1. DISTANCE

For two points $A, B$ in the plane, $|AB|$ is the distance between $A$ and $B$, or equivalently it is the length of the line segment $AB$. So $|AB|$ is always a nonnegative number. We have $|AB| = 0$ if and only if $A = B$.

**Axiom 1** (The triangle inequality). If $A, B, C$ are points in the plane then

$$|AC| \leq |AB| + |BC|.$$ 

Equality holds if and only if $A, B, C$ lie on a line and $B$ lies on the line segment $AB$.

**Exercise 1.** Prove the reverse triangle inequality: If $A, B, C$ are points in the plane, then

$$|AC| \geq ||AB| - |BC||.$$ 

(For a real number $x$, $|x|$ denotes its absolute value.)

**Axiom 2.** Given two triangles $\triangle ABC$ and $\triangle A'B'C'$ in the plane, the following three statements are equivalent:

1. $$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|CA|}{|C'A'|}.$$ 

2. $\angle CAB = \angle C'A'B'$ and $\angle ABC = \angle A'B'C'$.

3. $\angle CAB = \angle C'A'B'$ and $\frac{|AB|}{|A'B'|} = \frac{|AC|}{|A'C'|}$.
Two such triangles as in Axiom ?? are called *similar* and we denote this by \( \triangle ABC \cong \triangle A'B'C' \). Note that it is important how the vertices are labeled. The statement \( \triangle ABC \cong \triangle A'B'C' \) is *not* the same as \( \triangle ABC \cong \triangle B'C'A' \). However, the statements \( \triangle ABC \cong \triangle A'B'C' \), \( \triangle BCA \cong \triangle B'C'A' \) and \( \triangle ACB \cong \triangle A'C'B' \) are all equivalent.

If \( \triangle ABC \cong \triangle A'B'C' \) and \( |AB| = |A'B'| \), then \( |BC| = |B'C'| \) and \( |CA| = |C'A'| \) by Axiom ?? (1). In that case we say that \( \triangle ABC \) and \( \triangle A'B'C' \) are *congruent* and we denote this by \( \triangle ABC \cong \triangle A'B'C' \).

**Axiom 3.** Suppose that \( O, B, C \) are points in the plane. For every ray \( r \) starting at \( O \) there is a (unique) point \( A \) on \( r \) such that \( |OA| = |BC| \).

**Theorem 1.** Suppose \( l \) and \( l' \) are parallel lines, \( O \) does not lie on \( l \) or \( l' \), \( A, B \) points on \( l \), \( OA \) intersects \( l' \) in \( A' \) and \( OB \) intersects \( l' \) in \( B' \). Then we have

\[
\frac{|OA|}{|OA'|} = \frac{|OB|}{|OB'|} = \frac{|AB|}{|A'B'|}.
\]
Proof. We have $\angle BOA = \angle B'O'A'$. Also, $\angle OAB = \angle O'A'B'$ and $\angle OBA = \angle O'B'A'$ by an Axiom in the first handout. So $\triangle OAB$ and $\triangle O'A'B'$ are similar and

$$\frac{|OA|}{|OA'|} = \frac{|OB|}{|OB'|} = \frac{|AB|}{|A'B'|},$$

by Axiom ??.

A triangle $\triangle BOA$ is called isosceles, if $|OA| = |OB|$.

**Theorem 2** (Isosceles triangles). A triangle $\triangle BOA$ is isosceles if and only if $\angle OAB = \angle OBA$. 

$$\angle BOA = \angle B'O'A'. \quad \square$$
Proof. If $|OA| = |OB|$, then $\triangle OAB$ and $\triangle OBA$ are similar. Therefore, $\angle OAB = \angle OBA$.

Conversely, if $\angle OAB = \angle OBA$, then $\triangle OAB$ and $\triangle OBA$ are similar, so

$$\frac{|OA|}{|OB|} = \frac{|AB|}{|BA|} = 1.$$ 

\[\square\]

**Theorem 3** (Parallellograms). Suppose that $A, B, C, D$ are 4 distinct points. The following statements are equivalent:

1. $|AB| = |DC|$ and $|BC| = |AD|$.
2. $\uparrow AB$ and $\uparrow DC$ are parallel, and $\uparrow BC$ and $\uparrow AD$ are parallel.
3. $|AB| = |DC|$ and $\uparrow AB$ and $\uparrow DC$ are parallel.
4. $\angle DAB = \angle BCD$ and $\angle ABC = \angle CDA$.
5. $AC$ and $BD$ bisect each other.

Proof. (1)$\Rightarrow$(4) Because obviously $|BD| = |DB|$, $\triangle ABD$ and $\triangle CDB$ are similar. So $\angle DAB = \angle BCD$. By similar reasoning, we also get $\angle ABC = \angle CDA$.

(4)$\Rightarrow$(2) The sum of the angles in the quadrangle is $360^\circ$. Since $\angle DAB = \angle BCD$ and $\angle ABC = \angle CDA$, we have $\angle DAB + \angle CDA = 180^\circ$. Therefore, $\uparrow AB$ and $\uparrow DC$ must be parallel. Similarly $\uparrow AD$ and $\uparrow BC$ are parallel.
(2)⇒(3) $\angle DAB = 180^\circ - \angle ADC = \angle BCD$. Also $\angle ABC = \angle CDB$. Therefore, $\triangle ABD$ and $\triangle CDB$ are similar. In particular

$$\frac{|AB|}{|CD|} = \frac{|BC|}{|CB|} = 1.$$ 

(3)⇒(5) Suppose that $AD$ and $BC$ intersect in $P$. Then $\angle PAB = \angle PDC$ and $\angle PBA = \angle PCD$ because $\overline{AB}$ and $\overline{DC}$ are parallel. Also $\angle BPA = \angle DPC$. Therefore $\triangle PAB$ and $\triangle PCD$ are similar. We have

$$\frac{|PA|}{|PC|} = \frac{|PB|}{|PD|} = \frac{|AB|}{|CD|} = 1.$$ 

Hence

$$|PA| = |PC| \text{ and } |PB| = |PD|.$$ 

(5)⇒(1) Let $P$ be again the intersection of $AB$ and $BC$. We assume that $|PA| = |PC|$ and $|PB| = |PD|$. We also have $\angle BPA = \angle DPC$. So $\triangle PAB$ and $\triangle PCD$ are similar. Therefore

$$\frac{|AB|}{|CD|} = \frac{|PA|}{|PC|} = 1,$$ 

so $|AB| = |CD|$. Similarly, one deduces $|BC| = |DA|$. \qed

**Theorem 4 (Pythagoras).** Suppose that $\triangle BOA$ is a right triangle with $\angle BOA = 90^\circ$, $|OA| = a$, $|OB| = b$ and $|AB| = c$. Then

$$a^2 + b^2 = c^2$$

![Diagram](https://via.placeholder.com/150)

Proof. Let $P$ be the foot of the altitude from $O$. So $P$ lies on $AB$ such that $\angle OPB = \angle OPA = 90^\circ$. $\angle APB = \angle AOB = 90^\circ$ and $\angle POA = 90^\circ - \angle PAO = 90^\circ - \angle BAO = \angle ABO$. It follows that $\triangle APO$ and $\triangle AOB$ are similar. Therefore

$$\frac{|AP|}{|AO|} = \frac{|OA|}{|AB|},$$

so

$$|AP| = \frac{|AO|^2}{|AB|} = \frac{a^2}{c}.$$
Similarly, \( \triangle BPO \) and \( \triangle BOA \) are similar, so

\[
\frac{|BP|}{|BO|} = \frac{|OB|}{|AB|},
\]

so

\[
|BP| = \frac{|BO|^2}{|AB|} = \frac{b^2}{c}.
\]

We now have

\[
c = |AB| = |AP| + |PB| = \frac{a^2}{c} + \frac{b^2}{c},
\]

so

\[
a^2 + b^2 = c^2.
\]

\[\square\]

2. CIRCLES

A circle with center \( O \) and radius \( r \) is the set of all points \( A \) in the plane with \( |OA| = r \). If \( A, B \) are distinct points on a circle with center \( O \), then \( \triangle AOB \) is isosceles.

**Theorem 5.** Suppose that \( A \) and \( B \) lie on a circle with center \( O \). If \( P \) lies on the circle on the same side of \( AB \) as \( O \). Then \( \angle BPA = \frac{1}{2} \angle BOA \). If \( P \) is on the circle, but not on the same side of \( AB \) as \( O \), then \( \angle BPA = 180^\circ - \frac{1}{2} \angle BOA \).

**Proof.** The triangles \( \triangle OPA \) and \( \triangle OPB \) are isosceles. Thus, \( \angle OAB = \angle OBA = \alpha \), say, and \( \angle OBP = \angle OPB = \beta \), say. We have \( \angle AOP = 180^\circ - 2\alpha \), \( \angle POB = 180^\circ - 2\beta \).

case 1:
Suppose that $P$ and $O$ lie on the same side of $AB$ and $O$ lies inside $\triangle PAB$. and

$$\angle BPA = 360 - \angle AOP - \angle POB = 2\alpha + 2\beta$$

So

$$\angle BPA = \alpha + \beta = \frac{1}{2}(2\alpha + 2\beta) = \frac{1}{2}\angle BPA.$$

case 2:

Suppose that $P$ and $O$ lie on the same side of $AB$, but $P$ does not lie inside $\triangle PAB$. Assume that $P$ is closer to $B$ than to $A$ (the other case goes similarly). Now

$$\angle BOA = \angle POA - \angle POB = (180^\circ - 2\alpha) - (180^\circ - 2\beta) = 2\beta - 2\alpha.$$ 

and

$$\angle BPA = \angle BPO - \angle APO = \beta - \alpha = \frac{1}{2}\angle BOA.$$

case 3:
Suppose that $P$ and $O$ lie on opposite sides of $AB$. Then we have
\[ \angle BOA = \angle BOP + \angle POA = (180^\circ - 2\alpha) + (180^\circ - 2\beta) = 360^\circ - 2\alpha - 2\beta \]
and
\[ \angle BPA = \alpha + \beta = 180^\circ - \frac{1}{2}\angle BOA. \]
\[ \square \]

**Corollary 6.** Suppose that the line through the center $O$ of a circle intersects that circles in $A$ and $B$. If $P$ is any other point on the circle, then $\angle BPA$ is a right angle.

Proof.
\[ \angle BPA = \frac{1}{2}\angle BOA = \frac{1}{2}180^\circ = 90^\circ. \]
\[ \square \]