THE EULER CHARACTERISTIC

Suppose we have an outer n-gon, whose interior is divided into k polygons. Let the k inner polygons have $n_1, \ldots, n_k$ sides. Such a figure divides the plane into $k+1$ regions: there are k inner regions, and there is one infinite region which is the outside of the original n-gon. See the figures on the next page.

We now sum all the interior angles formed in such a figure. There are two ways of calculating this sum.

Firstly we may calculate the angle formed at each vertex in the figure. For a vertex in the interior of our n-gon, the angle will simply be $360°$. For one of the vertices of our original n-gon the angle will simply be the angle of the original n-gon at that vertex. Thus the sum of the angles is

$$360° \times (\text{number of inner vertices}) + (\text{sum of angles of the original } n\text{-gon}).$$

Since the sum of angles of an n-gon equals $180°n - 360°$, we conclude that the sum of angles is

$$(1)\quad 360° \times (\text{number of inner vertices} + n) - 180°n - 360° = 360°V - 180°n - 360°,$$

where V denotes the total number of Vertices.

Secondly we may compute this sum of angles by summing the angles in each one of the k inner polygons. The sum of the angles in the n_j-gon is $180°n_j - 360°$. Adding these with $j = 1, 2, \ldots, k$ we get that the sum of angles is

$$(2)\qquad 180°(n_1 + n_2 + \ldots + n_k) - 360°k.$$

We now equate (1) and (2), and divide both sides by $180°$. Thus we get

$$2V - n - 2 = n_1 + n_2 + \ldots + n_k - 2k.$$

We rewrite this as

$$(3)\qquad 2V + 2k + 2 = n_1 + n_2 + \ldots + n_k + n + 4.$$

Now $2k+2$ is just $2R$ where $R(= k+1)$ is the number of regions of the plane. Thus the LHS of (3) is $2V + 2R$ and we have expressed it very simply in terms of the geometry of our figure. We want to find a simple geometric meaning for the RHS.

Note that $n_1 + n_2 + \ldots + n_k$ counts all the line segments (edges) in our figure. But some edges are counted only once and some are counted twice. More precisely, the outer edges of our original n-gon are counted only once, but each inner edge is shared by two polygons and so is counted twice. If we add n to $n_1+n_2+\ldots+n_k$ then the n outer edges will also be counted twice. In other words $n_1 + n_2 + \ldots + n_k + n$ is simply twice the number of edges.

Letting E denote the number of edges, we see that (3) gives

$$2V + 2R = 2E + 4,$$

or, equivalently,

$$(4)\qquad V - E + R = 2.$$

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By projection (**explain more**) we see that (4) holds for convex polyhedra where in place of R we have F which is the number of faces of the polyhedron. This is known as the "'Euler Characteristic Formula"':

$$V - E + F = 2.$$

The Five Platonic Solids. These are the tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron.

The tetrahedron has 4 faces that are equilateral triangles. There are four vertices, and at each vertex meet three triangular faces. Thus $V = 4$, $E = 6$, $F = 4$ and $V - E + F = 2$.

The cube has 6 square faces. There are 8 vertices, and at each vertex meet three faces. Thus $V = 8$, $E = 12$ and $F = 6$ with $V - E + F = 2$.

The octahedron has 8 triangular faces. There are six vertices with four triangles meeting at each vertex. Thus $V = 6$, $E = 12$, and $F = 8$ with $V - E + F = 2$.

The dodecahedron has 12 pentagonal faces. There are 20 vertices with three pentagons meeting at each vertex. Thus $V = 20$, $E = 30$, and $F = 12$ with $V - E + F = 2$.

The icosahedron has 20 triangular faces. There are 12 vertices with five triangles meeting at each vertex. Thus $V = 12$, $E = 30$, and $F = 20$ with $V - E + F = 2$.

We will now show that these are all the convex polyhedra all of whose faces are regular k-gons, and such that at each vertex the same number of k-gons meet, say ℓ of these with $\ell \geq 3$.