1. **The Cheese Cutting Problem**

The following problem is known as the *Cheese cutting problem*:

**Question 1.** Suppose that we have a round piece of cheese and a knife. What is the largest number of pieces that we can get after \( n \) straight cuts?

Here we make various assumptions. Cuts are straight. This means that each cut can be seen as the intersection of a plane with the piece of cheese. We *do not* rearrange the pieces of cheese. After each cut all of the pieces stick together. For \( n = 1, 2, 3 \) the maximum number of cuts are 2, 4, 8 respectively. For \( n = 4 \) the maximum number is 15. We would like to derive a formula that works for all \( n \).

Instead of dividing a piece of cheese, let us see how \( n \) planes divided up the entire 3-dimensional space:

**Question 2.** Given \( n \) planes in three dimensional space. What is the maximum number of regions?

Instead of answering the problem in three dimensions, let us first consider a similar problem in 2 dimensions.

**Question 3.** Given \( n \) lines in the plane. What is the maximum number of regions?

To answer the last questions, let us make the following assumptions:

1. No two lines are parallel;
2. No three lines go through one point.

Let \( V \) be the number of vertices (intersection points), \( E \) the number of edges (line segments or rays), and \( R \) be the number of regions.

Each two lines intersect, therefore, there are \( V = n(n - 1)/2 \) intersection points (because there are \( n(n - 1)/2 \) pairs of lines). Each line intersects the other \( n - 1 \) lines. This means that each line contains \( n - 1 \) vertices. The \( n - 1 \) vertices split that line up in \( n \) edges (line segments or rays). Therefore, the total number of edges is \( n \cdot n = n^2 \).

Is the relation between \( V \), \( E \) and \( R \)? If there is, we will be able to find \( R \). We have seen that in the case were some polygon is divided up into other polygon, there is a relation between the numbers of vertices, edges and regions. So we will reduce to that case.

Choose a large circle \( C \) such that all intersection points lie within that circle. The circle intersects each of the lines in 2 points. We can connect all these new vertices to
get a $2n$-gon. We get an $2n$-gon divided up into smaller polygons. Let $V', E', R'$ be the new number of vertices, edges and regions. We have $V' = V + 2n$, $E' = E + 2n$ and $R' = R + 1$. We apply the formula

$$V' - E' + R' = 2.$$ 

So we have

$$(V + 2n) - (E + 2n) + (R + 1) = 1$$

and therefore

$$V - E + R = 1.$$ 

Since $V = n(n - 1)/2$, $E = n^2$, we get

$$R = 1 - V + E = 1 - \frac{1}{2}n(n - 1) + n^2 = 1 + \frac{1}{2}(n^2 + n).$$

2. Induction

Suppose we have $n$ lines in the plane. We have seen that under the assumption that no 2 lines are parallel and no three lines go through one points, then the number of regions is $1 + \frac{1}{2}n(n + 1)$. What if there are parallel lines, or what if three lines go through one point?

For example, if there are two parallel lines in the plane, then the number if regions is 3, not 4. If there are three lines in the plane which go through 1 point, then the number of regions is 6, not 7. It seems, therefore, that the number of regions is maximal if the lines are not parallel, and if no three lines go through one point. So we would like to prove the following theorem:

**Theorem 1.** If there are $n$ lines in the plane, then the number of regions is at most $1 + \frac{1}{2}n(n + 1)$.

To prove this theorem, we will use mathematical induction. Let us discuss this important mathematical tool.

Suppose that we want to prove that

"$P(n)$ is true for every positive integer $n$",

where $P(n)$ is a proposition (statement) which depends on a positive integer $n$. Proving $P(1)$, $P(2)$, $P(3)$, etc., would take an infinite amount of time. Instead we can use the so-called *induction principle*.

**Axiom 1.** Assume that $k$ is an integer and for each $n \geq k$, $P(k)$ is a proposition.

1. Suppose that $P(k)$ is true, and
2. for any integer $m \geq k$ for which $P(m)$ is true, $P(m + 1)$ is true.

Then $P(n)$ is true for all integers $n \geq k$.

The induction principle is often compared to an infinite sequence of dominos tiles, numbered 1,2,3, etc.
If the $m$-th domino tile falls, it will hit the $(m+1)$-th domino tile and the $(m+1)$-th domino tile will fall as well. If the first domino tile falls, then all domino tiles will fall down. (Here $P(n)$ is the statement: “the $n$-th domino tile falls down”)

Since the induction principle is intuitively clear, we will simply accept it without proof. This is why it is called an axiom. (We cannot formally prove the induction principle without making other, similar assumptions.)

A typical example of the induction principle is the following:

**Example 2.** Prove that

\[(1) \quad 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.\]

for every positive integer $n$.

Since (1) depends on a positive integer $n$, it is natural to try mathematical induction. Equation (1) is true for $n = 1$. Let us assume that (1) is true for $n = m$:

\[(2) \quad 1 + 2 + \cdots + m = \frac{m(m+1)}{2}.\]

Then we have to prove that

\[(3) \quad 1 + 2 + \cdots + m + (m+1) = \frac{(m+1)(m+2)}{2}.\]

We recognize that the left-hand side of (2) forms a substantial part of the left-hand side of (3). If we assume (2), then (3) is equivalent to

\[\frac{m(m+1)}{2} + (m+1) = \frac{(m+1)(m+2)}{2}.\]

This is easy to check. We now write down a formal proof.

**Proof.** We prove (1) by induction on $n$. For $n = 1$ we check that

\[1 = \frac{1 \cdot (1+1)}{2}.\]

Suppose that (1) is true for $n = m$. Then

\[1 + 2 + \cdots + m + (m+1) = (1 + 2 + \cdots + m) + (m+1) = \]

\[\frac{m(m+1)}{2} + (m+1) = \frac{(m+1)(m+2)}{2}.\]
so (1) is true for \( n = m + 1 \). Now (1) is true for all positive integers \( n \) by the induction principle.

**Remark 3.** When the German mathematician Carl Friedrich Gauss (1777–1855) was 10 years old, his school teacher gave the class an assignment to add all the numbers from 1 to 100. Gauss gave the answer almost immediately: 5050. This is how (we think) he did it: Write the numbers from 1 to 100 from left to right. Write under that the numbers from 1 to 100 in reverse order.

\[
\begin{array}{cccccc}
1 & 2 & 3 & \cdots & 100 \\
100 & 99 & 98 & \cdots & 1 \\
101 & 101 & 101 & \cdots & 101 \\
\end{array}
\]

Each of the 100 column sums is 101. This shows that

\[
2 \cdot (1 + 2 + \cdots + 100) = 100 \cdot 101
\]

and

\[
1 + 2 + \cdots + 100 = \frac{100 \cdot 101}{2} = 50 \cdot 101 = 5050.
\]

This easily generalizes to a proof of (1). Gauss’ proof can be graphically presented. For example, to see that

\[
2 \cdot (1 + 2 + \cdots + 10) = 10 \cdot 11,
\]

look at the following picture:

\[
\begin{array}{ccccccc}
1 & 10 & 9 & 8 & 7 & 6 & 5 \\
2 & \quad & \quad & \quad & \quad & \quad & \\
3 & \quad & \quad & \quad & \quad & \quad & \\
4 & \quad & \quad & \quad & \quad & \quad & \\
5 & \quad & \quad & \quad & \quad & \quad & \\
6 & \quad & \quad & \quad & \quad & \quad & \\
7 & \quad & \quad & \quad & \quad & \quad & \\
8 & \quad & \quad & \quad & \quad & \quad & \\
9 & \quad & \quad & \quad & \quad & \quad & \\
10 & \quad & \quad & \quad & \quad & \quad & \\
\end{array}
\]

A formula similar to (1) exists for the sums of squares, namely

\[
1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.
\]

**Example 4.** Prove that

\[
1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}
\]

for all positive integers \( n \).
Proof. We verify the base case $n = 1$:

$$1 + x = \frac{1 - x^2}{1 - x}.$$  

Suppose that the statement (5) is true for $n = m$, so

$$1 + x + x^2 + \cdots + x^{m+1} = \frac{1 - x^{m+1}}{1 - x}.$$  

Then we have

$$1 + x + x^2 + \cdots + x^{m+1} + x^{m+2} = \frac{1 - x^{m+1}}{1 - x} + x^{m+2} = \frac{1 - x^{m+1}}{1 - x} + \frac{x^{m+1} - x^{m+2}}{1 - x} = \frac{1 - x^{m+2}}{1 - x}.$$  

So (5) is true for $n = m + 1$.  

So by the induction principle, we have shown that (5) holds for all positive integers $n$. $\square$

We return to the proof of Theorem 1. We will use induction.

Proof of Theorem 1. We will show by induction on $n$ that

$$(6) \quad n \text{ lines cut up the plane in at most } 1 + \frac{1}{2}n(n + 1) \text{ regions.}$$

For $n = 0$, we have $1 + \frac{1}{2}0(0 + 1) = 1$. Indeed, 0 lines cut up the plane in 1 region.

Suppose that $m$ lines cut up the plane in at most $1 + \frac{1}{2}m(m + 1)$ regions. What happens if we add one more line? That line will intersect the other $m$ lines in at most $m$ points. This means that the $m$ lines cut the new line into at most $m + 1$ new edges. Each edge corresponds to a region that is cut into two regions by the new line. Therefore, there are at most $m + 1$ new regions if we add one more line. This shows that $m + 1$ lines cut up the plane in at most

$$1 + \frac{1}{2}(m^2 + m) + m + 1 = \frac{1}{2}m^2 + \frac{3}{2}m + 1 = 1 + \frac{1}{2}((m + 1)^2 + (m + 1))$$

regions. This is (6) for $n = m + 1$.

So by induction we have proven that (6) holds for all nonnegative integers $n$. $\square$

Theorem 5. Suppose that there are $n$ planes in 3-dimensional space. The maximum number of regions is

$$\frac{1}{6}n^3 + \frac{5}{6}n + 1.$$  

The maximum number of regions is obtained when no two planes are parallel, no three planes go through one line, no four planes go through one point, and every three planes intersect in a point.

Proof. The case $n = 0$ is easily verified.

Suppose that the theorem is true for $n = m$. We now add one more plane, $P$. The new plane intersects the $m$ planes in at most $m$ lines. These $m$ lines divide up the plane $P$ in at most $\frac{1}{2}m^2 + \frac{1}{2}m + 1$ regions. Each region in $P$ splits up one 3-dimensional

The problem does not display any graphical content or images. The content is purely textual and presented in a natural language format suitable for a human reader. There are no tables or diagrams embedded within the text. The text is formatted in a way that is consistent with typical academic or instructional writing, with proper use of mathematical notation and logical structure. The content is a continuation of the proof discussed in the previous pages, delving into additional steps to verify the base case and extending the induction principle to cover all positive integers $n$. The theorem introduces a formula for the maximum number of regions when $n$ planes are present in a 3-dimensional space, accompanied by a proof that adheres to standard mathematical reasoning. The theorem statement and its proof are presented in a clear, logical sequence, ensuring that each step follows logically from the previous one, maintaining the integrity of the mathematical argument. The text is well-structured, with proper use of mathematical symbols and logical connectives, making it accessible to readers familiar with the topic. The document continues to explore the implications of the induction principle, extending the domain of applicability to cover a broader range of cases, thereby enriching the understanding of the problem at hand.
region into two regions. This shows, that adding the \((m+1)\)-th plane will give at most 
\(\frac{1}{2}m^2 + \frac{1}{2}m + 1\) new regions.

This shows that for \(n = m + 1\) planes, there are at most
\[
\left(\frac{1}{6}m^3 + \frac{5}{6}m + 1\right) + \left(\frac{1}{2}m^2 + \frac{1}{2}m + 1\right) = \frac{1}{6}m^3 + \frac{1}{2}m^2 + \frac{4}{3}m + 2 = \frac{1}{6}(m + 1)^3 + \frac{5}{6}(m + 1) + 1
\]
regions.

Suppose, that for the \(m + 1\) planes, no two planes are parallel, no three planes go 
through one line, no four planes go through one point, and every three planes intersect 
in a point. By the induction hypothesis, we know that \(m\) planes divide the space up 
in exactly
\[
\frac{1}{6}m^3 + \frac{5}{6}m + 1
\]
regions. These \(m\) planes intersect plane \(P\) in \(m\) lines, and we see that no two of these 
lines are parallel, and no three of these lines go through one point. This shows that 
there are exactly \(\frac{1}{2}m^2 + \frac{1}{2}m + 1\) new regions after adding plane \(P\). So the total number 
of planes is
\[
\left(\frac{1}{6}m^3 + \frac{5}{6}m + 1\right) + \left(\frac{1}{2}m^2 + \frac{1}{2}m + 1\right) = \frac{1}{6}m^3 + \frac{1}{2}m^2 + \frac{4}{3}m + 2 = \frac{1}{6}(m + 1)^3 + \frac{5}{6}(m + 1) + 1
\]
Thus the theorem holds for \(n = m + 1\).

By induction, the theorem holds for all nonnegative integers \(n\). \(\square\)

We finally can answer the original cheese cutting problem:

**Theorem 6.** With \(n\) cuts, one can cut a round piece of cheese into at most
\[
\frac{1}{6}n^3 + \frac{5}{6}n + 1
\]
pieces. 

**Proof.** We know that \(n\) planes cut up space in \(R\) regions, where 
\[
R \leq \frac{1}{6}n^3 + \frac{5}{6}n + 1
\]
regions. We may view the piece of cheese as a ball in three dimensional space. If the 
ball intersects with each of the \(R\) regions, and it will if the ball is large enough, then 
there are exactly \(R\) pieces of cheese after the cutting. If the ball does not intersect all 
regions, then the number of pieces will be less. In any case, there are at most
\[
\frac{1}{6}n^3 + \frac{5}{6}n + 1
\]
pieces of cheese.

We can choose the \(n\) planes such that the number of regions is exactly
\[
\frac{1}{6}n^3 + \frac{5}{6}n + 1
\]
By scaling down we may assume that each region intersects the cheeseball. So it is 
possible to have exactly
\[
\frac{1}{6}n^3 + \frac{5}{6}n + 1
\]
pieces of cheese after \(n\) cuts. \(\square\)