie}(G')$ *is a p-power multiple of an S-weight on* $\mathrm{Lie}(G)$.

Proof. By Proposition A.2.5 the centralizer G^S in G for the S-action is smooth and connected, since $G^S \times S$ is the centralizer of S in the smooth connected affine group $G \rtimes S$ Applying Proposition A.2.8 to the surjective homomorphism $G \rtimes S \twoheadrightarrow G' \rtimes S$ implies that $G^S \to G'^S$ is surjective, so by Proposition A.2.5 the case of the trivial S-weight is settled.

Consider a nontrivial S-weight $a \in \mathrm{X}(S)$ that occurs in $\mathrm{Lie}(G')$. Let S_a be the codimension-1 subtorus $(\ker a)^0_{\mathrm{red}}$. We may replace G with G^{S_a}, G' with G'^{S_a}, and S with S/S_a to reduce to the case that $S = \mathrm{GL}_1$. Choosing $\lambda \in \mathrm{X}_*(S)$ such that the composition $a \circ \lambda \in \mathbf{Z}$ is positive, the a-weight space in $\mathrm{Lie}(G')$ is supported inside of $\mathrm{Lie}(U_{G'}(\lambda))$ (using terminology as in Remark 2.1.11 with abstract torus actions). By Proposition 2.1.9 (and Remark 2.1.11), the natural map $U_G(\lambda) \to U_{G'}(\lambda)$ is surjective. Thus, by Proposition 2.1.10 (and Remark 2.1.11) we may rename $U_G(\lambda)$ as G and rename $U_{G'}(\lambda)$ as G' to reduce the general problem (without reference to a) to the case that G is k-split unipotent (and nontrivial) and all weights in $\mathrm{X}(S) = \mathbf{Z}$ are > 0.

Since G is not k-wound, by Proposition B.3.2 it contains a nontrivial central vector group V. By Galois descent from k_s, there is such a V that is S-stable, so $V' := f(V)$ is an S-stable central vector group in G' (by Theorem B.2.5). Any S-weight on $\mathrm{Lie}(G')$ occurs in either

$\mathrm{Lie}(V')$ or $\mathrm{Lie}(G'/V')$, and $G/V \to G'/V'$ is an S-equivariant surjection with G/V also k-split (Proposition A.1.4). Hence, by induction on $\dim G$ we reduce to the case that G and G' are vector groups. By Theorem B.4.3 there is a decomposition $G = \prod L_i$ where $L_i = \mathbf{G}_{\mathrm{a}}$ on which S acts via $t.x = t^{n_i}x$ for some $n_i > 0$. Running through the same filtration argument again, we may assume $G = \mathbf{G}_{\mathrm{a}}$ with S-action $t.x = t^n x$ for some $n > 0$. We can also assume $G' \neq 1$, so $G' = \mathbf{G}_{\mathrm{a}}$ with S-action $t.x = t^{n'}x$ for some $n' > 0$. The map $f : G \twoheadrightarrow G'$ is a nonzero additive polynomial in one variable such that $f(t^n x) = t^{n'} f(x)$. Thus, $f(x) = cx^{p^e}$ for some $c \in k^\times$ and $e \geqslant 0$ with $n' = p^e n$. $\square$

Remark B.4.6 A proof of Proposition B.4.5 that avoids Theorem B.4.3 (and only requires G and G' to be smooth of finite type rather than affine) can be given by using Proposition A.7.14 and the exactness properties of p-Lie algebras of infinitesimal group schemes in characteristic $p > 0$ (as discussed at the end of Example A.7.16). The argument proceeds by reduction to the case when $H = \ker f$ is infinitesimal, for which one proves that each S-weight on $\mathrm{Lie}(G')$ is $p^i a$ for an S-weight a on $\mathrm{Lie}(G)$ and an integer $0 \leqslant i \leqslant h$ with h minimal so that $F_{H/k,h}$ vanishes. To prove this, one reduces to the case $h = 1$ and then factorizes $F_{G/k}$ through f and factorizes $F_{G'/k}$ through $f^{(p)}$.

p. 494: On line -13, replace "A key observation is" with " We shall prove".

p. 499: On lines -9 through -5, replace the entire text "assertion that ... as U." with "fact that $X_K \to X$ is open (see [EGA, IV$_2$, 2.4.10])."

p. 502: On line -14, replace "on k^n" with "in k^n". On line -12, replace "f involves each variable (since $f(0) = 0$)" with "f_{prin} involves each variable".

p. 506: On line 6 of the proof of Theorem C.2.3, replace "subgroups" with "subgroup".

p. 511: At the end of C.2 add the following:

C.2.11 The k-root system of G and a Tits system in $G(k)$. Let G be a smooth connected affine k-group and $\pi : G \to \overline{G} := G/\mathscr{R}_{u,k}(G)$ its maximal pseudo-reductive quotient. We fix a maximal k-split torus S of G, and a minimal pseudo-parabolic k-subgroup P of G containing S. Let $N = N_G(S)$ and $Z = Z_G(S)$, so $Z = N \cap P$ (by Theorem C.2.8). Our aim in the remainder of §C.2 is to construct a root system ${}_k\overline{\Phi}$ associated to (G, S) such that (i) $W({}_k\overline{\Phi})$ is naturally identified with $N(k)/Z(k)$, (ii) the set of parabolic subsets of ${}_k\overline{\Phi}$ is in natural bijective correspondence with the set of pseudo-parabolic k-subgroups of G containing S, and (iii) the hypothesis that G is pseudo-split in 3.3.12–3.3.16, 3.4.2 and 3.5.1 can be removed by using maximal k-split tori. We will also exhibit a Tits system associated to $(G(k), P(k), N(k))$; the definition of a Tits system will be recalled later.

Let T be a maximal k-torus of G containing S. The map π carries the tori S and T isomorphically onto their respective images $\overline{S}$ and $\overline{T}$ in $\overline{G}$. In this way, we identify $\mathrm{X}(\overline{S})$ and $\mathrm{X}(\overline{T})$ with $\mathrm{X}(S)$ and $\mathrm{X}(T)$ respectively. By definition the *root system* of G_{k_s} with respect to T_{k_s} is the (possibly non-reduced) root system $\Phi(\overline{G}_{k_s}, \overline{T}_{k_s})$ of $\overline{G}_{k_s}$ with respect to $\overline{T}_{k_s}$ viewed as a root system in $\mathrm{X}(T_{k_s})_{\mathbf{R}}$ via the identification of $\mathrm{X}(\overline{T}_{k_s})$ with $\mathrm{X}(T_{k_s})$. (The discussion immediately following Definition 3.2.5 shows that $\Phi(\overline{G}_{k_s}, \overline{T}_{k_s})$ is a root system in its $\mathbf{R}$-linear span.) We will often denote the root system $\Phi(\overline{G}_{k_s}, \overline{T}_{k_s})$ by $\overline{\Phi}$ in the sequel.

The k-groups $W = N_G(T)/Z_G(T)$ and $\overline{W} = N_{\overline{G}}(\overline{T})/Z_{\overline{G}}(\overline{T})$ are finite étale (Lemma A.2.9), and by Lemma 3.2.1 the natural homomorphism $W \to \overline{W}$ is a k-isomorphism. Since $Z_G(T)$ is smooth, the homomorphism $N_G(T)(k_s)/Z_G(T)(k_s) \to W(k_s)$ is an isomorphism. By Proposition 3.2.7, $W(k_s)$ restricts isomorphically onto the Weyl group of $\overline{\Phi}$.

According to Proposition C.2.10, the finite étale k-group $W(G, S) = N/Z$ is constant and $N(k)/Z(k) \to W(G, S)(k)$ is an isomorphism. Let $\Phi(\overline{G}, \overline{S}) = \overline{\Phi}|_{\overline{S}_{k_s}} - \{0\}$. Via the identification $\mathrm{X}(\overline{S}) = \mathrm{X}(\overline{S}_{k_s})$, $\Phi(\overline{G}, \overline{S})$ is the set of nonzero weights for the adjoint action of the k-split $\overline{S}$ on the Lie algebra of $\overline{G}$. Define ${}_kW = W(G, S)(k)$ and denote by ${}_k\overline{\Phi}$ the subset of $\mathrm{X}(S)$ corresponding to $\Phi(\overline{G}, \overline{S}) \subset \mathrm{X}(\overline{S})$ under the identification of $\mathrm{X}(\overline{S})$ with $\mathrm{X}(S)$, so ${}_k\overline{\Phi} \subset \Phi(G, S)$; in general this latter inclusion need not be an equality. When $\mathscr{R}_{u,k}(G) \subset Z_G(S)$ (for example, if G is pseudo-reductive) then ${}_k\overline{\Phi}$ equals the set $\Phi(G, S)$ of nontrivial S-weights on $\mathrm{Lie}(G)$, and we will often denote ${}_k\overline{\Phi}$ by ${}_k\Phi$.

By Lemma 2.2.3(1), Proposition 2.2.10, and Theorem C.2.3, ${}_k\overline{\Phi}$ is empty if and only if G does not contain proper pseudo-parabolic k-subgroups. It is clear that the natural action of ${}_kW$ on $\mathrm{X}(S)$ carries ${}_k\overline{\Phi}$ into itself. We will prove in Theorem C.2.13 that ${}_k\overline{\Phi}$ is a root system in its $\mathbf{R}$-span in $\mathrm{X}(S)_{\mathbf{R}}$, with ${}_kW$ restricting isomorphically onto $W({}_k\Phi)$. We call ${}_k\overline{\Phi}$ the *k-root system of G with respect to S* (or *relative root system*), and the elements of ${}_k\overline{\Phi}$ the *k-roots* of G with respect to S.

For $a \in {}_k\overline{\Phi}$, let $\mathscr{S}_a = \ker a$ and $S_a = (\ker a)^0_{\mathrm{red}}$, so $\mathscr{S}_a$ is a k-group scheme of multiplicative type and S_a is a torus. Beware that $\mathscr{S}_a$ may not be smooth. By Proposition A.8.10(1), there exists a closed k-subgroup scheme $Z_G(\mathscr{S}_a)$ in G representing the centralizer of $\mathscr{S}_a$ under the conjugation action on G. The k-group $Z_G(\mathscr{S}_a)$ is smooth by Proposition A.8.10(2), and we define G_a to be its identity component. Note that the k-subgroups G_a, $\mathscr{S}_a$, and S_a in G are unaffected by replacing a with $-a$. (In the main text with pseudo-reductive groups, notation such as G_a generally denotes the derived group of the centralizer of a suitable torus. In the remainder of §C.2 we consider rather general G, possibly not pseudo-reductive, and we use the identity component of a torus centralizer – such as G_a above – without passing to its derived subgroup.)

The following assertions are immediate from Proposition A.8.14. The image $\overline{G}_a$ of G_a in $\overline{G} = G/\mathscr{R}_{u,k}(G)$ is pseudo-reductive and equal to the identity component of the functorial centralizer of the natural $\mathscr{S}_a$-action on $\overline{G}$ (again, Proposition A.8.10(1),(2) provides the existence and smoothness of such a centralizer scheme). Moreover, the natural map $G_a/\mathscr{R}_{u,k}(G_a) \to \overline{G}_a$ is an étale unipotent isogeny, so it induces an isomorphism between the Lie algebras and an isomorphism $T \simeq \overline{T}$ between maximal k-tori. This identifies $\Phi((G_a/\mathscr{R}_{u,k}(G_a))_{k_s}, T_{k_s})$ and $\Phi((\overline{G}_a)_{k_s}, \overline{T}_{k_s})$ in $\mathrm{X}(T_{k_s}) = \mathrm{X}(\overline{T}_{k_s})$, and this common subset consists of those $\alpha \in \overline{\Phi}$ whose restriction to $S_{k_s} \simeq \overline{S}_{k_s}$ is an integral multiple of a (in $\mathrm{X}(S_{k_s}) = \mathrm{X}(S)$).

Lemma C.2.12 *The smooth connected k-group G_a has proper pseudo-parabolic k-subgroups, and all such pseudo-parabolic k-subgroups are minimal. There are exactly two such k-subgroups P_a and P'_a that contain S.*

Proof. The natural map $q : G_a^{\mathrm{pred}} := G_a/\mathscr{R}_{u,k}(G_a) \to \overline{G}_a$ is an étale unipotent isogeny (Proposition A.8.14(2)), so its finite étale kernel is *central* and hence contained in any pseudo-parabolic k-subgroup of G_a^{pred}. It is then easy to see (using Corollary 2.1.9 and lifting maximal k-split tori through an étale k-isogeny) that forming images and scheme-theoretic preimages under q defines an inclusion-preserving bijective correspondence between the sets of pseudo-parabolic k-subgroups of G_a^{pred} and $\overline{G}_a$. The same holds between G_a and G_a^{pred} due to Proposition 2.2.10, so to prove the existence and minimality of proper pseudo-parabolic k-subgroups in G_a it is equivalent to prove the same for $\overline{G}_a$.

Since $\overline{S}$ is a maximal k-split torus in $\overline{G}_a$, by Theorem C.2.3 every pseudo-parabolic k-subgroup of $\overline{G}_a$ containing $\overline{S}$ has the form $P_{\overline{G}_a}(\overline{\mu})$ for some $\overline{\mu} \in \mathrm{X}_*(\overline{S})$. But $\overline{S}_a := \pi(S_a)$ is a central torus of $\overline{G}_a$ with codimension 1 in $\overline{S}$, so it is the maximal central k-split torus in $\overline{G}_a$ (as $a \in {}_k\Phi$) and $P_{\overline{G}_a}(\overline{\mu})$ only depends on the image of $\overline{\mu}$ in $\mathrm{X}_*(\overline{S}/\overline{S}_a) \simeq \mathbf{Z}$. For example, this image vanishes if and only if $\overline{\mu}$ is central in $\overline{G}_a$, which is equivalent to the condition $P_{\overline{G}_a}(\overline{\mu}) = \overline{G}_a$ (Lemma 2.2.3(2)), so $\overline{G}_a$ admits a proper pseudo-parabolic k-subgroup. By Remark 2.1.7 we have $P_{\overline{G}_a}(\overline{\mu}) = P_{\overline{G}_a}(\overline{\mu}^n)$ for any integer $n \geqslant 1$, so all proper pseudo-parabolic k-subgroups of $\overline{G}_a$ (and hence of G_a) are minimal.

Fix a 1-parameter subgroup $\lambda_a : \mathrm{GL}_1 \to S$ such that $\langle a, \lambda_a \rangle > 0$, and let $\overline{\lambda}_a \in \mathrm{X}_*(\overline{S})$ correspond to λ_a. The pseudo-parabolic k-subgroup $P_a = P_{G_a}(\lambda_a)\mathscr{R}_{u,k}(G_a)$ of G_a contains S and has image $P_{\overline{G}_a}(\overline{\lambda}_a)$ in $\overline{G}_a$ (Corollary 2.1.9), so it is a proper k-subgroup due to the non-centrality of $\overline{\lambda}_a$ in $\overline{G}_a$ (Lemma 2.2.3(2)). Hence, P_a is a minimal pseudo-parabolic k-subgroup of G_a. Likewise, $P'_a := P_{G_a}(-\lambda_a)\mathscr{R}_{u,k}(G_a)$ is a minimal pseudo-parabolic k-subgroup of G_a containing S, and $P'_a \neq P_a$ (as we see by comparing the Lie algebras of their images $P_{\overline{G}_a}(\pm\overline{\lambda}_a)$ in $\overline{G}_a$). The preceding arguments show that $P_{\overline{G}_a}(\pm\overline{\lambda}_a)$ are the only proper pseudo-parabolic k-subgroups of $\overline{G}_a$ that contain $\overline{S}$, so P_a and P'_a are the only proper pseudo-parabolic k-subgroups of G_a that contain S. ☐

By Theorems C.2.5 and C.2.3, there exists $n_a \in N_{G_a}(S)(k)$ such that $P'_a = n_a P_a n_a^{-1}$. Note that n_a does not centralize S, since via its natural action on $\mathrm{X}_*(S)$ we have

$$P_{G_a}(n_a.\lambda_a)\mathscr{R}_{u,k}(G_a) = n_a P_a n_a^{-1} = P'_a \neq P_a = P_{G_a}(\lambda_a)\mathscr{R}_{u,k}(G_a).$$

Let $r_a \in {}_kW = N(k)/Z(k)$ denote the nontrivial image of n_a. Under the natural faithful linear action of the finite group ${}_kW$ on $\mathrm{X}(S)_{\mathbf{R}}$, the element r_a acts nontrivially and restricts to the identity on a hyperplane (since n_a centralizes the codimension-1 subtorus S_a in S), so r_a acts as a reflection. Since the codimension-1 subtorus S_a in S is central in G_a, it follows that the subgroup $W(G_a, S)(k)$ of ${}_kW$ has order 2 and is generated by the reflection r_a. In particular, r_a is independent of the choice of n_a.

Recall that every pseudo-parabolic k-subgroup of G containing S also contains $Z_G(S)$ (Proposition C.2.4) and hence contains T. Now let Q be such a pseudo-parabolic k-subgroup of G and $\overline{Q}$ be its image in $\overline{G}$. Then $\overline{Q}$ is a pseudo-parabolic k-subgroup of $\overline{G}$ containing $\overline{S}$. We will denote by $\Phi(\overline{Q}, \overline{S})$ the set of nontrivial weights of $\overline{S}$ on $\mathrm{Lie}(\overline{Q})(\subset \mathrm{Lie}(\overline{G}))$.

Let P be a minimal pseudo-parabolic k-subgroup of G containing S. The image $\overline{P}$ of P in $\overline{G}$ is a minimal pseudo-parabolic k-subgroup of $\overline{G}$ (by Proposition 2.2.10); it contains $\overline{S}$ and $\overline{T}$. Since the subset $\Phi(\overline{P}, \overline{S}) \subset \mathrm{X}(\overline{S}) = \mathrm{X}(\overline{S}_{k_s})$ consists of the nontrivial restrictions to $\overline{S}_{k_s}$ of the parabolic subset $\Phi(\overline{P}_{k_s}, \overline{T}_{k_s})$ in $\overline{\Phi} = \Phi(\overline{G}_{k_s}, \overline{T}_{k_s})$, it is clear that $\Phi(\overline{G}, \overline{S}) = \Phi(\overline{P}, \overline{S}) \cup -\Phi(\overline{P}, \overline{S})$. The minimality of $\overline{P}$ in $\overline{G}$ gives more: this union is *disjoint.* Indeed, by Proposition C.2.4 we have $\overline{P} = Z_{\overline{G}}(\overline{S}) \cdot \mathscr{R}_{us,k}(\overline{P})$ with $Z_{\overline{G}}(\overline{S})$ pseudo-reductive (Proposition 1.2.4) and $\mathscr{R}_{us,k}(\overline{P}) \cap Z_{\overline{G}}(\overline{S}) = 1$ (because this intersection is smooth and connected, by Proposition A.2.5), so $\overline{P} = Z_{\overline{G}}(\overline{S}) \ltimes \mathscr{R}_{us,k}(\overline{P})$. Hence, it suffices to show that a pair of opposite nontrivial $\overline{S}$-weights cannot both arise on the Lie algebra of $\mathscr{R}_{us,k}(\overline{P})$. Theorem C.2.3 applied to $\overline{P}$ implies $\overline{P} = P_{\overline{G}}(\overline{\mu}) = Z_{\overline{G}}(\overline{\mu}) \ltimes U_{\overline{G}}(\overline{\mu})$ for some $\overline{\mu} : \mathrm{GL}_1 \to \overline{S}$, so $\mathscr{R}_{us,k}(\overline{P}) = U_{\overline{G}}(\overline{\mu})$ and hence $\Phi(\overline{P}, \overline{S})$ is the set of $\overline{a} \in \Phi(\overline{G}, \overline{S})$ such that $\langle \overline{a}, \overline{\mu} \rangle > 0$. Thus, no such opposite nontrivial weights exist.

Since $\Phi(\overline{P}, \overline{S})$ and $-\Phi(\overline{P}, \overline{S})$ are disjoint, and $Z_{\overline{G}}(\overline{S})$ is contained in $\overline{P}$ (Proposition C.2.4), the intersection of $\Phi(\overline{P}_{k_s}, \overline{T}_{k_s})$ and $-\Phi(\overline{P}_{k_s}, \overline{T}_{k_s})$ is the root system $\Phi(Z_{\overline{G}}(\overline{S})_{k_s}, \overline{T}_{k_s})$ of $Z_{\overline{G}}(\overline{S})_{k_s}$ with respect to $\overline{T}_{k_s}$. On the other hand, since the subset $\Phi(\overline{P}_{k_s}, \overline{T}_{k_s})$ of $\overline{\Phi}$ is parabolic, by Proposition 2.2.8 there is a positive system of roots $\overline{\Phi}^+$ of $\overline{\Phi}$ contained in $\Phi(\overline{P}_{k_s}, \overline{T}_{k_s})$ and a subset Δ_0 of the basis Δ of $\overline{\Phi}^+$ such that $\Phi(\overline{P}_{k_s}, \overline{T}_{k_s})$ is the union of $\overline{\Phi}^+$ and the set $[\Delta_0]$ of roots that are integral linear combination of elements in Δ_0. Then the intersection $\Phi(\overline{P}_{k_s}, \overline{T}_{k_s}) \cap -\Phi(\overline{P}_{k_s}, \overline{T}_{k_s})$ is the subset $[\Delta_0]$, and we conclude that the root system of $Z_{\overline{G}}(\overline{S})_{k_s}$ with respect to $\overline{T}_{k_s}$ is $[\Delta_0]$ and Δ_0 is a basis of this root system. Therefore, Δ_0 is precisely the subset of Δ consisting of positive simple roots whose restriction to $\overline{S}_{k_s}$ is trivial. Let ${}_k\Delta$ be the subset of $\mathrm{X}(S)$ obtained from $(\Delta - \Delta_0)|_{\overline{S}_{k_s}}$ using the identification of $\mathrm{X}(S)$ with $\mathrm{X}(\overline{S}_{k_s})$.

For $w \in {}_kW = W(G,S)(k) = N(k)/Z(k)$ and a pseudo-parabolic k-subgroup Q of G containing S, define $w \circ Q := nQn^{-1}$ for a representative $n \in N(k)$ of w (the choice of which does not matter, since $Z \subseteq Q$ by Proposition C.2.4). This is a left action of ${}_kW$ on the collection of such Q. Recall that ${}_k\overline{\Phi} = \Phi(\overline{G}, \overline{S})$ inside of $\mathrm{X}(\overline{S}) = \mathrm{X}(S)$.

Theorem C.2.13 *The set* ${}_k\overline{\Phi}$ *is a root system* (*in its* $\mathbf{R}$-*linear span in* $\mathrm{X}(S)_{\mathbf{R}}$) *with a basis given by* ${}_k\Delta$, *and the natural action of* ${}_kW$ *on the* $\mathbf{R}$-*linear span of* ${}_k\overline{\Phi}$ *identifies* ${}_kW$ *with the Weyl group* $W({}_k\overline{\Phi})$ *of* ${}_k\overline{\Phi}$.

The assignment $Q \mapsto {}_k\overline{\Phi}_Q := \Phi(Q/\mathscr{R}_{u,k}(G), \overline{S})$ *is a bijection from the set of pseudo-parabolic* k-*subgroups of* G *containing* S *onto the set of parabolic subsets of* ${}_k\overline{\Phi}$, *with* $Q \subseteq Q'$ *if and only if* ${}_k\overline{\Phi}_Q \subseteq {}_k\overline{\Phi}_{Q'}$. *The positive system of roots* ${}_k\overline{\Phi}_P$ *has basis* ${}_k\Delta$.

This result generalizes Proposition 3.3.15(1),(2) and Proposition 3.5.1(1),(3).

Proof. Passage to the pseudo-reductive case goes as follows. By Lemma 3.2.1 the natural maps $W(G_{k_s}, T_{k_s}) \to W(\overline{G}_{k_s}, \overline{T}_{k_s})$ and $W(G, S) \to W(\overline{G}, \overline{S})$ are isomorphisms, as is

$W(G_a, S) \to W(\overline{G}_a, \overline{S})$. Also, essentially by definition, the identification $\mathrm{X}(S) = \mathrm{X}(\overline{S})$ carries ${}_k\overline{\Phi}$ onto $\Phi(\overline{G}, \overline{S})$. In view of the bijective correspondence in Proposition 2.2.10 between the sets of pseudo-parabolic k-subgroups of G and of $\overline{G}$ (respecting containment of T and $\overline{T}$), and the fact that a pseudo-parabolic k-subgroup containing S necessarily contains $Z_G(S) \supseteq T$, we may replace (G, P, T, S, Δ) with $(\overline{G}, \overline{P}, \overline{T}, \overline{S}, \Delta)$ to reduce to the case of pseudo-reductive G. With G now pseudo-reductive, we shall write Φ, ${}_k\Phi$, and ${}_k\Phi_Q$ rather than $\overline{\Phi}$, ${}_k\overline{\Phi}$, and ${}_k\overline{\Phi}_Q$ respectively.

Step 1. We next reduce to using $G' = \mathscr{D}(G)$ in place of G. The main task is to set up bijective correspondences between constructions for (G, S) and (G', S') with a suitable maximal k-split torus S' in G'.

By Lemma 1.2.5(iii), the maximal k-torus T is an almost direct product of the maximal central k-torus T_0 in G and the maximal k-torus $T' = T \cap G'$ in G'. Semisimplicity of the isogeny category of k-tori then implies that the k-subtorus S is an almost direct product of the maximal k-split central k-torus S_0 in G and the k-torus $S' = (S \cap G')^0_{\mathrm{red}}$ that is necessarily a maximal k-split k-torus in G'. The commutativity of G/G' implies that the root spaces in $\mathrm{Lie}(G)_{k_s}$ for the action of T_{k_s} lie in $\mathrm{Lie}(G')_{k_s}$. The finite-index inclusion $\mathrm{X}(T_{k_s}) \hookrightarrow \mathrm{X}(T'_{k_s}) \oplus \mathrm{X}(T_{0 k_s})$ carries $\Phi := \Phi(G_{k_s}, T_{k_s})$ into $\mathrm{X}(T'_{k_s})$ with image $\Phi' := \Phi(G'_{k_s}, T'_{k_s})$, and it is equivariant with respect to the natural homomorphism $W(G'_{k_s}, T'_{k_s}) \to W(G_{k_s}, T_{k_s})$. In particular, there is a natural correspondence between parabolic subsets of Φ and Φ', so Δ corresponds to a basis Δ' for a positive system of roots in Φ' (carrying Δ_0 to the analogous Δ'_0). Also, if $\alpha \in \Phi$ restricts to $\alpha' \in \Phi'$ then the natural map $W((G'_{k_s})_{\alpha'}, T'_{k_s}) \to W((G_{k_s})_{\alpha}, T_{k_s})$ between groups of order 2 is injective and hence an equality, so r_α restricts to $r_{\alpha'}$.

By replacing T_{k_s} with S in the preceding arguments, the finite-index inclusion $\mathrm{X}(S) \hookrightarrow \mathrm{X}(S') \oplus \mathrm{X}(S_0)$ caries ${}_k\Phi$ onto ${}_k\Phi'$ (and ${}_k\Delta$ onto ${}_k\Delta'$) equivariantly with respect to the natural homomorphism $W(G', S') \to W(G, S)$, and if $a \in {}_k\Phi$ restricts to $a' \in {}_k\Phi'$ then the reflection $r_a \in {}_kW$ restricts to the reflection $r_{a'} \in {}_kW'$ on $\mathrm{X}(S')$ (and restricts to the identity on $\mathrm{X}(S_0)$). But any pseudo-parabolic k-subgroup of G containing the almost direct product $S = S' \cdot S_0$ must contain $Z = Z_G(S)$ (Proposition C.2.4) and has the form $P_G(\lambda)$ for some $\lambda \in \mathrm{X}_*(S')$ (due to Remark 2.1.7 and Theorem C.2.3). Thus, the equality $G = Z \cdot G'$ (which follows from Proposition 1.2.6) implies that $Q \mapsto Q \cap G'$ and $Q' \mapsto ZQ'$ are inverse bijections between the set of pseudo-parabolic k-subgroups of G containing S and the set of pseudo-parabolic k-subgroups of G' containing S'. We may now replace (G, S, T, P, Δ) with $(G', S', T', P \cap G', \Delta')$, so G is perfect (Proposition 1.2.6).

Step 2. Since G is perfect, $G^{\mathrm{red}}_{\overline{k}}$ is semisimple, so there is no nontrivial central torus in $G_{\overline{k}}$. Thus, Φ spans $\mathrm{X}(T_{k_s})_{\mathbf{Q}}$ and ${}_k\Phi$ spans $\mathrm{X}(S)_{\mathbf{Q}}$ (by Corollary A.8.11, applied to a torus in the intersection of the kernels of all elements of Φ, or of ${}_k\Phi$). Thus, for each $a \in {}_k\Phi$ the reflection r_a is uniquely determined by the property that it negates $a \neq 0$ and preserves ${}_k\Phi$ (see [Bou2, VI, §1.1, Lemma 1]). For every $b \in \mathrm{X}(S)_{\mathbf{Q}}$ we have $r_a(b) - b = n_{b,a} a$ for some $n_{b,a} \in \mathbf{Q}$ since r_a is a reflection in a, so ${}_k\Phi$ is a root system if and only if $n_{b,a} \in \mathbf{Z}$ for all $a, b \in {}_k\Phi$. To establish this integrality, we will use the scheme-theoretic kernel $\mathscr{S}_a = \ker a$ and the identity component $G_a = Z_G(\mathscr{S}_a)^0$ of its scheme-theoretic centralizer.

Recall that $\Phi_a := \Phi((G_a)_{k_s}, T_{k_s})$ consists of the roots $\alpha \in \Phi$ such that the restriction $\alpha|_{S_{k_s}} \in \mathrm{X}(S_{k_s}) = \mathrm{X}(S)$ is an integral multiple of a. For any $n \in N_{G_a}(S)(k)$ we have $Z_{G_a}(S) \supseteq n^{-1}Tn \supseteq S$. Since T and $n^{-1}Tn$ are maximal k-tori of $Z_{G_a}(S)$, by conjugacy of such tori over k_s (Proposition A.2.10) there exists $z \in Z_{G_a}(S)(k_s)$ such that $z^{-1}n^{-1}T_{k_s}nz = T_{k_s}$; i.e., $nz \in N_{G_a}(T)(k_s)$. The conjugation action of nz on T_{k_s} keeps S_{k_s} stable, and restricted to S_{k_s} it coincides with the conjugation action of n. This implies, in particular, that we can find $w \in N_{G_a}(T)(k_s)/Z_{G_a}(T)(k_s)$ which stabilizes S_{k_s} inside T_{k_s} and acts on S_{k_s} by the reflection r_a. The group $N_{G_a}(T)(k_s)/Z_{G_a}(T)(k_s)$ is naturally identified with the Weyl group of the root system $\Phi_a (\subseteq \Phi)$. Writing w as a product of reflections in roots belonging to Φ_a, we easily see that for any $\beta \in \Phi$, $w(\beta) - \beta = \sum_{\alpha\in\Phi_a} s_\alpha \alpha$ with integers $s_\alpha = s_\alpha(\beta)$.

For any $b \in {}_k\Phi$ there exists $\beta \in \Phi$ whose restriction to S_{k_s} is b. Thus, $w(b) - b = \sum_{\alpha\in\Phi_a} s_\alpha \alpha|_{S_{k_s}}$ for integers s_α. But the restriction to S_{k_s} of any root in Φ_a is an integral multiple of a, and the restriction of $w(\beta)$ to S_{k_s} is $r_a(b)$. Hence, $r_a(b) - b$ is an integral multiple of a as required. This proves that ${}_k\Phi$ is a root system spanning $\mathrm{X}(S)_{\mathbf{Q}}$, and for each $a \in {}_k\Phi$ the associated reflection is r_a. In particular, $W({}_k\Phi) \subseteq {}_kW$ inside of $\mathrm{Aut}(\mathrm{X}(S))$.

Step 3. Now we turn to the analysis of ${}_k\Delta$ and ${}_kW$. Every nontrivial S-weight on $\mathrm{Lie}(G)$ is the restriction of a nontrivial T_{k_s}-weight on $\mathrm{Lie}(G)_{k_s}$, and each $\beta \in \Phi$ has the form $\sum_{\alpha\in\Delta} n_\alpha \alpha$ with integers n_α that are either all $\geqslant 0$ or all $\leqslant 0$. Hence, each $b \in {}_k\Phi$ has the form $\sum_{a\in{}_k\Delta} n_a a$ with *integers* n_a that are either all $\geqslant 0$ or all $\leqslant 0$. In particular, ${}_k\Delta$ spans $\mathrm{X}(S)_{\mathbf{Q}}$ (so $\#{}_k\Delta \geqslant \dim S$) and if ${}_k\Delta$ is linearly independent then its elements must be non-divisible in ${}_k\Phi$. Thus, linear independence of ${}_k\Delta$ would imply that it is a basis for a positive system of roots in ${}_k\Phi$ (by [Bou2, VI, §1.7, Cor. 3]), so the subgroup $W({}_k\Phi)$ of ${}_kW$ is generated by the reflections r_a for $a \in {}_k\Delta$. It remains to prove that $\#{}_k\Delta = \dim S$, the inclusion $W({}_k\Phi) \subseteq {}_kW$ is an equality, and $Q \mapsto \Phi(Q, S)$ is a bijection from the set of pseudo-parabolic k-subgroups of G containing S to the set of parabolic subsets of ${}_k\Phi$.

To determine $\#{}_k\Delta$, we first describe the "$*$-action" of $\Gamma := \mathrm{Gal}(k_s/k)$ on Δ. For $\gamma \in \Gamma$, the subset $\gamma(\Phi^+)$ is a positive system of roots in Φ, so there is an element $w_\gamma \in W(k_s)$ such that $w_\gamma(\gamma(\Phi^+)) = \Phi^+$. (Note that since the Weyl group $W(k_s)$ acts simply transitively on the set of positive systems of roots in Φ, w_γ is unique.) Then, clearly, $w_\gamma(\gamma(\Delta)) = \Delta$. The correspondence $\alpha \mapsto \gamma * \alpha := w_\gamma(\gamma(\alpha))$ defines an action of Γ on Δ which is called the $*$-*action.*

In the sequel, for simplicity, we will denote $\mathscr{R}_{u,k}(P)$ by U and denote $Z_G(S)$ by Z. (By Corollary 2.2.5, $U = \mathscr{R}_{us,k}(P)$.) Since P is minimal and $P = P_G(\lambda) = Z_G(\lambda) \ltimes U_G(\lambda)$ for some $\lambda \in \mathrm{X}_*(S)$ (by Theorem C.2.3), Proposition C.2.4 gives that $P = Z \ltimes U$. We will now show that for every $\gamma \in \Gamma$, there exists a representative n_γ of w_γ in $N_Z(T)(k_s)$. Since $T \subseteq P$, clearly P_{k_s} is a pseudo-parabolic k_s-subgroup of G_{k_s} containing T_{k_s}. The pseudo-parabolic k_s-subgroups of P_{k_s} are precisely the pseudo-parabolic k_s-subgroups of G_{k_s} which are contained in P_{k_s} (Proposition 3.5.8), and each of them contains U_{k_s}. The quotient map $P_{k_s} \to P_{k_s}/U_{k_s} \simeq Z_{k_s}$ induces a bijective correspondence between the set of pseudo-parabolic

k_s-subgroups of P_{k_s} and the set of pseudo-parabolic k_s-subgroups of Z_{k_s} (Proposition 2.2.10). Under this correspondence, the minimal pseudo-parabolic k_s-subgroups of P_{k_s} containing T_{k_s} correspond to the minimal pseudo-parabolic k_s-subgroups of Z_{k_s} containing T_{k_s}.

On the other hand, there is a natural bijective correspondence between the set of minimal pseudo-parabolic k_s-subgroups of Z_{k_s} containing T_{k_s} and the set of positive systems of roots in the root system $\Phi(Z_{k_s}, T_{k_s})$ of Z_{k_s}, and the group $N_Z(T)(k_s)$ acts transitively on both of these sets (Proposition 3.3.15). There is a similar bijective correspondence between the set of minimal pseudo-parabolic k_s-subgroups of P_{k_s} containing T_{k_s} and the set of positive systems of roots in the root system Φ which are contained in the parabolic set $\Phi_P := \Phi(P_{k_s}, T_{k_s}) = \Phi^+ \cup [\Delta_0]$ (Propositions 3.5.1(3) and 3.3.15(1)). We conclude that $N_Z(T)(k_s)(\subset N_G(T)(k_s))$ acts transitively on the set of positive system of roots in Φ which are contained in Φ_P. The parabolic set Φ_P is stable under Γ since P is defined over k. Hence, for $\gamma \in \Gamma$, $\gamma(\Phi^+)$ is a positive system of roots contained in Φ_P, so there is an $n_\gamma \in N_Z(T)(k_s)$ that carries $\gamma(\Phi^+)$ onto Φ^+. From the uniqueness of w_γ, we see that n_γ represents it.

The subset Δ_0 can be described as the set of simple roots α such that $-\alpha$ lies in Φ_P. From this description it is obvious that Δ_0 is stable under the $*$-action. As n_γ lies in $N_Z(T)(k_s)$, and Δ_0 is a basis of the root system $\Phi(Z_{k_s}, T_{k_s})$, w_γ lies in the group generated by the reflections r_α for $\alpha \in \Delta_0$. Hence, for any $\alpha \in \Delta$ and $\gamma \in \Gamma$ the difference $\gamma * \alpha - \gamma(\alpha) = w_\gamma(\gamma(\alpha)) - \gamma(\alpha)$ belongs to the integral span of Δ_0. This implies that the restrictions of $\gamma * \alpha$ and $\gamma(\alpha)$ to S_{k_s} are equal. But the restriction of $\gamma(\alpha)$ to S_{k_s} coincides with the restriction of α. Thus all roots belonging to a Γ-orbit in Δ under the $*$-action restrict to the same element in $\mathrm{X}(S_{k_s}) = \mathrm{X}(S)$. We will prove that the number of Γ-orbits in $\Delta - \Delta_0$ under the $*$-action is $\dim S$, so the inequality $\#_k\Delta \geqslant \dim S$ is an equality. It will then follow that ${}_k\Delta$ is a basis of the root system ${}_k\Phi$ and that a pair of elements of $\Delta - \Delta_0$ have the same restriction to S_{k_s} if and only if they lie in the same Γ-orbit under the $*$-action.

To prove that the number of Γ-orbits in $\Delta - \Delta_0$ under the $*$-action is $\dim S$, we will relate it to the set of pseudo-parabolic k-subgroups of G containing the minimal pseudo-parabolic k-subgroup P. This set can be parameterized in two different ways. First, we have:

Lemma C.2.14 *The map $Q \mapsto \Phi(Q, S)$ is a bijection from the set of pseudo-parabolic k-subgroups of G containing S onto the set of parabolic subsets of ${}_k\Phi$. For any such Q and Q', $Q \subseteq Q'$ if and only if $\Phi(Q, S) \subseteq \Phi(Q', S)$. In particular, $\Phi(P, S)$ is a positive system of roots in ${}_k\Phi$ and the set of Q that contain P is parameterized by the set of subsets of the basis of $\Phi(P, S)$.*

Proof. By Theorem C.2.3, any such Q has the form $Q = P_G(\lambda)$ for some $\lambda \in \mathrm{X}_*(S)$, so $\Phi(Q, S) = ({}_k\Phi)_{\lambda \geqslant 0}$. Thus, $\Phi(Q, S)$ is a parabolic subset of ${}_k\Phi$ and every parabolic subset of ${}_k\Phi$ arises from some Q (by Proposition 2.2.8). To prove that $Q \subseteq Q'$ if $\Phi(Q, S) \subseteq \Phi(Q', S)$ (the converse being obvious), fix a maximal k-torus T of G containing S. Since Q and Q' contain $Z_G(S)$ (Proposition C.2.4), they contain T. Thus, $Q_{k_s} \subseteq Q'_{k_s}$ if and only if $\Phi(Q_{k_s}, T_{k_s}) \subseteq \Phi(Q'_{k_s}, T_{k_s})$ inside of Φ (Proposition 3.5.1). As $Q = P_G(\lambda)$, $\Phi(Q_{k_s}, T_{k_s}) = \Phi_{\lambda_{k_s} \geqslant 0}$. Since λ_{k_s} is valued in S_{k_s}, for any $\alpha \in \Phi$ with restriction $a = \alpha|_{S_{k_s}} \in \mathrm{X}(S_{k_s}) = \mathrm{X}(S)$

we have $\langle \alpha, \lambda_{k_s} \rangle = \langle a, \lambda \rangle$. Hence, $\Phi(Q_{k_s}, T_{k_s})$ is the set of nonzero T_{k_s}-weights on $\mathrm{Lie}(Q_{k_s})$ contained in the preimage of $\{0\} \cup ({}_k\Phi)_{\lambda \geqslant 0} = \{0\} \cup \Phi(Q,S)$ under the restriction map $\mathrm{X}(T_{k_s}) \to \mathrm{X}(S_{k_s}) = \mathrm{X}(S)$. Thus, if $\Phi(Q,S) \subseteq \Phi(Q',S)$ then $Q_{k_s} \subseteq Q'_{k_s}$, so $Q \subseteq Q'$.

To prove the last assertion, let ${}_k\Phi^+ = \Phi(P,S)$ be the positive system of roots in ${}_k\Phi$ determined by the minimal pseudo-parabolic k-subgroup P. The description of parabolic sets containing ${}_k\Phi^+$ given by Proposition 2.2.8(2) (also see [Bou2, VI, §1.6, Lemma 3]) implies that the set of such parabolic sets is in bijective correspondence with the set of subsets of the basis of ${}_k\Phi^+$. $\square$

On the other hand, given any pseudo-parabolic k-subgroup Q of G containing P (and hence T), the set $\Phi_Q := \Phi(Q_{k_s}, T_{k_s})$ is a parabolic subset of Φ such that: (i) Φ_Q contains Φ_P, and (ii) Φ_Q is stable under the natural action of Γ on $\mathrm{X}(T_{k_s})$. By Proposition 3.5.2, $Q \mapsto \Phi_Q$ is a bijection from the set of such Q onto the set of parabolic subsets of Φ satisfying (i) and (ii). A parabolic subset of Φ containing $\Phi_P = \Phi^+ \cup [\Delta_0]$ is of the form $\Phi^+ \cup [\Delta_0 \cup \Delta']$, for a unique subset Δ' of $\Delta - \Delta_0$. Such a parabolic subset is stable under the action of w_γ, for every $\gamma \in \Gamma$, since n_γ lies in $N_Z(T)(k_s) (\subseteq Z(k_s) \subset P(k_s))$. Hence, $\Phi^+ \cup [\Delta_0 \cup \Delta']$ is stable under the natural action of Γ on $\mathrm{X}(T_{k_s})$ if and only if for every $\gamma \in \Gamma$, $w_\gamma(\gamma(\Phi^+ \cup [\Delta_0 \cup \Delta'])) = \Phi^+ \cup [\Delta_0 \cup \Delta']$. But from the definition of the $*$-action it is clear that $w_\gamma(\gamma(\Phi^+ \cup [\Delta_0 \cup \Delta'])) = \Phi^+ \cup [\Delta_0 \cup \gamma * \Delta']$. Therefore, $\Phi^+ \cup [\Delta_0 \cup \Delta']$ is stable under the natural action of Γ on $\mathrm{X}(T_{k_s})$ if and only if Δ' is stable under the $*$-action of Γ.

Putting everything together, the set of pseudo-parabolic k-subgroups of G containing P is parameterized in two ways: by the set of subsets of the basis of ${}_k\Phi^+ = \Phi(P,S)$ (there are $2^{\dim S}$ such subsets), and by the set of subsets of $\Delta - \Delta_0$ which are stable under the $*$-action of Γ. If r denotes the number of Γ-orbits in $\Delta - \Delta_0$ (under the $*$-action), then the number of Γ-stable subsets is clearly 2^r. Thus, $2^r = 2^{\dim S}$, so $r = \dim S$.

STEP 4. Since ${}_k\Delta$ is the basis of a positive system of roots in ${}_k\Phi$, the subset

$$\overline{C} = \{\lambda \in \mathrm{X}_*(S)_{\mathbf{R}} \mid \langle a, \lambda \rangle \geqslant 0 \text{ for all } a \in {}_k\Delta\}$$

is a fundamental domain for the action of $W({}_k\Phi)$ on $\mathrm{X}_*(S)_{\mathbf{R}}$ [Bou2, V, §3.3, Thm. 2].

For $\lambda \in \mathrm{X}_*(S) \cap \overline{C}$, we claim that the pseudo-parabolic k-subgroup $P_G(\lambda)$ contains P. By Propositions 2.2.10 and 3.5.1(3) it is equivalent to show that every $\beta \in \Phi(P_{k_s}, T_{k_s}) = \Phi^+ \cup [\Delta_0]$ satisfies $\langle \beta, \lambda_{k_s} \rangle \geqslant 0$. Since $\langle \beta, \lambda_{k_s} \rangle$ only depends on the restriction of β to S_{k_s}, and more specifically this pairing vanishes for $\beta \in [\Delta_0]$, we may and do restrict our attention to $\beta \in \Phi^+$, and even to $\beta \in \Delta - \Delta_0$. Thus, it suffices to prove $\langle a, \lambda \rangle \geqslant 0$ for all $a \in {}_k\Delta$. This in turn is immediate from the condition $\lambda \in \overline{C}$.

Any pseudo-parabolic k-subgroup Q of G containing S is of the form $Q = P_G(\mu)$ for some $\mu \in \mathrm{X}_*(S)$ (Theorem C.2.3), and since μ can be carried into $\overline{C}$ by the action of some element of $W({}_k\Phi)$ we see that: (i) some member Q' of the $W({}_k\Phi)$-orbit of Q contains P and (ii) if Q is a minimal pseudo-parabolic k-subgroup then $Q' = P$. This proves that $W({}_k\Phi)$ acts transitively on the set $\mathscr{P}$ of minimal pseudo-parabolic k-subgroups of G containing S. Since ${}_kW$ acts simply transitively on $\mathscr{P}$ (see the proof of Proposition C.2.10), and $W({}_k\Phi) \subseteq {}_kW$, we conclude that $W({}_k\Phi) = {}_kW$. $\square$

A *Tits system* is a 4-tuple $(\mathscr{G}, B, N, \Sigma)$ where $\mathscr{G}$ is an abstract group, B and N are subgroups, and $\Sigma \subseteq N/(B \cap N)$ is a subset such that the following four axioms are satisfied: (T1) $B \cup N$ generates $\mathscr{G}$ and $B \cap N$ is normal in N, (T2) the elements of Σ have order 2 in the quotient group $W := N/(B \cap N)$ and generate W, (T3) for all $\sigma \in \Sigma$ and $w \in W$, $\sigma B w \subseteq BwB \cup B\sigma w B$ (using any representatives for σ and w in N, the choices of which do not matter), and (T4) $\sigma B \sigma \not\subseteq B$ for all $\sigma \in \Sigma$ (which is equivalent to $\sigma B \sigma \neq B$, since $\sigma^2 = 1$ in W). We refer the reader to [Bou2, IV, §2] for the basic properties of Tits systems. By [Bou2, IV, §2.5, Rem. (1)], Σ is uniquely determined by $(\mathscr{G}, B, N)$.

The following theorem was announced by Borel and Tits in [BoTi2] without proof.

Theorem C.2.15 *Let G be a smooth connected affine group over a field k, S a maximal k-split torus in G, $N = N_G(S)$, and P a minimal pseudo-parabolic k-subgroup of G containing S. Let ${}_k\Delta$ be the basis of the positive system of roots ${}_k\overline{\Phi}^+$ in ${}_k\overline{\Phi}$ associated to P, and let $R = \{r_a \mid a \in {}_k\Delta\}$. The 4-tuple $(G(k), P(k), N(k), R)$ is a Tits system.*

This is the *standard Tits system* associated to (G, S, P). We will provide a proof of Theorem C.2.15 after some general preparations involving a notion of "root group" associated to any $a \in \Phi(G, S)$.

C.2.16 Root groups in G. For any $a \in \mathrm{X}(S) - \{0\}$, let $\langle a \rangle$ be the semigroup consisting of positive integral multiples of a. By Proposition 3.3.6 (applied to the semigroup $A = \langle a \rangle$) there is a unique smooth connected unipotent k-subgroup $U_a = H_{\langle a \rangle}(G)$ in G that is normalized by S and for which $\mathrm{Lie}(U_a)$ is the span of the weight spaces in $\mathrm{Lie}(G)$ for the weights in $\Phi(G, S)$ that are positive integral multiples of a. Moreover, U_a contains every smooth connected k-subgroup $H \subseteq G$ normalized by S such that S acts on $\mathrm{Lie}(H)$ only with weights that are positive integral multiples of a. Let $\mathscr{S}_a = \ker a$ and $G_a = Z_G(\mathscr{S}_a)^0$ be as in C.2.11. For $\lambda_a \in \mathrm{X}_*(S)$ such that $\langle a, \lambda_a \rangle > 0$ we have $U_{G_a}(\lambda_a) \subseteq U_a$. This is an equality because $U_a \subset G_a$ by Corollary A.8.11 (forcing $\mathrm{Lie}(U_{G_a}(\lambda_a)) = \mathrm{Lie}(U_a)$). In particular, if $H \subset G$ is a smooth closed k-subgroup normalized by S then $U_a \bigcap H$ is smooth and connected (by Remark 2.1.11 and Proposition 2.1.8).

We call U_a the *root group of G associated to* $a \in \mathrm{X}(S) - \{0\}$ (even if $a \notin {}_k\overline{\Phi}$, or $a \notin \Phi(G, S)$!). The U_a of most interest are for $a \in \Phi(G, S)$, but in some later considerations with general connected smooth affine k-groups it is convenient to allow the possibility $a \notin \Phi(G, S)$. The above dynamic description of U_a is compatible with any extension of the ground field, so by using scalar extension to k_s we see that $(U_a)_{k_s}$ is normalized by $Z_G(S)(k_s)$ and hence U_a is normalized by $Z_G(S)$ for any $a \in \mathrm{X}(S) - \{0\}$. Since $U_a = U_{G_a}(\lambda_a)$, the root group U_a is a k-split smooth connected unipotent group (Proposition 2.1.10); this also follows from Lemma 3.3.8. When G is pseudo-reductive and pseudo-split with S a k-split maximal k-torus in G, the above recovers Definition 2.3.13 for $a \in \Phi(G, S)$.

Proof of Theorem C.2.15. The Bruhat decomposition (Theorem C.2.8) implies axiom (T1), with $P(k) \cap N(k) = Z_G(S)(k)$. By Theorem C.2.13, the quotient group $N(k)/(P(k) \cap N(k)) = {}_kW$ is generated by R. This is axiom (T2). To prove axiom (T4), for $r = r_a \in R$ (with

$a \in {}_k\Delta \subset {}_k\overline{\Phi} \subset \Phi(G,S)$) it suffices to prove that $rP(k)r \neq P(k)$. Clearly rPr contains $rU_ar = U_{r(a)} = U_{-a}$, so it suffices to prove that $U_{-a}(k)$ is not contained in $P(k)$.

By applying Proposition 3.3.10 to the semigroup A consisting of negative integral multiples of a we see that $U_{-a} \cap P$ is a smooth connected k-subgroup of the k-split unipotent group U_{-a}. By Lemma 3.3.8, $U_{-a} \cap P$ is also k-split. To show that it is a proper k-subgroup of U_{-a}, note that $-a$ is a root of the maximal pseudo-reductive quotient $\overline{G} = G/\mathscr{R}_{u,k}(G)$ but not of the image $\overline{P}$ of P in $\overline{G}$. Thus, consideration of the S-equivariant surjection $\mathrm{Lie}(G) \to \mathrm{Lie}(\overline{G})$ implies that $\mathrm{Lie}(U_{-a})$ is not contained in $\mathrm{Lie}(P)$, so indeed $U_{-a} \cap P \neq U_{-a}$.

For the verification of (T4), it now suffices to show that if $U \hookrightarrow U'$ is an inclusion between k-split smooth connected unipotent k-groups and $U(k) = U'(k)$ then $U = U'$. Note that U and U' are affine spaces since both of them are connected k-split unipotent groups. Thus, the case of infinite k is settled by Zariski-density considerations. If k is finite of cardinality q then we may instead argue by dimension, since $\#U(k) = q^{\dim U}$ and $\#U'(k) = q^{\dim U'}$.

It remains to prove axiom (T3). That is, for $r := r_a$ (with $a \in {}_k\Delta$) and $w \in {}_kW = W(G,S)(k)$, we claim that

$$(*) \qquad rP(k)\{w, rw\}P(k) \subseteq P(k)\{w, rw\}P(k).$$

We first reduce the proof of $(*)$ to the case where G is pseudo-reductive. The natural homomorphism $W(G,S) \to W(\overline{G},\overline{S})$ is a k-isomorphism (Lemma 3.2.1), so from the Bruhat decomposition (Theorem C.2.8) we see that $\pi : G \to \overline{G}$ maps any two distinct double cosets of $P(k)$ in $G(k)$ into two distinct double cosets of $\overline{P}(k)$ in $\overline{G}(k)$. Hence, it suffices to prove $(*)$ with G replaced by the maximal pseudo-reductive quotient $\overline{G}$. Accordingly, we assume now that G is pseudo-reductive, so ${}_k\overline{\Phi} = \Phi(G,S)$ and we shall denote this as ${}_k\Phi$. As we will see below, the proof of $(*)$ for reductive groups given in [Bo2, p. 237] works for pseudo-reductive groups.

Let $\lambda : \mathrm{GL}_1 \to S$ be a 1-parameter k-subgroup such that $P = P_G(\lambda) = Z_G(\lambda) \ltimes U_G(\lambda)$, so by Proposition C.2.4 and the minimality of P we have $Z_G(\lambda) = Z_G(S) =: Z$ and $U_G(\lambda) = \mathscr{R}_{us,k}(P)$ (since $Z_G(\lambda)$ is pseudo-reductive and $U_G(\lambda)$ is k-split). Thus, ${}_k\Phi^+ = \Phi(P,S) = ({}_k\Phi)_{\lambda \geqslant 0} = ({}_k\Phi)_{\lambda > 0}$, so $\langle a, \lambda \rangle > 0$. More generally, a k-root $b \in {}_k\Phi$ is *positive* (i.e., it lies in ${}_k\Phi^+$) if and only if $\langle b, \lambda \rangle > 0$. If necessary, replacing w with rw, we may (and we will) assume that $w^{-1}(a)$ is positive.

Now let $U = \mathscr{R}_{us,k}(P) = U_G(\lambda)$, so for $G_a := Z_G(\ker a)^0$ the root group U_a is equal to $U_{G_a}(\lambda) = G_a \cap U$. Let $S_a = (\ker a)^0_{\mathrm{red}}$. As a is nondivisible in $\Phi(G,S)$ (since $a \in {}_k\Delta$), an element of $\Phi(G,S)$ is a rational multiple of a if and only if it is an integral multiple of a. Thus, the containment $G_a \subseteq Z_G(S_a)$ between smooth connected k-groups is an equality via comparison of their Lie algebras. Choose a 1-parameter k-subgroup $\lambda_a : \mathrm{GL}_1 \to S_a$ such that $\langle b, \lambda_a \rangle > 0$ for all $b \in {}_k\Delta - \{a\}$. Note that $\langle a, \lambda_a \rangle = 0$, so $G_a = Z_G(\lambda_a)$. Let $V = U_G(\lambda_a)$, so $P_G(\lambda_a) = Z_G(\lambda_a) \ltimes U_G(\lambda_a) = G_a \ltimes V$ and hence $U = U_a \ltimes V$. We also noted above that $P = Z \ltimes U$ for $Z := Z_G(S)$. Since G_a contains Z as well as a representative of $r = r_a$ (by definition of r_a), it follows that V is normalized by r and hence,

$$rP(k)\{w, rw\}P(k) = Z(k)rU(k)\{w, rw\}P(k) = Z(k)V(k)rU_a(k)\{w, rw\}P(k).$$

Thus,

$$rP(k)\{w, rw\}P(k) \subseteq P(k)G_a(k)wP(k).$$

For the pseudo-parabolic k-subgroup $P_a := P_{G_a}(\lambda) = G_a \cap P = Z \ltimes U_a$ of G_a, the Bruhat decomposition (Theorem C.2.8) for $G_a(k)$ with respect to $P_a(k)$ implies that

$$G_a(k) = Z(k)U_a(k)\{1, r\}U_a(k),$$

so

$$rP(k)\{w, rw\}P(k) \subseteq P(k)\{1, r\}U_a(k)wP(k).$$

But the positivity of $w^{-1}(a)$ implies that $w^{-1}U_aw \subset P$, so

$$rP(k)\{w, rw\}P(k) \subseteq P(k)\{w, rw\}P(k).$$

□

Remark C.2.17 Since ${}_kW = W({}_k\overline{\Phi})$ acts simply transitively on the set of minimal pseudo-parabolic k-subgroups of G containing S, if P is such a k-subgroup corresponding to a positive system of roots ${}_k\overline{\Phi}^+$ then for a suitable $w_0 \in {}_kW$ the k-subgroup $w_0 \circ P$ corresponds to the positive system of roots $-{}_k\overline{\Phi}^+$. Now assume that S commutes with $\mathscr{R}_{u,k}(G)$; i.e., $\mathscr{R}_{u,k}(G) \subset Z_G(S)$. Then $P = P_G(\lambda) = Z_G(\lambda) \ltimes U_G(\lambda)$ for some $\lambda \in \mathrm{X}_*(S)$ (by Theorem C.2.3), and $w_0 \circ P = P_G(-\lambda)$. Thus, $P \cap w_0 \circ P = Z_G(\lambda) = Z_G(S) =: Z$ (by considering the weights of the S-action on Lie algebras), so $Z(k) \subseteq \bigcap_{w \in {}_kW} wP(k)w^{-1} \subseteq P(k) \cap w_0P(k)w_0^{-1} = Z(k)$; i.e., $Z(k) = \bigcap_{w \in {}_kW} wP(k)w^{-1}$. For $N := N_G(S)$ we have $Z(k) = P(k) \cap N(k)$ by the Bruhat decomposition of $G(k)$ relative to the minimal P (Theorem C.2.8), so if $Z_G(S)$ contains $\mathscr{R}_{u,k}(G)$ then $\bigcap_{n \in N(k)} nP(k)n^{-1} = P(k) \cap N(k)$; i.e., in such cases the Tits system in Theorem C.2.15 is saturated (in the sense of [Bou2, IV, Exer. 2.5]). Since $P(k)$ is a semidirect product of its subgroup $Z(k) = P(k) \cap N(k)$ against the nilpotent subgroup $U_G(\lambda)(k)$, the Tits system is split.

We will now prove the following generalization of Proposition 3.4.2.

Proposition C.2.18 *Let G be a smooth connected affine k-group, S a maximal k-split torus in G, $N = N_G(S)$, and $Z = Z_G(S)$. Let ${}_k\overline{\Phi}$ be the root system of G with respect to S. Assume that no nonzero weight of S on* $\mathrm{Lie}(\mathscr{R}_{u,k}(G))$ *is an integral multiple of a root in ${}_k\overline{\Phi}$. Choose $a \in {}_k\overline{\Phi}$ and let $U_{\pm a}$ be the root groups associated to $\pm a$.*

If $2a \notin {}_k\overline{\Phi}$ then U_a is commutative and if moreover $\mathrm{char}(k) = p > 0$ *then U_a is p-torsion. If $2a \in {}_k\overline{\Phi}$ then $(U_a, U_a) \subset U_{2a}$ and U_{2a} is contained in the center of U_a.*

For a nontrivial element $u \in U_a(k)$, the following properties are satisfied:

- (*i*) *There exist unique $u', u'' \in U_{-a}(k)$ such that $m(u) := u'uu''$ normalizes S. The action of $m(u)$ on $\mathrm{X}(S)$ is the reflection r_a, and moreover $u', u'' \neq 1$.*
- (*ii*) *For any field extension K of k and any $z \in Z_G(S)(K)$, if zuz^{-1} lies in $U_a(k)$, then $zu'z^{-1}, zu''z^{-1} \in U_{-a}(k)$ and $m(zuz^{-1}) = zm(u)z^{-1}$.*
- (*iii*) *If a is a non-multipliable root then $u' = u''$, $u' = m(u)^{-1} \cdot u \cdot m(u)$, and $m(u)^2 \in S(k)$.*

It is not generally true, even when G is reductive (and non-split), that $u' = u''$.

Proof. Let G_a be the identity component of the centralizer of $\mathscr{S}_a = \ker a$ in G, so G_a contains $U_{\pm a}$ and its k-unipotent radical $\mathscr{R}_{u,k}(G_a)$ is equal to $(\mathscr{R}_{u,k}(G) \cap G_a)^0$ (Proposition A.8.14(2)). Since no nonzero weight of S on the Lie algebra of $\mathscr{R}_{u,k}(G)$ is an integral multiple of a, we conclude that $\mathscr{R}_{u,k}(G_a) \subset Z_{G_a}(S)$. But $Z_G(S) \subseteq G_a$, so we may (and do) replace G with G_a. Now ${}_k\Delta = \{a\}$, ${}_k\overline{\Phi} = \Phi(G, S)$, the Weyl group ${}_kW$ is a group of order 2 generated by r_a, and $P := Z \ltimes U_{-a}$ is a minimal pseudo-parabolic k-subgroup of G.

Since ${}_k\overline{\Phi}$ is a root system, the only positive integral multiple of a, other than a, which can be in ${}_k\overline{\Phi}$ is $2a$. By Proposition 3.3.5, if $2a \in {}_k\overline{\Phi}$ then $(U_a, U_a) \subset U_{2a}$ and U_{2a} is contained in the center of U_a; likewise, if $2a \notin {}_k\overline{\Phi}$ then U_a is commutative. Furthermore, if $\mathrm{char}(k) = p > 0$ and $2a \notin {}_k\overline{\Phi}$ then the commutative group U_a is p-torsion by Example 3.3.3.

Now consider a nontrivial element $u \in U_a(k)$. Let $P' = r_a P r_a = Z \ltimes U_a$ be the pseudo-parabolic k-subgroup "opposite" to P. By the Bruhat decomposition (Theorem C.2.8),

$$G(k) = P(k) \cup P(k) r_a P(k) = P(k) \cup U_{-a}(k)(N(k) - Z(k))U_{-a}(k).$$

Since $U_a(k) \cap P(k)$ is trivial, u must lie in $U_{-a}(k)(N(k) - Z(k))U_{-a}(k)$, and therefore we can find $u', u'' \in U_{-a}(k)$ and $n \in N(k) - Z(k)$ such that $u = u'^{-1} n u''^{-1}$. Then $m(u) := u'uu'' = n \in N(k)$ and it maps onto r_a in the Weyl group ${}_kW$.

To see that u' and u'' are unique, we first note that for any $u', u'' \in U_{-a}(k)$ such that $n' := u'uu'' \in N(k)$, necessarily $n' \notin Z(k)$ since otherwise we would have $u = u'^{-1} n' u''^{-1} \in U_{-a}(k)Z(k) = P(k)$, a contradiction because $u \in U_a(k)$ and $U_a \cap P = 1$. We have arranged that $G = G_a$, so n' must represent r_a. Since any two representatives of r_a in $N(k)$ coincide modulo $Z(k)$, the uniqueness of u' and u'' is reduced to the assertion that $U_{-a}(k)Z(k)U_a(k)$ is a direct product set (via multiplication) inside of $G(k)$. This in turn is a special case of Proposition 2.1.8(3). To prove that $u', u'' \neq 1$, we just have to show that no $n \in N(k) - Z(k)$ can have the form $u'u$ or uu' for $u' \in U_{-a}(k)$. If $u'u = n$ then $u' = nu^{-1} = (nu^{-1}n^{-1})n \in U_{-a}(k)n$, forcing $n \in U_{-a}(k)$, an absurdity since $u \notin U_{-a}(k)$; the case $uu' = n$ is likewise ruled out, so we have proved assertion (i).

To prove assertion (ii), consider $z \in Z_G(S)(K)$ such that zuz^{-1} lies in $U_a(k)$. Then $m(zuz^{-1}) = u'_z(zuz^{-1})u''_z$ for some $u'_z, u''_z \in U_{-a}(k)$, so

$$z^{-1}m(zuz^{-1})z = z^{-1}u'_z z \cdot u \cdot z^{-1}u''_z z = (z^{-1}u'_z z \cdot u'^{-1})m(u)(u''^{-1} \cdot z^{-1}u''_z z).$$

Hence,

$$(m(u)^{-1} \cdot z^{-1}m(zuz^{-1})z)(u''^{-1} \cdot z^{-1}u''_z z)^{-1} = m(u)^{-1}(z^{-1}u'_z z \cdot u'^{-1})m(u).$$

The element on the left side of this equality lies in $P(K)$, whereas the element on the right side lies in $U_a(K)$. Therefore, the elements on both sides are 1. This implies that $zu'z^{-1} = u'_z$, $zu''z^{-1} = u''_z$ and $m(zuz^{-1}) = zm(u)z^{-1}$, so we have proved (ii).

We will now prove assertion (iii), so consider a non-multipliable root a. The root group U_a is commutative for such a, and if $\mathrm{char}(k) = p > 0$ then U_a is also p-torision. If $\mathrm{char}(k) = 2$ then the elements u', u and u'' are of order 2, so $u''uu' = m(u)^{-1}$ and hence from the uniqueness of u' and u'' we infer that $u' = u''$. This implies that $m(u)^{-1} = m(u)$, so $m(u)^2 = 1$. Now assume $\mathrm{char}(k) \neq 2$, so there exists $s \in S(\overline{k})$ such that $a(s) = -1$.

Conjugating the equation $u'uu'' = m(u)$ by s, we obtain $u'^{-1}u^{-1}u''^{-1} = sm(u)s^{-1}$, which in turn implies that $u''uu' = sm(u)^{-1}s^{-1}$. From the uniqueness of u' and u'' we deduce that $u' = u''$, so $sm(u)^{-1}s^{-1} = m(u)$. Therefore, $m(u)^2 = m(u)sm(u)^{-1} \cdot s^{-1} \in S(k)$. Allowing $\mathrm{char}(k)$ to be arbitrary, since $m(u) = u'uu'$ we see that

$$u' \cdot (m(u)^{-1}u'm(u)) \cdot (m(u)^{-1}um(u)) = m(u)$$

with $m(u)^{-1}u'm(u) \in U_a(k)$ and $m(u)^{-1}um(u) \in U_{-a}(k)$ because $m(u)$ normalizes S and acts as r_a on $\mathrm{X}(S)$. Thus, $m(m(u)^{-1}u'm(u)) = m(u)$ and $u' = m(u)^{-1}um(u)$. □

A subset Ψ of $\Phi(G,S)$ is *saturated* if the subsemigroup A of $\mathrm{X}(S)$ spanned by Ψ does not contain 0 and $\Psi = A \cap \Phi(G,S)$. If $\Phi := \Phi(G,S)$ is a root system (for example, if $\mathscr{R}_{u,k}(G) \subset Z_G(S)$), Φ^+ a positive system of roots in Φ, and $\Psi \subseteq \Phi^+$ a closed subset, then Ψ is saturated by Proposition 2.2.7. However, we will be interested in allowing for the possibility that $\Phi(G,S)$ is not a root system, so saturatedness will be a more useful concept below than closedness.

Particularly interesting examples of saturated subsets of $\Phi(G,S)$ are obtained as follows. Let $\lambda : \mathrm{GL}_1 \to S$ be a 1-parameter k-subgroup. Then

$$\Phi(G,S)_{\lambda>0} := \{a \in \Phi(G,S) \mid \langle a, \lambda\rangle > 0\}$$

is a saturated subset of $\Phi(G,S)$. Another example of a saturated subset which will be of interest to us is the following. Let $a, b \in \Phi(G,S)$ be two linearly independent elements, and let (a,b) denote the set of elements of $\Phi(G,S)$ of the form $ma + nb$ for positive integers m and n. Clearly (a,b) is a saturated subset.

We seek to construct a smooth connected unipotent k-subgroup U_Ψ attached to any saturated Ψ, and to describe it as a direct span (in any order) of suitable root groups. However, if $\Phi(G,S)$ is not a root system then we cannot expect a direct spanning result to hold when using only root groups U_c attached to $c \in \Psi$. The problem is that there may be distinct elements in a saturated Ψ that are not divisible in Ψ but share a common positive integral multiple in Ψ; e.g., perhaps $c \in \mathrm{X}(S) - \{0\}$ with $2c, 3c, 6c \in \Psi$ but $c \notin \Psi$. (This problem cannot arise if $\Phi(G,S)$ is a root system, as then the only possible non-trivial linear dependence relations in Ψ are of the form $c' = 2c''$.) In such a situation, the root group U_c contains U_{nc} for $n = 2, 3, 6$ where U_{2c} and U_{3c} each contain U_{6c}. In general, a *half-line* in $\mathrm{X}(S)_{\mathbf{Q}}$ is the set of $\mathbf{Q}_{>0}$-multiples of a nonzero element, and if $\{c_i\}$ is a non-empty subset of $\Phi(G,S)$ lying in a half-line L of $\mathrm{X}(S)_{\mathbf{Q}}$ (so the codimension-1 tori $(\ker c_i)^0_{\mathrm{red}}$ coincide) then their *greatest common divisor* is the unique $c \in \mathrm{X}(S)$ such that $c_i = n_i c$ with integers $n_i > 0$ satisfying $\gcd(n_i) = 1$.

Proposition C.2.19 *Let G be a smooth connected affine group over a field k, S a maximal k-split k-torus in G.*

1. *For any saturated subset Ψ of $\Phi(G,S)$, there is a unique S-stable smooth connected unipotent k-subgroup U_Ψ such that* $\mathrm{Lie}(U_\Psi)$ *is the span of the subspaces* $\mathrm{Lie}(U_c)$ *for $c \in \Psi$. It is normalized by $Z_G(S)$, and directly spanned in any order by the subgroups*

$U_c \cap U_\Psi$ for the greatest common divisors c of the non-empty intersections of Ψ with half-lines in $\mathrm{X}(S)_{\mathbf{Q}}$.

(2) *For any pseudo-parabolic k-subgroup P of G containing S, the subset Ψ of ${}_k\overline{\Phi} \subseteq \Phi(G,S)$ (via the identification $\mathrm{X}(S) = \mathrm{X}(\overline{S})$) consisting of $a \in {}_k\overline{\Phi}$ such that $-a \notin {}_k\overline{\Phi}_P := \Phi(P/\mathscr{R}_{u,k}(G), \overline{S})$ is saturated, and $\mathscr{R}_{u,k}(P) = U_\Psi \mathscr{R}_{u,k}(G)$ for any such Ψ.*

(3) *A pseudo-parabolic k-subgroup P of G is minimal if and only if all k-split k-tori in $P/\mathscr{R}_{u,k}(P)$ are central.*

Before we prove this result, we record its relation with earlier results in the pseudo-split pseudo-reductive case. Part (1) applied to pseudo-split pseudo-reductive G recovers Corollaries 3.3.12 and 3.3.13(1), and if $\Phi(G,S)$ is a root system then the greatest common divisors that occur in the direct spanning description are precisely the non-divisible elements of Ψ. Part (2) applied to pseudo-split pseudo-reductive G recovers Proposition 3.5.1(2) and the direct spanning description in Corollary 3.3.16 for k-unipotent radicals of pseudo-parabolic k-subgroups of such G. Finally, part (3) for pseudo-split pseudo-reductive G recovers Propositions 3.3.15(3) and 3.5.1(4) because if H is a pseudo-reductive k-group with a central maximal k-torus then it is commutative (Proposition 1.2.3).

Proof. To prove assertion (1), let A be the subsemigroup of $\mathrm{X}(S)$ generated by Ψ. Then as Ψ is saturated, $0 \notin A$ and $A \cap \Phi(G,S) = \Psi$. Let $U_\Psi = H_A(G)$, where the subgroup $H_A(G)$ is as in Proposition 3.3.6. Since $0 \notin A$, U_Ψ is a smooth connected unipotent k-subgroup (Proposition 3.3.6). To prove that $Z_G(S)$ normalizes U_Ψ, we note that the description of $H_A(G)$ makes sense without reference to the maximality of S as a k-split torus in G, and is compatible with any extension of the ground field. Thus, using scalar extension to k_s implies that $(U_\Psi)_{k_s}$ is normalized by $Z_G(S)(k_s)$, and hence U_Ψ is normalized by $Z_G(S)$. The rest of assertion (1) follows easily from Theorem 3.3.11 applied to the smooth connected solvable k-group $U_\Psi = H_A(G)$ equipped with its natural action by S (taking as the disjoint union decomposition $\Psi = \coprod \Psi_i$ where the Ψ_i are the non-empty intersections of Ψ with half-lines in $\mathrm{X}(S)_{\mathbf{Q}}$).

To prove assertion (2), we fix a $\lambda \in \mathrm{X}_*(S)$ such that $P = P_G(\lambda)\mathscr{R}_{u,k}(G)$. (Such a λ exists by Theorem C.2.3 applied to P.) By Corollary 2.1.9, for $\Psi := \Phi(G,S)_{\lambda>0}$ the overlap $\Psi \cap {}_k\overline{\Phi}$ is as desired, so to prove the rest of (2), we can (and do) replace G with the pseudo-reductive quotient $G/\mathscr{R}_{u,k}(G)$ and replace Ψ with the set of $a \in \Phi(G,S)$ such that $-a \notin \Phi(P,S)$. Now $P = P_G(\lambda)$ and $\mathscr{R}_{u,k}(P) = U_G(\lambda)$. Since $U_\Psi = U_G(\lambda)$ for $\Psi = \Phi(G,S)_{\lambda>0}$, this completes the proof of (2).

By Remark 3.5.6 and Proposition 3.5.8 (and Proposition 2.2.10), to prove assertion (3) we can rename $P/\mathscr{R}_{u,k}(P)$ as G to reduce to showing that if G is pseudo-reductive then it contains no proper pseudo-parabolic k-subgroup if and only if its k-split k-tori are all central. This equivalence is Lemma 2.2.3(1). □

If we impose the additional hypothesis on the pair (G,S) in Proposition C.2.19 that $\mathscr{R}_{u,k}(G) \subset Z_G(S)$ (equivalently, S acts trivially on $\mathrm{Lie}(\mathscr{R}_{u,k}(G))$), so $\Phi(G,S) = {}_k\overline{\Phi}$ and hence $\Phi(G,S)$ is a root system, then there is an "open cell" description of G (recovering

Corollary 3.3.16 when G is pseudo-split and pseudo-reductive): for any positive system of roots ${}_k\overline{\Phi}^+$ in the k-root system, the multiplication map

$$U_{{}_k\overline{\Phi}^+} \times Z_G(S) \times U_{-{}_k\overline{\Phi}^+} \to G$$

is an open immersion. Indeed, the equality $\Phi(G,S) = {}_k\overline{\Phi}$ allows us to carry over the proof of Corollary 3.3.16 by using Proposition C.2.19(1) in place of Corollary 3.3.12.

The next proposition generalizes Corollary 3.3.13(2) and is immediate from the following ingredients: Proposition 3.3.5, the definition of the root groups U_a ($a \in \mathrm{X}(S) - \{0\}$), and the construction of U_Ψ in Proposition C.2.19(1) for the saturated subset $\Psi = (a,b)$ in $\Phi(G,S)$ with independent $a, b \in \Phi(G,S)$.

Proposition C.2.20 *Let G be a smooth connected affine group over a field k, S a maximal k-split torus. For any linearly independent $a, b \in \Phi(G,S)$, let (a,b) be the set of elements of $\Phi(G,S)$ of the form $ma + nb$, with m, n positive integers. Then*

$$(U_a, U_b) \subseteq U_{(a,b)} = \prod_i (U_{c_i} \cap U_{(a,b)}),$$

where the direct spanning is taken with respect to the enumeration $\{c_i\}$ in any order of the greatest common divisors of the non-empty intersections of (a,b) with half-lines in $\mathrm{X}(S)_{\mathbf{Q}}$. In particular, if (a,b) is empty, then U_a and U_b commute.

When $\Phi(G,S)$ is a root system, the c_i in Proposition C.2.20 are precisely the non-divisible elements of (a,b).

In general, it is natural to wonder if the reduced root system ${}_k\overline{\Phi}'$ consisting of non-multipliable roots in the k-root system ${}_k\overline{\Phi}$ coincides with the root system of a split connected reductive k-subgroup of G containing S as a maximal torus. The following theorem provides an affirmative answer under some hypotheses on the S-action on $\mathrm{Lie}(\mathscr{R}_{u,k}(G))$. For (possibly non-split) connected reductive G it recovers [Thm. 7.2, BoTi1], and for pseudo-split pseudo-reductive G it recovers part of Theorem 3.4.6. For reductive G, the proof given below is simpler than the proof of [Thm. 7.2, BoTi1].

Theorem C.2.21 *Let G be a smooth connected affine group over an infinite field k, and S a maximal k-split torus in G. Assume $S \neq 1$. Let ${}_k\overline{\Phi}$ be the root system of G with respect to S, and let ${}_k\overline{\Phi}'$ be the root system of non-multipliable roots in ${}_k\overline{\Phi}$. Let Δ' be a basis of ${}_k\overline{\Phi}'$. Assume that no nonzero weight of S on $\mathrm{Lie}(\mathscr{R}_{u,k}(G))$ is an integral multiple of a root in ${}_k\overline{\Phi}$, and that U_a commutes with U_{-b} for all distinct $a, b \in \Delta'$.*

For each $a \in \Delta'$, let E_a be a smooth connected 1-dimensional k-subgroup of U_a that is normalized by S (i.e., E_a is a k-linear subspace of U_a of dimension 1). There is a unique k-split connected reductive k-subgroup F of G containing S and every E_a. The root system of F with respect to S is ${}_k\overline{\Phi}'$.

In the formulation of the theorem, we are invoking the consequence of Proposition C.2.18 that (since $2a \notin {}_k\overline{\Phi}$, as $a \in {}_k\overline{\Phi}'$) U_a is commutative and moreover is p-torsion when $\mathrm{char}(k) =$

$p > 0$. The k-group U_a admits a unique linear structure that linearizes the S-action (use Theorem B.4.3 when $\mathrm{char}(k) > 0$). It is also helpful to remember that if G is not pseudo-reductive, ${}_k\overline{\Phi}$ is generally smaller than the set $\Phi(G, S)$ of nontrivial S-weights on $\mathrm{Lie}(G)$. We also note that by Proposition C.2.20, for distinct $a, b \in \Delta'$ the subgroups U_a and U_{-b} commute when $\Phi(G, S)$ does not contain any elements of the form $ma - nb$, with positive integers m and n. This occurs, for example, when $\Phi(G, S) = {}_k\overline{\Phi}$.

Proof. For each $a \in \Delta'$, we fix $u_a \in E_a(k) - \{1\}$. By Proposition C.2.18(iii), there is a unique $v_a \in U_{-a}(k)$ such that $n_a := m(u_a) = v_a u_a v_a$ normalizes S, and the induced action of n_a on $\mathrm{X}(S)$ is the reflection r_a in a. This proposition also gives that $n_a^2 \in S(k)$ and $v_a = n_a^{-1} u_a n_a$. The 1-dimensional smooth connected k-subgroup $E_{-a} := n_a^{-1} E_a n_a$ of U_{-a} is stable under the conjugation action of S, so it is a 1-dimensional k-linear subgroup. Clearly $v_a \in E_{-a}(k)$.

We claim that the k-subgroup E_{-a} does not depend on the choice of u_a. Since S acts on $E_a \simeq \mathbf{G}_{\mathrm{a}}$ through $a \neq 1$, if u'_a is another choice then $u'_a = s u_a s^{-1}$ for some $s \in S(\overline{k})$. Thus, Proposition C.2.18(ii) yields that $n'_a := m(u'_a) = s n_a s^{-1}$, so

$$(E'_{-a})_{\overline{k}} := {n'_a}^{-1}(E_a)_{\overline{k}}\, n'_a = s n_a^{-1} s^{-1} (E_a)_{\overline{k}}\, s n_a s^{-1} = n_a^{-1}(n_a s n_a^{-1} s^{-1})(E_a)_{\overline{k}}\,(s n_a s^{-1} n_a^{-1}) n_a.$$

Since $n_a s n_a^{-1} s^{-1} \in S(\overline{k})$ and E_a is normalized by S, the right side is $n_a^{-1}(E_a)_{\overline{k}} n_a = (E_{-a})_{\overline{k}}$. Thus, the k-subgroups E'_{-a} and E_{-a} coincide over $\overline{k}$, so they are equal over k.

STEP 1. Let F_a be the smooth connected k-subgroup of G generated by S and $E_{\pm a}$. We will first prove that F_a is a k-split connected reductive k-subgroup of G whose root system with respect to the maximal torus S is $\{\pm a\}$. The key point is to give a concrete subgroup of $F_a(k)$ that is Zariski-dense in F_a. Since $n_a = v_a u_a v_a \in F_a(k)$, the subset

$$\Gamma_a = E_a(k)\{1, n_a\} S(k) E_a(k)$$

is contained in $F_a(k)$. We will now prove that Γ_a is a subgroup of $F_a(k)$ that is Zariski-dense in F_a. It is clear that Γ_a contains 1 and is stable under inversion in $G(k)$ (as n_a normalizes S and $n_a^2 \in S(k)$), so we just have to show that Γ_a is stable under multiplication. Since it is stable under left and right multiplication by $E_a(k)$ and $S(k)$, and $n_a^2 \in S(k)$, to prove that Γ_a is a subgroup it suffices to show that $n_a E_a(k) n_a^{-1}$ is contained in Γ_a.

By the transitivity of the conjugation action of $S(\overline{k})$ on $E_a(\overline{k}) - \{1\}$, each nontrivial $u \in E_a(k)$ has the form $u = s u_a s^{-1}$ for some $s \in S(\overline{k})$. For such an s, $a(s) \in k^{\times}$ since $u, u_a \in E_a(k)$. Thus, the conjugation action of s on $E_{\pm a}$, and so also of $n_a s n_a^{-1}$ on $E_{\pm a}$, is k-rational. Now

$$n_a u n_a^{-1} = n_a s u_a s^{-1} n_a^{-1} = n_a s n_a^{-1} \cdot n_a u_a n_a^{-1} \cdot (n_a s n_a^{-1})^{-1},$$

so it suffices to prove that $n_a u_a n_a^{-1} \in E_a(k) n_a E_a(k)$. Indeed, the conjugation action of $n_a s n_a^{-1}$ keeps $E_a(k)$ stable (since it is a k-rational action), and the conjugate

$$c_s := n_a s n_a^{-1} \cdot n_a \cdot (n_a s n_a^{-1})^{-1}$$

of n_a under $n_a s n_a^{-1}$ is equal to the product $n_a \cdot s n_a s^{-1} n_a^{-1}$ that lies in $n_a S(k)$ (because, by Proposition C.2.18(ii), $s n_a s^{-1} = m(s u_a s^{-1}) = m(u) \in G(k)$, forcing $s n_a s^{-1} n_a^{-1} \in S(k)$).

The formula $n_a = v_a u_a v_a$ with $v_a := n_a^{-1} u_a n_a$ yields

$$n_a = n_a v_a n_a^{-1} \cdot n_a u_a n_a^{-1} \cdot n_a v_a n_a^{-1} = u_a \cdot n_a u_a n_a^{-1} \cdot u_a,$$

so $n_a u_a n_a^{-1} = u_a^{-1} n_a u_a^{-1} \in E_a(k) n_a E_a(k) \subseteq \Gamma_a$. This proves that Γ_a is a subgroup. Since $S(k)$, $E_a(k)$, and $n_a^{-1} E_a(k) n_a = E_{-a}(k)$ are Zariski-dense in S, E_a, and E_{-a} respectively, we conclude that Γ_a is Zariski-dense in F_a.

By Proposition A.8.10(1),(2), the scheme-theoretic centralizer of $\ker a$ in G is smooth with Lie algebra spanned by S-weight spaces in $\mathrm{Lie}(G)$ for the trivial weight and $\pm a$ (as a is non-multipliable in $\Phi(G,S)$). Thus, the identity component G_a of this centralizer is generated by $Z_G(S)$ and $U_{\pm a}$. The multiplication map

$$U_{-a} \times Z_G(S) \times U_a \longrightarrow G_a$$

is an open immersion (by Proposition 2.1.9(3), since $U_a = U_{G_a}(\lambda)$ for $\lambda \in \mathrm{X}_*(S)$ satisfying $\langle a, \lambda \rangle > 0$). Since $n_a^{-1} U_a n_a = U_{-a}$, the map

$$U_a \times Z_G(S) \times U_a \longrightarrow G_a$$

defined by $(u', z, u'') \mapsto u' n_a z u''$ is an isomorphism onto the open subscheme

$$\Omega_a := U_a n_a Z_G(S) U_a$$

of G_a. But $P_a := Z_G(S) \ltimes U_a$ is a minimal pseudo-parabolic k-subgroup of G_a containing S, so the Bruhat decomposition (Theorem C.2.8) gives that $\Omega_a(k) \cap P_a(k)$ is empty. In particular, the set $E_a(k) S(k) E_a(k)$ ($\subseteq P_a(k)$) is disjoint from $\Omega_a(k)$, so $\Gamma_a \cap \Omega_a(k) = E_a(k) n_a S(k) E_a(k)$.

The formation of closures is of local nature in any topological space, so by the Zariski-density of Γ_a in F_a we conclude that the subset $\Gamma_a \cap \Omega_a(k) = E_a(k) n_a S(k) E_a(k)$ is Zariski-dense in $F_a \cap \Omega_a$. The Zariski-closure of $E_a(k) n_a S(k) E_a(k)$ in Ω_a is clearly $E_a n_a S E_a$, so the open subscheme $F_a \cap \Omega_a$ of F_a is equal to $E_a n_a S E_a$. In particular, $\dim F_a = 2 + \dim S$. But $n_a \in F_a(k)$ and $E_{-a} = n_a^{-1} E_a n_a$, so

$$F_a \cap U_{-a} Z_G(S) U_a = F_a \cap n_a^{-1} \Omega_a = n_a^{-1}(F_a \cap \Omega_a) = n_a^{-1} E_a n_a S E_a = E_{-a} S E_a.$$

Since $U_{-a} Z_G(S) U_a$ is a direct product (as a scheme), we conclude that $F_a \cap Z_G(S) = S$ (hence S is a maximal torus of F_a), $F_a \cap U_a = E_a$, and $F_a \cap U_{-a} = E_{-a}$.

The derived group of any solvable smooth connected affine group H is unipotent, so in any such H the normalizer of a maximal torus is equal to its centralizer. Since $n_a \in F_a(k)$ normalizes S but does not centralize S, it follows that F_a is not solvable. Thus, the connected solvable codimension-1 subgroups $B_a := S \ltimes E_a$ and $B'_a := S \ltimes E_{-a}$ of F_a are Borel k-subgroups of F_a. Since $B_a \cap B'_a = S$ is a torus, we conclude that F_a is reductive. The root system of F_a with respect to S is clearly $\{\pm a\}$.

STEP 2. Let F be the smooth connected k-subgroup generated by the F_a for varying $a \in \Delta'$. By our hypothesis, for distinct $a, b \in \Delta'$ the root groups U_a and U_{-b} commute with each other, so E_a and E_{-b} commute with each other. The split k-torus S is maximal in each F_a, and we claim that S is maximal in F and that F is reductive (hence split). These two properties of F follow from [Thm. 5.4, St]. For the convenience of the reader, we give a self-contained presentation of Steinberg's argument, and also establish the uniqueness of F. To get started, we give a concrete "open cell" in F and analyze the S-weights on $\mathrm{Lie}(F)$.

For a smooth locally closed subscheme X of G, we will denote its geometrically reduced closure by $\overline{X}$. Let U (resp. V) be the smooth connected k-subgroup of F generated by the E_a (resp. E_{-a}) for $a \in \Delta'$. The subgroups U and V are normalized by S and by Proposition 3.3.5 the weights of S on $\mathrm{Lie}(U)$ (resp., $\mathrm{Lie}(V)$) are contained in the semigroup A^+ (resp., A^-) generated by Δ' (resp., $-\Delta'$). We are expecting $V \times S \times U$ to be an open subscheme of F via multiplication, so we are led to analyze the image C of the multiplication map $V \times SU \to F$. This image is an orbit of the group $V \times SU$ (as well as of the group $VS \times U$), so it is open in its closure $\overline{C}$. We will now show that $\overline{C} = F$, so C is open in F. We first show by induction on $r \geqslant 0$ that for any sequence of elements $a, a_1, \dots, a_r \in \Delta'$,

$$(*) \qquad E_aE_{-a_1}\cdots E_{-a_r} \subseteq \overline{C}.$$

For $r = 0$, this is obvious. Now consider $r > 0$. If $a \neq a_1$ then $E_aE_{-a_1} = E_{-a_1}E_a$, whereas if $a = a_1$ then $E_aE_{-a_1} \subset F_a = \overline{E_{-a}SE_a}$. Thus, in both cases $(*)$ follows from the induction assumption. We have $E_aV \subset \overline{C}$ by $(*)$ since $V = E_{-a_1}\cdots E_{-a_r}$ for some sequence $a_1, \dots, a_r \in \Delta'$. It follows that $E_a \cdot \overline{C} \subset \overline{C}$, and clearly $E_{-a} \cdot \overline{C} \subset \overline{C}$ and $S \cdot \overline{C} \subset \overline{C}$. Since S and the subgroups E_a and E_{-a} ($a \in \Delta'$) generate F, we conclude that $F \cdot \overline{C} \subset \overline{C}$, so $\overline{C} = F$. As noted above, this implies that C is an open subscheme of F.

Now we fix a 1-parameter k-subgroup $\lambda : \mathrm{GL}_1 \to S$ such that $\langle a, \lambda \rangle > 0$ for all $a \in \Delta'$, and $Z_G(\lambda) = Z_G(S)$. Clearly $U \subseteq U_F(\lambda)$, $V \subseteq U_F(-\lambda)$, and $Z_F(\lambda) = Z_F(S)$. Moreover, the multiplication map

$$U_F(-\lambda) \times Z_F(S) \times U_F(\lambda) \longrightarrow F$$

is an open immersion (Proposition 2.1.8(3)); let Ω_F denote its image. Since $C = VSU$ ($\subseteq \Omega_F$) is an open subscheme of F, C is dense in Ω_F. But U, S, and V are closed in $U_F(\lambda)$, $Z_F(S)$, and $U_F(-\lambda)$ respectively, so $C = VSU$ is closed in Ω_F. We conclude that $VSU = C = \Omega_F = U_F(-\lambda)Z_F(S)U_F(\lambda)$, so $U = U_F(\lambda)$, $V = U_F(-\lambda)$, and $S = Z_F(S)$. The last equality implies that S is a maximal torus of F. Since the multiplication map

$$V \times S \times U \longrightarrow F$$

is an open immersion, $\mathrm{Lie}(F) = \mathrm{Lie}(V) + \mathrm{Lie}(S) + \mathrm{Lie}(U)$. Therefore, the nonzero weights of S on $\mathrm{Lie}(F)$ are all contained in $A^+ \cup A^-$.

Step 3. We will now prove that F is reductive. The subgroup $\mathscr{N}$ of $F(k)$ generated by the elements n_a ($a \in \Delta'$) maps onto the Weyl group ${}_kW$ of the root system ${}_k\overline{\Phi}'$. Hence, the set of S-weights on $\mathrm{Lie}(F)$ and the set of $S_{\overline{k}}$-weights on $\mathrm{Lie}(\mathscr{R}_u(F_{\overline{k}}))$ are stable under ${}_kW$. Let $w = r_{a_s}\cdots r_{a_1}$ (with $a_1, \dots, a_s \in \Delta'$) be the longest element of ${}_kW$, so $n = n_{a_s}\cdots n_{a_1}$ is a representative of w in $\mathscr{N}$. The element w is of order 2 and $w(\Delta') = -\Delta'$, so $w(A^+) = A^-$ and $nUn^{-1} = V$. Now we observe (using the definition of r_a and the fact that Δ' is a linearly independent set) that for $a \in \Delta'$, $r_a(A^+) \cap A^-$ is precisely the set of negative integral multiples of a. Also, by our assumption that no nonzero S-weight on $\mathrm{Lie}(\mathscr{R}_{u,k}(G))$

is an integral multiple of a root in ${}_k\overline{\Phi}$, for any $a \in \Delta'$ the only positive integral multiple of a which is a weight on $\mathrm{Lie}(G)$ (or on $\mathrm{Lie}(F)$) is a.

Assume that $\mathscr{R}_u(F_{\overline{k}}) \neq 1$; we seek a contradiction. Since $Z_F(S) = S$, every weight of $S_{\overline{k}}$ on the Lie algebra of $\mathscr{R}_u(F_{\overline{k}})$ is nonzero, so it belongs to $A^+ \cup A^-$. As the set of $S_{\overline{k}}$-weights on $\mathrm{Lie}(\mathscr{R}_u(F_{\overline{k}}))$ is stable under the Weyl group, there is an $S_{\overline{k}}$-weight of this Lie algebra, say a, which belongs to A^+. Let $w_0 = 1$ and for all positive integers $j \leqslant s$ define $w_j = r_{a_j} \cdots r_{a_1}$, so $w = w_s$. Note that for all $j \leqslant s$, $w_j(a)$ is an $S_{\overline{k}}$-weight on $\mathrm{Lie}(\mathscr{R}_u(F_{\overline{k}}))$, so it lies in $A^+ \cup A^-$. Moreover, $w_0(a) = a \in A^+$ and $w_s(a) \in A^-$. Let $j_0 \leqslant s$ be the smallest positive integer such that $w_{j_0}(a)$ lies in A^-, and let $a_0 = w_{j_0 - 1}(a)$. Then $a_0 \in A^+$ and $r_{a_{j_0}}(a_0) \in A^-$. Therefore, a_0 is a positive integral multiple of the simple root a_{j_0}. But the only positive integral multiple of a_{j_0} which is an S-weight on $\mathrm{Lie}(G)$ is a_{j_0}, so $a_0 = a_{j_0} \in \Delta'$.

We shall next prove that $(E_{a_0})_{\overline{k}} \subset \mathscr{R}_u(F_{\overline{k}})$. Once this is proved, then as $(E_{a_0})_{\overline{k}} \subset (F_{a_0})_{\overline{k}}$, it would follow that $(E_{a_0})_{\overline{k}} \subseteq \mathscr{R}_u((F_{a_0})_{\overline{k}}) = 1$ (by the reductivity of F_{a_0}). This evident contradiction would then imply that F is reductive. To establish the containment $(E_{a_0})_{\overline{k}} \subset \mathscr{R}_u(F_{\overline{k}})$ we note that since (i) a_0 is a $S_{\overline{k}}$-weight on $\mathrm{Lie}(\mathscr{R}_u(F_{\overline{k}}))$, (ii) no positive integral multiple of a_0 other than a_0 is a weight on $\mathrm{Lie}(G)$, and (iii) the weight space in $\mathrm{Lie}(F_{\overline{k}})$ for the weight a_0 is contained in $\mathrm{Lie}(U_{\overline{k}})$, by Proposition 3.3.6 (applied to the group $\mathscr{R}_u(F_{\overline{k}})$ and the semigroup $A\,(\subset \mathrm{X}(S))$ consisting of positive integral multiples of a_0) there is a unique (nontrivial) smooth connected $\overline{k}$-subgroup of $\mathscr{R}_u(F_{\overline{k}})$ which is normalized by $S_{\overline{k}}$ and whose Lie algebra is the weight space in $\mathrm{Lie}(\mathscr{R}_u(F_{\overline{k}}))$ for the weight a_0, and this subgroup is contained in $U_{\overline{k}}$ (since $U = H_{A^+}(F)$, using the notation of Proposition 3.3.6). The same reasoning shows that $U_{\overline{k}}$ contains a unique nontrivial smooth connected subgroup $\mathscr{U}_{a_0}$ that is maximal for the properties of being normalized by $S_{\overline{k}}$ and having a_0 as the unique $S_{\overline{k}}$-weight on its Lie algebra. We will show that $\mathscr{U}_{a_0}$ equals the 1-dimensional subgroup $(E_{a_0})_{\overline{k}}$. This will clearly imply that $(E_{a_0})_{\overline{k}} \subset \mathscr{R}_u(F_{\overline{k}})$.

Now to show that $\mathscr{U}_{a_0} = (E_{a_0})_{\overline{k}}$, we fix a 1-parameter k-subgroup $\lambda_0 : \mathrm{GL}_1 \to S$ such that $\langle a_0, \lambda_0 \rangle = 0$ and $\langle a, \lambda_0 \rangle > 0$ for all $a \in \Delta' - \{a_0\}$. Consideration of the Lie algebra shows that $\mathscr{U}_{a_0}$ is centralized by λ_0. The conjugation action

$$(t, u) \mapsto \lambda_0(t) u \lambda_0(t)^{-1}$$

of λ_0 on U extends to a map $f : \mathbf{A}^1_k \times U \to U$, and we claim that $\mathscr{U}_{a_0} = f(\{0\} \times U)_{\overline{k}}$. Clearly $f(\{0\} \times U)_{\overline{k}}$ is a smooth connected subgroup of $U_{\overline{k}}$ that is normalized by $S_{\overline{k}}$, and all $S_{\overline{k}}$-weights on its Lie algebra must lie in A^+ (since the same holds for the S-weights on $\mathrm{Lie}(U)$). Since $f(\{0\} \times U)$ is centralized by λ_0, it follows that any $S_{\overline{k}}$-weight on its Lie algebra is a positive rational multiple of a_0 that lies in A^+. The only such weight is a_0, so $f(\{0\} \times U)_{\overline{k}} \subseteq \mathscr{U}_{a_0}$. But $f(\{0\} \times U)_{\overline{k}}$ contains the centralizer for the λ_0-action on $U_{\overline{k}}$, so we get the reverse inclusion $\mathscr{U}_{a_0} \subseteq f(\{0\} \times U)_{\overline{k}}$. Therefore, $\mathscr{U}_{a_0} = f(\{0\} \times U)_{\overline{k}}$.

Since $U = E_{a_1} \cdots E_{a_n}$ for some sequence $a_1, \ldots, a_n \in \Delta'$, and $f(\{0\} \times E_{a_i}) = 1$ for all $a_i \neq a_0$ whereas $f(\{0\} \times E_{a_0}) = E_{a_0}$, we see that $f(\{0\} \times U) \subseteq E_{a_0}$. Thus $\mathscr{U}_{a_0} \subseteq (E_{a_0})_{\overline{k}}$, and hence $\mathscr{U}_{a_0} = (E_{a_0})_{\overline{k}}$. We have seen above that this leads to a contradiction and so completes the proof that F is reductive.

Step 4. To prove that the root system of F with respect to the k-split maximal k-torus S is ${}_k\overline{\Phi}'$, first note that $\Phi(F, S)$ clearly contains Δ', so it contains ${}_kW \cdot \Delta' = {}_k\overline{\Phi}'$. To establish the reverse containment, choose $a \in \Phi(F, S)$ and let $U_a \simeq \mathbf{G}_{\mathrm{a}}$ be the corresponding root group in F. The map $h : F \to \overline{G} := G/\mathscr{R}_{u,k}(G)$ has unipotent kernel that must be infinitesimal (due to the reductivity of F), and it restricts to the natural isomorphism $S \simeq \overline{S}$ onto the maximal k-split torus $\overline{S}$ in $\overline{G}$. Thus, h carries $U_a = \mathbf{G}_{\mathrm{a}}$ onto an $\overline{S}$-stable $\mathbf{G}_{\mathrm{a}}$ in $\overline{G}$. Since this isogeny between $\mathbf{G}_{\mathrm{a}}$'s is equivariant with respect to the isomorphism $S \simeq \overline{S}$, if $p \geqslant 1$ is the characteristic exponent of k then $p^r a \in \Phi(\overline{G}, \overline{S}) = {}_k\Phi$ for some $r > 0$. Thus, a is a rational multiple of an element of ${}_k\overline{\Phi}'$. Since $\Phi(F, S)$ is reduced (as F is reductive and k-split), the containment ${}_k\overline{\Phi}' \subseteq \Phi(F, S)$ must therefore be an equality.

Now only the uniqueness of F remains to be proved. Suppose H is a split connected reductive k-subgroup of G that contains S and E_a for every $a \in \Delta'$. Note that S must be a maximal k-torus in H, since H is k-split and S is maximal as a k-split torus in G. Choose $a \in \Delta'$. From the uniqueness assertion in Proposition C.2.18(i) (along with the existence assertion applied to (H, S, a)), the element $n_a = m(u_a)$ introduced at the beginning of this proof must lie in H. Hence, the open subscheme $E_a n_a S E_a$ of F_a is contained in H, so F_a is contained in H for all $a \in \Delta'$. Therefore, $F \subseteq H$, so ${}_k\overline{\Phi}' \subseteq \Phi(H, S)$. The above proof that $\Phi(F, S) = {}_k\overline{\Phi}'$ may now be applied with H in place of F to establish that $\Phi(H, S) = {}_k\overline{\Phi}'$. Hence, F and H have common dimension $\#{}_k\overline{\Phi}' + \dim S$, so $H = F$ by connectedness. □

In the following proposition, for simplicity, we will use the phrase "k-pseudo-simple" in place of the phrase "pseudo-simple over k".

Proposition C.2.22 *Let G be a pseudo-reductive k-group, S a maximal k-split torus in G, and ${}_k\Phi$ the k-root system of G with respect to S. There is a natural bijective correspondence between the set of irreducible components of ${}_k\Phi$ and the set of k-isotropic k-pseudo-simple normal subgroups of G.*

Proof. It follows from Theorem C.2.3 that S contains a maximal k-split torus of any smooth normal k-subgroup of G. Now we record some consequences of Proposition 1.2.6. Any k-pseudo-simple subgroup of G is perfect, and so is contained in $\mathscr{D}(G)$ (which is itself perfect). Also, the k-root system of G coincides with that of $\mathscr{D}(G)$ with respect to the maximal k-split torus of the latter contained in S. Thus, we may (and do) replace G with $\mathscr{D}(G)$ to arrange that G is perfect.

Let $\{N_i\}_{i \in I}$ be the set of k-pseudo-simple normal subgroups of G. According to Proposition 3.1.8, the subgroups N_i pairwise commute and the product homomorphism

$$\pi : N = \prod_{i \in I} N_i \longrightarrow G$$

is surjective with central kernel which contains no nontrivial smooth connected k-subgroup. For $i \in I$, let S_i be the (unique) maximal k-split torus of N_i contained in S. Let A be the set of $i \in I$ such that N_i is anisotropic over k. Then $S_i = 1$ precisely for $i \in A$, and hence

S is an isogeneous quotient of the maximal k-split torus $\prod_{i \in I-A} S_i$ of N. This implies that for every $i \in A$, $N_i \subseteq Z_G(S)$ and the k-root system of N_i is empty. Moreover, π induces an injective homomorphism of $\mathrm{X}_*(\prod_{i \in I-A} S_i) = \bigoplus_{i \in I-A} \mathrm{X}_*(S_i)$ into $\mathrm{X}_*(S)$ whose image is a subgroup of finite index.

For any $\lambda \in \bigoplus_{i \in I-A} \mathrm{X}_*(S_i)$, the restriction of π to $U_N(\lambda)$ is an isomorphism onto $U_G(\pi \circ \lambda)$ since the kernel of π is central (forcing $\ker \pi \subseteq Z_N(\lambda)$). Hence, the set of root groups of G relative to S is the (disjoint) union of the set of root groups of N_i relative to S_i over all $i \in I - A$. Thus, ${}_k\Phi$ is the direct sum of the k-root systems $\Phi(N_i, S_i)$ for $i \in I - A$. It therefore suffices to show that for $i \in I - A$, the k-root system $\Phi(N_i, S_i)$ of N_i is irreducible. That is, we can replace G with a k-isotropic N_{i_0} (and S with S_{i_0}) to reduce to the case that G is k-pseudo-simple and k-isotropic. In this case it suffices to prove that the non-empty root system ${}_k\Phi$ is irreducible.

Let Φ_0 be an irreducible component of ${}_k\Phi$, and let G_0 be the nontrivial smooth connected k-subgroup generated by the root groups U_a for $a \in \Phi_0$. Then G_0 is clearly normalized by $Z_G(S)$, and by Proposition C.2.20 it is normalized by the root groups U_b for all $b \in {}_k\Phi$ and even centralized by the U_b for $b \notin \Phi_0$. It follows that G_0 is a normal subgroup of G. Since G_0 is nontrivial, smooth, and connected, the k-pseudo-simplicity of G implies that $G_0 = G$. This forces ${}_k\Phi = \Phi_0$ because if there exists $b \in {}_k\Phi - \Phi_0$ then the nontrivial smooth connected k-group U_b centralizes $G_0 = G$, contradicting that G is k-pseudo-simple. $\square$

We close this section with the following theorem.

Theorem C.2.23 *Let k be an infinite field, G a k-isotropic pseudo-simple k-group, S a maximal k-split torus of G, and P a minimal pseudo-parabolic k-subgroup of G containing S. Let $U = \mathscr{R}_{us,k}(P)$, and let $G(k)^+$ be the normal subgroup of $G(k)$ generated by $U(k)$.*

(i) Any infinite normal subgroup of $G(k)$ is Zariski-dense in G. In particular, $G(k)^+$ is Zariski-dense in G and the center C of $G(k)^+$ is central in G.

(ii) The abstract group $G(k)^+$ is perfect.

(iii) A subgroup of $G(k)$ that is normalized by $G(k)^+$ either contains $G(k)^+$ or is contained in the center of $G(k)$. In particular, the abstract group $G(k)^+/C$ is simple.

Proof. Since G is k-pseudo-simple, it does not contain a proper normal smooth connected k-subgroup. In particular, G is perfect and hence (by Proposition A.2.11) $G(k)$ is Zariski-dense in G. But then a normal subgroup of $G(k)$ is either finite and hence central in G, or is infinite and hence Zariski-dense in G. This proves (i).

Let ${}_k\Phi^+ (\subset {}_k\Phi)$ be the positive system of k-roots determined by the minimal pseudo-parabolic k-subgroup P, so $U_a \subseteq U$ for all $a \in {}_k\Phi^+$. Let P' be the minimal pseudo-parabolic k-subgroup containing S that corresponds to the positive system of k-roots $-{}_k\Phi^+$, and let $U' = \mathscr{R}_{us,k}(P')$. There exists $g \in N_G(S)(k)$ such that $P' = gPg^{-1}$, so $U' = gUg^{-1}$ and hence $U'(k) \subseteq G(k)^+$.

Fix $a \in {}_k\Phi^+$, and define $S_a = (\ker a)^0_{\mathrm{red}}$. For a nontrivial $u \in U_a(k)$, let $m(u) = u'uu''$ be as in Proposition C.2.18(i). The action of $m(u)$ on $\mathrm{X}(S)$ is the reflection r_a in a, so the conjugation action of $m(u)$ on S is trivial on S_a and the induced action on S/S_a is by inversion. For $s \in S(k)$, clearly $m(sus^{-1}) = sm(u)s^{-1}$, so the conjugation action of $m(sus^{-1})m(u)^{-1} = s \cdot m(u)s^{-1}m(u)^{-1}$ on U_a is the conjugation action of s^2. Since u' and u'' lie in $U'(k)$, $m(u)$ and $m(sus^{-1}) = sm(u)s^{-1}$ belong to $G(k)^+$. Thus, for $s \in S(k)$ the conjugation action of s^2 on U_a equals the conjugation action of an element of $G(k)^+$. From this it follows (via the infinitude of k, and the description of the S-action on suitable vector group subquotients of U_a over k) that the derived group of $G(k)^+$ contains $U_a(k)$ for all $a \in {}_k\Phi^+$. But these $U_a(k)$ generate $U(k)$ (due to the direct spanning property in Proposition C.2.19(1), where the greatest common divisors are actually roots since $\Phi(G, S)$ is a root system), so the derived group of $G(k)^+$ contains $U(k)$. Since $G(k)^+$ is the normal subgroup of $G(k)$ generated by $U(k)$, we conclude that $G(k)^+$ is perfect. Thus, we have proved (ii).

Since G is k-pseudo-simple, by Proposition C.2.22 the k-root system ${}_k\Phi$ is irreducible. Thus, when the k-Weyl group $({}_kW)(k) = W({}_k\Phi)$ is viewed as a Coxeter group in the natural way, its associated Coxeter graph is connected (due to the relationship between the Dynkin diagram of a root system and the Coxeter graph of the corresponding Weyl group; see [Bou2, VI, §4.2]). Also, $P = Z_G(S) \ltimes U$ and U is unipotent, so $P(k) = Z_G(S)(k) \ltimes U(k)$ and $U(k)$ is solvable. Now (iii) follows from [Bou2, IV, §2.7, Thm. 5] provided that the normal subgroup $Z := \bigcap_{g \in G(k)} gP(k)g^{-1}$ in $G(k)$ is the center $\mathscr{C}$ of $G(k)$. Since the opposite pseudo-parabolic k-subgroup P' is $G(k)$-conjugate to P, Z lies in $P(k) \cap P'(k) = Z_G(S)(k)$, so Z is a normal subgroup of $G(k)$ that is not Zariski-dense. Hence, (i) implies that Z is finite and normalized by G, so it is central. On the other hand, $\mathscr{C}$ lies in $Z_G(S)(k)$ and hence in $P(k)$. Thus, $\mathscr{C}$ lies in every $gP(k)g^{-1}$ $(g \in G(k))$ and hence in Z. This proves that $Z = \mathscr{C}$. $\square$

p. 513: On line -10, replace V^S with V_S. On lines -8 through -10, remove the final sentence "Its boundary ... this fact)." in the paragraph preceding Definition C.3.3. On line -4 replace "depends on the choice of" with "does not depend on the", and on line -3 insert ", as it says $U \subseteq U_G(\lambda)$" immediately to the right of "G".

p. 516: On line 12, replace "such" with "special", and on lines -6, -5, replace the expression "automorphisms: ${}^\gamma Z := k_s \otimes_{\gamma, k_s} Z \subseteq \gamma^*(G_{k_s}) = G(k_s)$." with the expression "automorphisms, which is to say,

$${}^\gamma Z := k_s \otimes_{\gamma, k_s} Z \subseteq k_s \otimes_{\gamma, k_s} G_{k_s} \simeq G_{k_s}$$

as k_s-schemes" (using a displayed expression as indicated above).

p. 516: The statement of Theorem C.3.8 should be replaced with the following new statement, along with the subsequent sentence given below:

Theorem C.3.8 *Let G be a pseudo-reductive group over a field k, H a smooth closed k-subgroup, and U a k-split smooth connected unipotent k-subgroup normalized by H. There exists a pseudo-parabolic k-subgroup P of G containing H such that $U \subseteq \mathscr{R}_{us,k}(P)$.*

Note that we do not require H to be connected.

p. 517: Replace γ^* with γ everywhere on this page: once on each of lines 10, 19, 20, 21, and twice on line 24. Also, on line -10, insert "unipotent" at the end of the line.

Immediately after the first sentence in the last paragraph of the proof of Theorem C.3.8 insert the following text:

As H normalizes U, if S is a special torus then for every $h \in H(k_s)$ the torus $S_h := hSh^{-1}$ is also special, and the optimal 1-parameter k_s-subgroup λ_S is carried to the optimal 1-parameter subgroup λ_{S_h} under conjugation by h. Therefore, Q is normalized by $H(k_s)$, so the Zariski-density of $H(k_s)$ in H_{k_s} implies that H_{k_s} normalizes Q. Then by Proposition 3.5.7, $H_{k_s} \subseteq Q$.

Replace the last sentence of the proof of Theorem C.3.8 with the following:

The containments $H_{k_s} \subseteq Q = P_{k_s}$ and $U_{k_s} \subseteq \mathscr{R}_{us,k_s}(Q) = \mathscr{R}_{us,k}(P)_{k_s}$ then imply that H is contained in P and U is contained in $\mathscr{R}_{us,k}(P)$.

p. 518: At the end of line 4, append the following:

Corollary C.3.9 implies that when this k-anisotropicity condition holds, the pseudo-reductive k-group G does not contain $\mathbf{G}_{\mathrm{a}}$ as a k-subgroup. Equivalently, by Proposition B.3.2, in such k-anisotropic cases every smooth connected unipotent k-subgroup U in G is k-wound in the sense of Definition B.2.1. Over imperfect fields it can happen, even in the connected semisimple case, that such nontrivial k-wound U exist.

For example, let k be a local function field of characteristic $p > 0$ and D a central division algebra over k of rank p^2. By local class field theory such D exist and the field $k' = k^{1/p}$ embeds into D over k. This identifies $\mathrm{R}_{k'/k}(\mathrm{GL}_1)$ as a k-subgroup of the algebraic unit group $\underline{D}^{\times}$ over k, so the k-anisotropic adjoint connected semisimple quotient $\underline{D}^{\times}/\mathrm{GL}_1$ contains the $(p-1)$-dimensional commutative smooth connected k-group $U := \mathrm{R}_{k'/k}(\mathrm{GL}_1)/\mathrm{GL}_1$ as a k-subgroup. The k-group U is unipotent since it is p-torsion, and Example B.2.8 gives a direct proof that U is k-wound.

Even with the additional requirement that G is simply connected (in addition to being connected, semisimple, and k-anisotropic) it can happen that G contains a nontrivial smooth connected (k-wound) unipotent k-subgroup. Examples of such G are given in §6.4 of the paper *Formules pour l'invariant de Rost* by P. Gille and A. Quéguiner-Mathieu.

p. 519: On the second line of section C.4 (the line beginning with "is valid more generally"), remove the word "connected".

p. 520: On line 9, replace "k-algebras A" with "k-algebras A of finite type". On line 13, replace "maximal ideals $\mathfrak{m}$ of A" with "points of Spec(A)". In the second paragraph, on line -3 remove "aspect".

p. 522: 5 lines above Proposition C.4.5, replace "commutator" with "commutator map". On line -2, replace "start" with "beginning".

p. 526: Insert the following new bibliography entries:

[EKM] R. Elman, N. Karpenko, A. Merkurjev, *The algebraic and geometric theory of quadratic forms*, AMS Colloq. Publ. **56**, Providence, 2008.

[St] R. Steinberg, *The isomorphism and isogeny theorems for reductive algebraic groups*, J. Algebra **216** (1999), 366–383."

p. 527: For the index entry for $\Phi(G,T)$, replace "68" with "xx, 68". Insert index entries for "$\Phi^+_>, \Phi^+_<$" (see just below Remark 9.3.2), "${}_k\overline{\Phi}$" (see the fourth paragraph of C.2.11 above), and "$K_>, K_<$" (see end of Proposition 6.3.2, paragraph 2 of its proof, and item (iv) on p. 210).

To make the index entries appear in the correct places (among the list of symbols early in the index), the labels within the index command should be as follows: for "$\Phi^+_>, \Phi^+_<$" use the index label `␣aaafigp`, for "${}_k\overline{\Phi}$" use the index label `␣aaafibar`, and for "$K_>, K_<$ use the index label `␣aaakg`.

[These index labels include a single blank space at the start to force the index entries to appear among the list of symbols at the start of the index, and not later on in the index.]

p. 528: In the list of symbols, please insert a new entry $\underline{W}$ (immediately above the entry $W(G,T)$) and link it to Definition A.1.2.

p. 530: Insert index entry for "non-reduced root system" with references to Example 3.2.8, §9.2 up through and including 9.2.5, 9.3.10, §9.4 from 9.4.3 onwards, and §10.1 up through and including 10.1.6.

p. 532: The index entries "root system, standard pseudo-reductive group" and "standard pseudo-reductive group, root system" should be referring to Remark 3.2.9 (on whatever page that comes out to be with the next printing), not whatever it is tagged to on page 381 (which has nothing to do with that topic). So please move these index references from whatever they are doing on page 381 to instead refer to Remark 3.2.9.

p. 545: For [BoTi2], replace "conjugasion" with "conjugaison". For [Chev], replace "Appliqées" with "Appliquées".

* *

Dear Reader: The authors welcome your comments and corrections. Please send them to `gprasad@umich.edu` or to `conrad@math.stanford.edu`.