Inverse degrees and the Jacobian conjecture

Harm Derksen
Mathematisches Institut
Universität Basel
Rheinsprung 21
CH-4051 BASEL
Switzerland
E-mail: hderksen@math.unibas.ch
(old address replaced)

Abstract

Bass, Connell and Wright proved that if the Jacobian Conjecture is true, then there exists a number $C(d)$ such that for every $k$-algebra $A$ and every invertible polynomial map $F : A^n \to A^n$ with $\det(J(F)) = 1$ the degree of $F^{-1}$ is bounded by $C(d)$. A year later Bass proved the converse. In this paper we give a short proof of this last result.

In this paper $k$ is a field of characteristic 0. If $F = (F_1, \ldots, F_n) : k^n \to k^n$ is a polynomial map, then we define the degree of $F$ as the maximum degree of the polynomials $F_1, \ldots, F_n$. This paper is about the equivalence of the following two conjectures:

**Conjecture 1** $JC(n)$: If $F : k^n \to k^n$ is a polynomial map with $\det(J(F)) = 1$ then $F$ is invertible, e.g. there exists a polynomial map $G : k^n \to k^n$ such that $G \circ F = F \circ G = id$.

This conjecture is known as the Jacobian conjecture (c.f. [3, 4, 6]), and it was first formulated by O.H. Keller in [5].

**Conjecture 2** $BI(n)$: For every finite dimensional $k$-algebra $A$ the following holds:
If $F : A^n \to A^n$ is an invertible polynomial map such that $\det(J(F)) = 1$, then the degree of $F^{-1}$ is bounded by a constant $C(d)$, depending only on $d$ (and $n$), and not on the choice of $A$.

If $A$ is a field, then one can take $C(d) = d^{n-1}$. This result is due to Gabber (c.f. [2, Ch. I, Cor. (1.4)]) if $A$ is a reduced ring, then it is easy to reduce it to the case that $A$ is a field. For a reduced ring $A$ we can also take $C(d) = d^{n-1}$ as the upper bound. If $A$ is allowed to have nilpotent elements then it is not known whether there can be given an upper bound for the degree of the inverse. In [2, Ch. I, Prop. (1.2)] Bass, Connell and Wright proved that $JC(n)$ implies $BI(n)$. Bass proved in [1] the converse. We will now give a short proof of this implication:

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Theorem 1 If the conjecture BI(n) is true then the jacobian conjecture JC(n) is true.

Proof: Assume BI(n) holds and assume that \( F : k^n \to k^n \) is a polynomial map satisfying \( \det(J(F)) = 1 \). Without loss of generality we may assume that \( F(0) = 0 \). Now \( F \) is locally invertible in the neighbourhood of 0, e.g. there exist \( G_1, G_2, \ldots, G_n \in k[[X_1, \ldots, X_n]] \) such that \( F_i(G_1, \ldots, G_n) = X_i \) for all \( i \). Write

\[
F = F_{(1)} + F_{(2)} + \ldots + F_{(d)} \text{ and } G = G_{(1)} + G_{(2)} + G_{(3)} + \ldots
\]

where \( F_{(i)} \) and \( G_{(i)} \) are homogeneous of degree \( 1 \) for all \( i \). Let us define \( \hat{F} : k[t]^n \to k[t]^n \) by

\[
\hat{F} = t^{-1}F(tX) = F_{(1)} + tF_{(2)} + \ldots + t^{d-1}F_{(d)}
\]

and likewise define

\[
\hat{G} = t^{-1}G(tX) = G_{(1)} + tG_{(2)} + t^2G_{(3)} + \ldots
\]

Since \( J(\hat{F}) = J(t^{-1}F(tX)) = J(F)(tX) \), we have \( \det(J(\hat{F})) = \det(J(F))(tX) = 1 \). Now \( \hat{G} \) is the formal inverse of \( \hat{F} \) because

\[
\hat{F} \circ \hat{G} = t^{-1}F(tt^{-1}G(tX)) = t^{-1}F(G(tX)) = t^{-1}tX = X
\]

Choose \( l > C(d) \) arbitrary. We now calculate modulo \( t^l \). Define \( \overline{F} : (k[t]/(t^l))^n \to (k[t]/(t^l))^n \) and \( \overline{G} \) to be \( \hat{F} \) respectively \( \hat{G} \) modulo \( t^l \). Again we get \( \det(J(\overline{F})) = 1 \) and \( \overline{F} \circ \overline{G} = X \). So \( \overline{G} \) is the inverse of \( \overline{F} \), hence the degree of \( \overline{G} \) is bounded by \( C(d) \). Since

\[
\overline{G} = G_{(1)} + tG_{(2)} + \ldots + t^{l-1}G_{(l)}
\]

it follows that \( t^{l-1}G_{(l)} = 0 \). The fact that \( t^{l-1} \neq 0 \) (in \( k[t]/(t^l) \)) forces \( G_{(l)} = 0 \). We can conclude that \( G_{(l)} = 0 \) for all \( l > C(d) \). So \( \overline{G} \) is also a polynomial map and this proves that \( F \) is invertible. \( \square \)

References


