HANDOUT , MATH 174, FALL 2006

1. DISTANCE

For two points A, B in the plane, $|AB|$ is the distance between A and B, or equivalently it is the length of the line segment AB. So $|AB|$ is always a nonnegative number. We have $|AB| = 0$ if and only if $A = B$.

Axiom 1 (The triangle inequality)**.** *If A, B, C are points in the plane then*

$$|AC| \leq |AB| + |BC|.$$

Equality holds of and only if A, B, C lie on a line and B lies on the line segment AB.

Exercise 1. Prove the *reverse* triangle inequality: If A, B, C are points in the plane, then

$$|AC| \geq \big|\, |AB| - |BC| \,\big|.$$

(For a real number x, $|x|$ denotes its absolute value.)

Axiom 2. *Given two triangles $\triangle ABC$ and $\triangle A'B'C'$ in the plane, the following three statements are equivalent:*

(1)

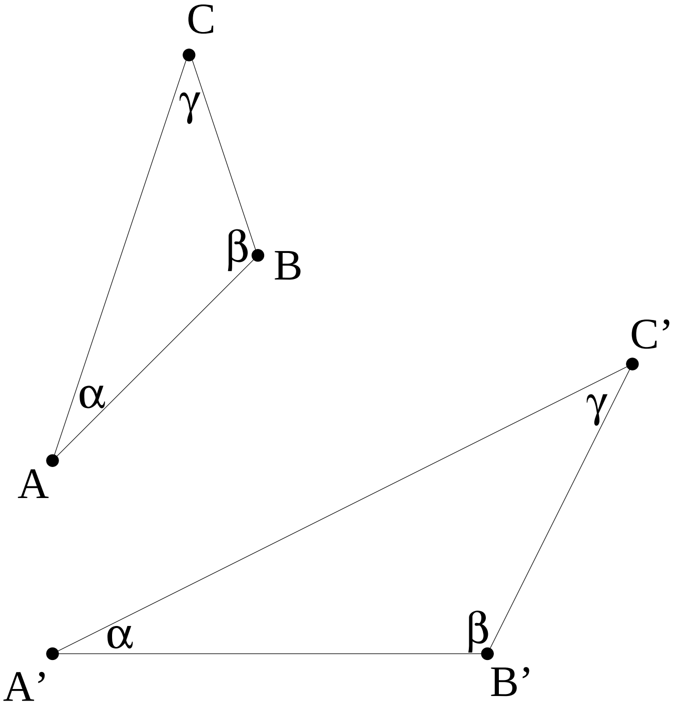
$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|CA|}{|C'A'|}.$$

(2)

$$\angle CAB = \angle C'A'B' \text{ and } \angle ABC = \angle A'B'C'.$$

(3)

$$\angle CAB = \angle C'A'B' \text{ and } \frac{|AB|}{|A'B'|} = \frac{|AC|}{|A'C'|}$$



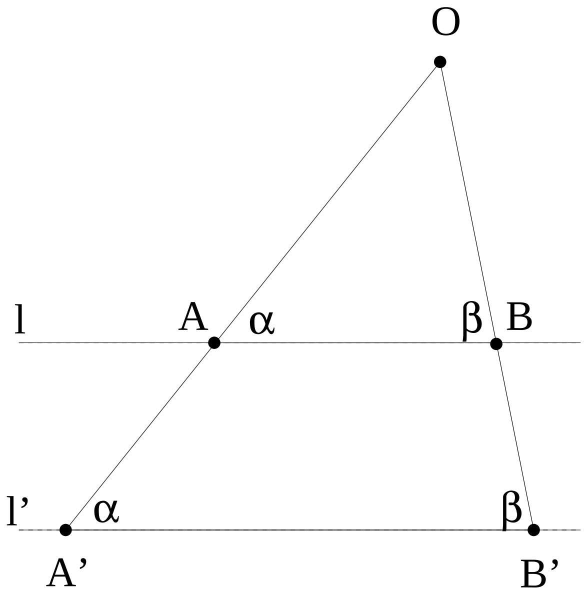
Two such triangles as in Axiom **??** are called *similar* and we denote this by $\triangle ABC \approx \triangle A'B'C'$. Note that it is important how the vertices are labeled. The statement $\triangle ABC \approx \triangle A'B'C'$ is *not* the same as $\triangle ABC \approx \triangle B'C'A'$. However, the statements $\triangle ABC \approx \triangle A'B'C'$, $\triangle BCA \approx \triangle B'C'A'$ and $\triangle ACB \approx A'C'B'$ are all equivalent.

If $\triangle ABC \approx \triangle A'B'C'$ and $|AB| = |A'B'|$, then $|BC| = |B'C'|$ and $|CA| = |C'A'|$ by Axiom **??** (1). In that case we say that $\triangle ABC$ and $\triangle A'B'C'$ are *congruent* and we denote this by $\triangle ABC \cong \triangle A'B'C'$.

Axiom 3. *Suppose that O, B, C are points in the plane. For every ray r starting at O there is a (unique) point A on r such that $|OA| = |BC|$.*

Theorem 1. *Suppose l and l' are parallel lines, O does not lie on l or l', A, B points on l, $\underline{OA}$ intersects l' in A' and $\underline{OB}$ intersects l' in B'. Then we have*

$$\frac{|OA|}{|OA'|} = \frac{|OB|}{|OB'|} = \frac{|AB|}{|A'B'|}.$$



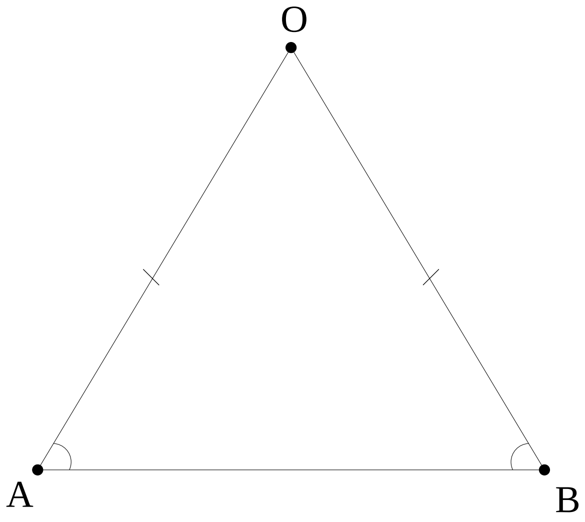
Proof. We have $\angle BOA = \angle B'O'A'$. Also, $\angle OAB = \angle OA'B'$ and $\angle OBA = \angle OB'A'$ by an Axiom in the first handout. So $\triangle OAB$ and $\triangle OA'B'$ are similar and

$$\frac{|OA|}{|OA'|} = \frac{|OB|}{|OB'|} = \frac{|AB|}{|A'B'|}.$$

by Axiom **??**. □

A triangle $\triangle BOA$ is called isosceles, if $|OA| = |OB|$.

Theorem 2 (Isosceles triangles)**.** *A triangle $\triangle BOA$ is isosceles if and only if $\angle OAB = \angle OBA$.*



Proof. If $|OA| = |OB|$, then $\triangle OAB$ and $\triangle OBA$ are similar. Therefore, $\angle OAB = \angle OBA$.

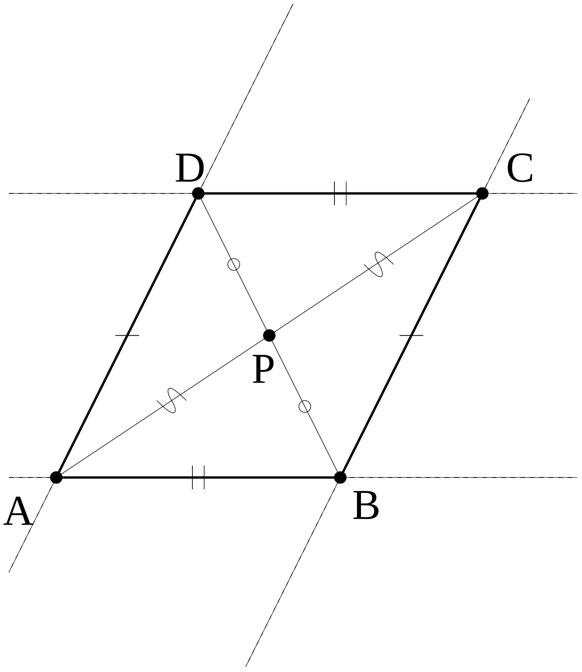
Conversely, if $\angle OAB = \angle OBA$, then $\triangle OAB$ and $\triangle OBA$ are similar, so

$$\frac{|OA|}{|OB|} = \frac{|AB|}{|BA|} = 1.$$

□

Theorem 3 (Parallellograms). *Suppose that A, B, C, D are 4 distinct points. The following statements are equivalent:*

(1) $|AB| = |DC|$ *and* $|BC| = |AD|$.
(2) $\uparrow AB$ *and* $\uparrow DC$ *are parallel, and* $\uparrow BC$ *and* $\uparrow AD$ *are parallel.*
(3) $|AB| = |DC|$ *and* $\uparrow AB$ *and* $\uparrow DC$ *are parallel.*
(4) $\angle DAB = \angle BCD$ *and* $\angle ABC = \angle CDA$.
(5) AC *and* BD *bisect each other.*



Proof. (1)⇒(4) Because obviously $|BD| = |DB|$, $\triangle ABD$ and $\triangle CDB$ are similar. So $\angle DAB = \angle BCD$. By similar reasoning, we also get $\angle ABC = \angle CDA$.

(4)⇒(2) The sum of the angles in the quadrangle is 360°. Since $\angle DAB = \angle BCD$ and $\angle ABC = \angle CDA$, we have $\angle DAB + \angle CDA = 180°$. Therefore, $\uparrow AB$ and $\uparrow DC$ must be parallel. Similarly $\uparrow AD$ and $\uparrow BC$ are parallel.

(2)⇒(3) $\angle DAB = 180^\circ - \angle ADC = \angle BCD$. Also $\angle ABC = \angle CDB$. Therefore, $\triangle ABD$ and $\triangle CDB$ are similar. In particular

$$\frac{|AB|}{|CD|} = \frac{|BC|}{|CB|} = 1.$$

(3)⇒(5) Suppose that AD and BC intersect in P. Then $\angle PAB = \angle PDC$ and $\angle PBA = \angle PCD$ because $\underline{AB}$ and $\underline{DC}$ are parallel. Also $\angle BPA = \angle DPC$. Therefore $\triangle PAB$ and $\triangle PCD$ are similar. We have

$$\frac{|PA|}{|PC|} = \frac{|PB|}{|PD|} = \frac{|AB|}{|CD|} = 1.$$

Hence

$$|PA| = |PC| \text{ and } |PB| = |PD|.$$

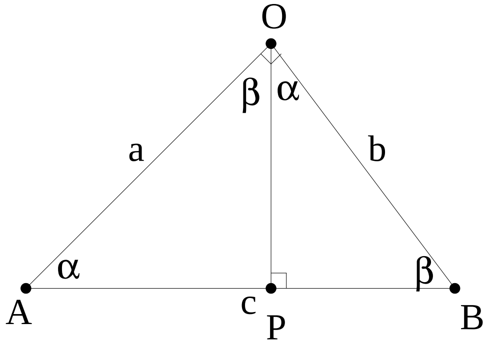
(5)⇒(1) Let P be again the intersection of AB and BC. We assume that $|PA| = |PC|$ and $|PB| = |PD|$. We also have $\angle BPA = \angle DPC$. So $\triangle PAB$ and $\triangle PCD$ are similar. Therefore

$$\frac{|AB|}{|CD|} = \frac{|PA|}{|PC|} = 1,$$

so $|AB| = |CD|$. Similarly, one deduces $|BC| = |DA|$. □

Theorem 4 (Pythagoras)**.** *Suppose that* $\triangle BOA$ *is a right triangle with* $\angle BOA = 90^\circ$, $|OA| = a$, $|OB| = b$ *and* $|AB| = c$. *Then*

$$a^2 + b^2 = c^2$$



Proof. Let P be the foot of the altitude from O. So P lies on AB such that $\angle OPB = \angle OPA = 90^\circ$. $\angle APB = \angle AOB = 90^\circ$ and $\angle POA = 90^\circ - \angle PAO = 90^\circ - \angle BAO = \angle ABO$. It follows that $\triangle APO$ and $\triangle AOB$ are similar. Therefore

$$\frac{|AP}{|AO} = \frac{|AO|}{|AB|},$$

so

$$|AP| = \frac{|AO|^2}{|AB|} = \frac{a^2}{c}.$$

Similarly. $\triangle BPO$ and $\triangle BOA$ are similar, so

$$\frac{|BP|}{|BO|} = \frac{|OB|}{|AB|},$$

so

$$|BP| = \frac{|BO|^2}{|AB|} = \frac{b^2}{c}.$$

We now have

$$c = |AB| = |AP| + |PB| = \frac{a^2}{c} + \frac{b^2}{c},$$

so

$$a^2 + b^2 = c^2.$$

□

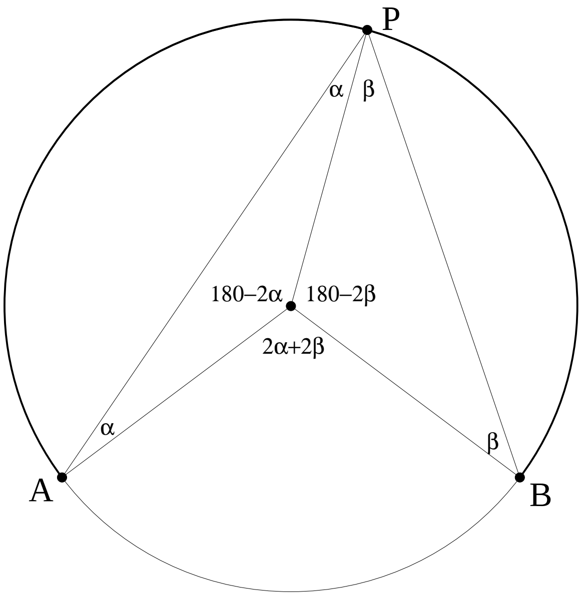
2. CIRCLES

A circle with *center* O and radius r is the set of all points A in the plane with $|OA| = r$. If A, B are distinct points on a circle with center O, then $\triangle AOB$ is isosceles.

Theorem 5. *Suppose that A and B lie on a circle with center O. If P lies on the circle on the same side of $\underline{AB}$ as O. Then $\angle BPA = \frac{1}{2}\angle BOA$. If P is on the circle, but not on the same side of $\underline{AB}$ as O, then $\angle BPA = 180° - \frac{1}{2}\angle BOA$.*

Proof. The triangles $\triangle OPA$ and $\triangle OPB$ are isosceles. Thus, $\angle OAB = \angle OBA = \alpha$, say, and $\angle OBP = \angle OPB = \beta$, say. We have $\angle AOP = 180° - 2\alpha$, $\angle POB = 180° - 2\beta$.

case 1:



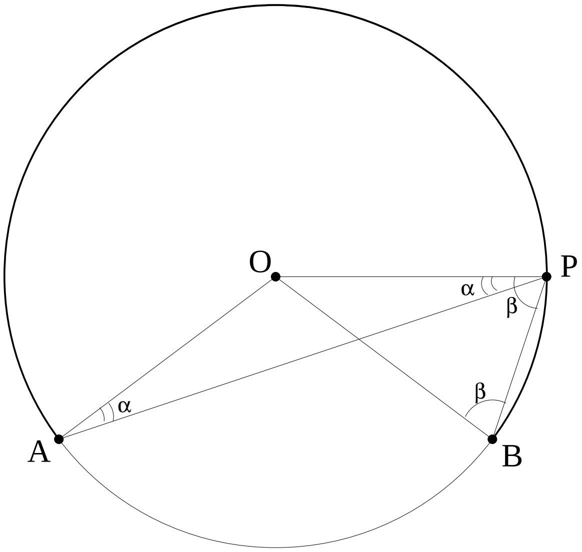
Suppose that P and O lie on the same side of $\underline{AB}$ and O lies inside $\triangle PAB$., and

$$\angle BPA = 360 - \angle AOP - \angle POB = 2\alpha + 2\beta$$

So

$$\angle BPA = \alpha + \beta = \tfrac{1}{2}(2\alpha + 2\beta) = \tfrac{1}{2}\angle BPA.$$

case 2:



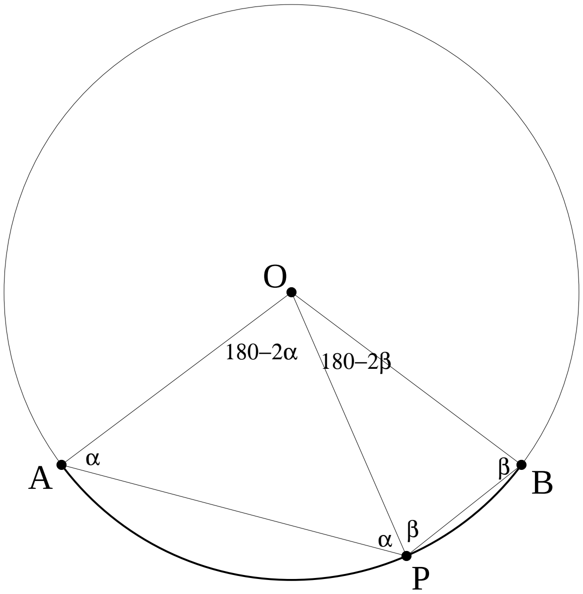
Suppose that P and O lie on the same side of $\underline{AB}$, but P does not lie onside $\triangle PAB$. Assume that P is closer to B than to A (the other case goes similarly). Now

$$\angle BOA = \angle POA - \angle POB = (180^\circ - 2\alpha) - (180^\circ - 2\beta) = 2\beta - 2\alpha.$$

and

$$\angle BPA = \angle BPO - \angle APO = \beta - \alpha = \tfrac{1}{2}\angle BOA.$$

case 3:



Suppose that P and O lie on opposite sides of $\underline{AB}$. Then we have

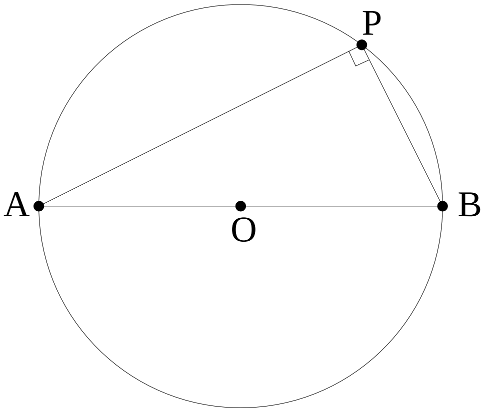
$$\angle BOA = \angle BOP + \angle POA = (180^\circ - 2\alpha) + (180^\circ - 2\beta) = 360^\circ - 2\alpha - 2\beta$$

and

$$\angle BPA = \alpha + \beta = 180^\circ - \tfrac{1}{2}\angle BOA.$$

□

Corollary 6. *Suppose that the line through the center O of a circle intersects that circles in A and B. If P is any other point on the circle, then $\angle BPA$ is a right angle.*



Proof.

$$\angle BPA = \tfrac{1}{2}\angle BOA = \tfrac{1}{2}180^\circ = 90^\circ.$$

□