A formal power series is an infinite sum
\[ A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \]
where the coefficients lie in \( \mathbb{Q}, \mathbb{R} \) or in \( \mathbb{C} \) (or some other field, for example the field \( \mathbb{F}_p \) of integers modulo a prime \( p \)). Here \( x \) is just a formal variable. One can add, multiply power series. If 
\[ B(x) = b_0 + b_1 x + b_2 x^2 + \cdots \]
then
\[ A(x) + B(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots \]
and
\[ A(x)B(x) = a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots \]
If \( A(x) \) is a power series with \( A(0) = a_0 \neq 0 \), then we can also divide by \( A(x) \). If 
\[ B(x) = b_0 + b_1 x + b_2 x^2 + \cdots \]
then the equation
\[ A(x)B(x) = (a_0 + a_1 x + a_2 x^2 + \cdots)(b_0 + b_1 x + b_2 x^2 + \cdots) = 1 \]
translates to 
\[ a_0 b_0 = 1, \quad a_0 b_1 + a_1 b_0 = 0, \quad a_0 b_2 + a_1 b_1 + a_2 b_0 = 0 \]
and so forth, so we can solve \( b_0 = 1/a_0, \quad b_1 = -a_1 b_0/a_0, \quad b_2 = (-a_1 b_1 + a_2 b_0)/a_0 \) and so forth.

For a formal power series with complex coefficients \( A(x) \), and a complex number \( c \), \( A(c) \) may not make sense because the series
\[ a_0 + a_1 c + a_2 c^2 + \cdots \]
may not converge. For example the series
\[ A(x) = 0! + 1! x + 2! x^2 + 3! x^3 + \cdots \]
does not converge for \( x = c \) unless \( c = 0 \).

For a sequence \( a_0, a_1, a_2, \ldots \) we define its generating function as the power series
\[ A(x) = a_0 + a_1 x + a_2 x^2 + \cdots \]
Formal manipulation of the generating function \( A(x) \) can be used to prove results about the sequence \( a_0, a_1, \ldots \).

**Example 1.** Consider the Fibonacci sequence defined by \( F_0 = F_1 = 1 \) and \( F_{n+1} = F_n + F_{n-1} \) for \( n \geq 1 \). Let
\[ F(x) = F_0 x + F_1 x^2 + F_2 x^3 + \cdots \]
From the definition of the Fibonacci numbers follows that
\[ F(x)(1-x-x^2) = x, \]
so
\[ F(x) = \frac{x}{1-x-x^2}. \]
We can factor \((1 - x - x^2) = (1 - \lambda_1 x)(1 - \lambda_2 x)\) where \(\lambda_1, \lambda_2\) are the roots of \(x^2 - x - 1\). By the abc-formula we know that
\[
\lambda_1 = \frac{1 + \sqrt{5}}{2}, \quad \lambda_2 = \frac{1 - \sqrt{5}}{2}.
\]
So we have
\[
F(x) = \frac{x}{1 - x - x^2} = \frac{a_1}{1 - \lambda_1 x} + \frac{a_2}{1 - \lambda_2 x}.
\]
for some \(a_1, a_2 \in \mathbb{R}\) by Lemma 2 below. If we multiply both sides with \(1 - x - x^2\) we get
\[
x = a_1(1 - \lambda_2 x) + a_2(1 - \lambda_1 x)
\]
So we get \(a_1 + a_2 = 0\) and \(a_1\lambda_2 + a_2\lambda_1 = -1\). So \(a_1(\lambda_2 - \lambda_1) = -a_1\sqrt{5} = -1\). We find \(a_1 = \sqrt{5}\) and \(a_2 = -\sqrt{5}\). So we have
\[
F_0 x + F_1 x^2 + \cdots = F(x) = \frac{1}{\sqrt{5}(1 - \lambda_1 x)} - \frac{1}{\sqrt{5}(1 - \lambda_2 x)} = \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \lambda_1^n x^n - \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \lambda_2^n x^n = \sum_{n=0}^{\infty} \frac{\lambda_1^n - \lambda_2^n}{\sqrt{5}} x^n
\]
This shows that
\[
F_n = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\sqrt{5}}
\]
Lemma 2 (partial fractions). Suppose that \(a(x), b_1(x), b_2(x)\) are polynomials such that \(\deg(a(x)) < \deg(b_1(x)) + \deg(b_2(x))\) and \(\gcd(b_1(x), b_2(x)) = 1\). Then there exist polynomials \(a_1(x), a_2(x)\)
with \(\deg(a_1(x)) < \deg(b_1(x)), \deg(a_2(x)) < \deg(b_2(x))\) and
\[
\frac{a(x)}{b_1(x)b_2(x)} = \frac{a_1(x)}{b_1(x)} + \frac{a_2(x)}{b_2(x)}.
\]
We can also use generating functions for finite sequences. In this case, the generating function is a polynomial.

Example 3. For example the generating function for the sequence for
\[
\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}
\]
is
\[
\binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n} x^n = (1 + x)^n.
\]
If we compare the coefficient of \(x^n\) in \((1 + x)^n(1 + x)^n = (1 + x)^{2n}\), then we get
\[
\sum_{i=0}^{n} \binom{n}{i} \binom{n}{n - i} = \binom{2n}{n}.
\]
Since \(\binom{n}{n-i} = \binom{n}{i}\) we get
\[
\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}.
\]
Example 4. A triangulation of a convex \(n\)-gon is a partition of the area of the \(n\)-gon into triangles such that the vertices of each triangle is a vertex of the \(n\)-gon. How many distinct triangulations does a convex 10-gon have? How about a regular \(n\)-gon?
Let $A_n$ be the number of triangulations of an $n$-gon. Let us find the value of $A_n$ for small $n$.

We have $A_3 = 1$ and $A_4 = 2$:

$A_5 = 5$:

$I$

$A_6 = 14$:

It becomes more and more clear that it may not be feasible to write down all triangulations of an 10-gon. As $n$ gets larger, we need a more systematic way of counting the possibilities to make sure that we are not forgetting any case.

Let $P$ and $Q$ be two fixed adjacent vertices of the $n$-gon. For each triangulation, there is a unique vertex $R$ of the $n$-gon ($R \neq P, Q$) such that $PQR$ is a triangle in the triangulation. For example, for $n = 8$ there are the following cases:

For fixed $R$, the complement of the triangle $PQR$ within the $n$-gon is a union of an $m$-gon and a $(n + 1 - m)$-gon. The $m$-gon has $A_m$ triangulations, and the $(n + 1 - m)$-gon has
\(A_{n+1-m} \) triangulations. This gives \(A_mA_{n+1-m} \) triangulations for this particular choice of \(R\). From this we see the equation:

\[
A_n = A_2A_{n-1} + A_3A_{n-2} + \cdots + A_{n-1}A_2.
\]

where we define \(A_2 = 1\). In particular,

\[
\begin{align*}
A_7 &= 1 \cdot 14 + 1 \cdot 5 + 2 \cdot 2 + 5 \cdot 1 + 14 \cdot 1 = 42 \\
A_8 &= 1 \cdot 42 + 1 \cdot 14 + 2 \cdot 5 + 5 \cdot 2 + 14 \cdot 1 + 42 \cdot 1 = 132 \\
A_9 &= 1 \cdot 132 + 1 \cdot 42 + 2 \cdot 14 + 5 \cdot 5 + 14 \cdot 2 + 42 \cdot 1 + 132 \cdot 1 = 429 \\
A_{10} &= 1 \cdot 429 + 1 \cdot 132 + 2 \cdot 42 + 5 \cdot 14 + 14 \cdot 5 + 42 \cdot 2 + 132 \cdot 1 + 429 \cdot 1 = 1430.
\end{align*}
\]

If one defines \(C_n = A_{n+2} \) for all \(n \geq 2\), then \(C_n\) are the so-called Catalan numbers. The recurrence relation for \(A_n\) gives the following recurrence relation for \(C_n\):

\[
C_n = C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{n-1}C_0.
\]

Define \(C(x) = C_0 + C_1x + C_2x^2 + \cdots\). Then the left hand side is the coefficient of \(x^n\) in \(C(x)\) and the right hand side is the coefficient of \(x^n\) in \(xC(x)^2\). So we have \(C(x) - 1 = xC(x)^2\) or \(xC(x)^2 - C(x) + 1 = 0\). If we solve for \(C(x)\) we get

\[
C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}.
\]

We have

\[
\sqrt{1 + y} = \left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)y + \left(\frac{1}{2}\right)y^2 + \cdots
\]

where

\[
\binom{1}{k} = \frac{\frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2} \cdots \frac{3-2k}{2}}{k!} = (-1)^{k-1} \frac{(2k-2)!}{2^{2k-1}(k-1)!k!}.
\]

So

\[
\sqrt{1 - 4x} = \sum_{k=0}^{\infty} (-1)^{k-1} \frac{(2k-2)!}{2^{2k-2}(k-1)!} (-4)^k x^k = \sum_{k=0}^{\infty} -\frac{2^{2k-1}}{k} x^k
\]

and

\[
C(x) = \frac{1 - \sqrt{1 - 4x}}{2x} = \sum_{k=0}^{\infty} \frac{2^k}{k+1} x^k
\]

This proves that

\[
C_n = \binom{2n}{n+1}.
\]

So

\[
A_{10} = C_8 = \frac{\binom{16}{8}}{9} = 1430.
\]

**Exercise 1.** *Show that for \(n > 0\) we have

\[
\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots + \binom{n}{2\lfloor n/2 \rfloor} = 2^{n-1}.
\]
Exercise 2. *** Give and prove a simple formula for
\[ \binom{4n}{0} + \binom{4n}{4} + \binom{4n}{8} + \cdots \binom{4n}{4n}. \]

Exercise 3. ** Suppose that \(0 \leq m \leq n\). Show that
\[ \sum_{i=\max(0,k-n)}^{\min(k,m)} \binom{m}{i} \binom{n}{k-i} = \binom{m+n}{k}. \]

Exercise 4. *** Give and prove a simple formula for
\[ \sum_{i=0}^{n} \frac{\binom{n}{i}}{i+1}. \]

Exercise 5. ** Define a sequence \(a_0, a_1, a_2, \ldots\) by \(a_0 = 0\), \(a_1 = 1\) and \(a_{n+1} = 5a_n - 6a_{n-1}\). Give and prove a formula for \(a_n\).

Exercise 6. * What is
\[ \sum_{n=0}^{\infty} F_n \left( \frac{1}{2} \right)^n? \]

Exercise 7. **** Define a sequence \(b_0, b_1, b_2, \ldots\) by \(b_0 = 1\) and
\[ b_{n+1} = \binom{n}{0} b_0 + \binom{n}{1} b_1 + \cdots + \binom{n}{n} b_n. \]
Show that
\[ \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n = e^{e^x-1}. \]

Exercise 8. **** There are \(n+1\) lights labelled 0, 1, 2, \ldots, \(n\). In the beginning, all the lights are off except light 0; In every round we do the following: we switch light \(i\) from on to off or from off to on if light \(i - 1\) was on after the previous round, and light \(i\) stays the same if light \(i - 1\) was off after the previous round for all \(i \geq 1\). For which \(n\) is it true that after \(n\) rounds, all the lights are off, except lights 0 and \(n\)?

Exercise 9. **** In a normal die, the numbers on the sides are 1, 2, 3, 4, 5, 6. Suppose that we have two dice. Show that it is possible to relabel the sides of the two dice such that (1) Every side is labelled by a positive integer, (2) the labels are different from the labels of normal dice, i.e., the labels of each of the dice is not 1, 2, 3, 4, 5, 6 (3) the probabilities of the outcomes 2, 3, 4, 5, \ldots, 12 are exactly the same as for two normal dice.

Exercise 10. **** The Chebychev polynomials are defined by \(T_0(x) = 1\), \(T_1(x) = x\) and \(T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)\) for \(n \geq 1\). Give a simple formula for
\[ \sum_{n=0}^{\infty} T_n(x)y^n. \]
and give a formula for \(T_n(x)\).