PROBLEM SET 1: INDUCTION

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This problem set is due on Thursday, September 20 at 3pm at the beginning of the class. You have to hand in exactly four problems on this problem set (but you may solve many more problems without handing them in). Don’t forget to put your name on the solutions (and the names of the people you cooperated with). First try the easier problems before you try the harder ones. Some harder recommended problems are 7, 9, 10 and 16 (and we may discuss some of these next time), but you can choose any 4 problems you like.

Let \( \mathbb{N} = \{1, 2, \cdots \} \) be the natural numbers (starting with 1!). One important property of \( \mathbb{N} \) is the induction principle. It is in fact rather an axiom than a theorem, i.e., we simply assume that the induction principle for natural numbers is true.

**Truth 1.** Suppose we have a statement which depends on a natural number \( n \in \mathbb{N} \). Suppose that

1. the statement is true for \( n = 1 \), and
2. if for a natural number \( m \), the statement is true for \( n = m \), then the statement is also true for \( n = m + 1 \).

Then the statement is true for all natural numbers \( n \).

Many mathematical problems involve induction proofs. Usually applying the induction principle is the easy part of the problem. However, sometimes it can be tricky to find the right induction statement. Typical applications are identities which involve an indeterminate natural number \( n \).

Induction is needed often in for example Putnam problems. Most of the time (but not always) it is obvious from the formulation of the problem that some kind of induction is needed.

1. **IDENTITIES**

**Example 1.** Show that

\[
\sum_{i=1}^{n} i = \frac{n(n + 1)}{2}.
\]

for all \( n \in \mathbb{N} \).
Proof. We check (1) for \( n = 1 \):
\[
\sum_{i=1}^{1} i = 1 = \frac{1(1 + 1)}{2}.
\]
Suppose that (1) is true for \( n = m \), i.e.,
\[
\sum_{i=1}^{m} i = \frac{m(m + 1)}{2}.
\]
then
\[
\sum_{i=1}^{m+1} i = \sum_{i=1}^{m} i + (m + 1) = \frac{m(m + 1)}{2} + (m + 1) = \frac{(m + 1)((m + 1) + 1)}{2}
\]
so (1) is also true for \( n = m + 1 \). Using the induction principle, (1) holds for all natural numbers \( n \in \mathbb{N} \). \( \square \)

**Problem 1.** *Proof that*
\[
1^2 + 2^2 + \cdots + n^2 = \frac{n(n + 1)(2n + 1)}{6}
\]
for all \( n \in \mathbb{N} \).

**Problem 2.** *Show that*
\[
1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2
\]
for all \( n \in \mathbb{N} \).

**Problem 3.** ***Give (and prove) a formula for*
\[
1^4 + 2^4 + \cdots + n^4.
\]

**Problem 4** (MMPC 29, 1). **Sometimes one finds in an old park a tetrahedral pile of cannon balls, that is, a pile each layer of which is a tightly packed triangular layer of balls.

(a). How many cannon balls are in a tetrahedral pile of cannon balls of \( N \) layers?
(b). How high is a tetrahedral pile of cannon balls of \( N \) layers? (Assume each cannon ball is a sphere of radius \( R \).)

Let us define the binomial coefficients by
\[
\binom{n}{k} := \frac{n(n-1)\cdots(n-k+1)}{k!}.
\]

**Problem 5.** *
Show that
\[
\sum_{i=0}^{n} \binom{i}{k} = \binom{n+1}{k+1}.
\]
for all \( n \in \mathbb{N} \).
Problem 6. (a) **** Let $f$ be a polynomial of degree $d$ with real coefficients. Show that
\[ f(n) = \sum_{k=0}^{d} a_k \binom{n}{k}. \]
where
\[ a_k = \sum_{i=0}^{k} \binom{k}{i} (-1)^{i+k} f(i) \]
for $k = 0, 1, \ldots, d$.
(b) (trivial) Use Problem 5 to deduce
\[ \sum_{i=0}^{n} f(i) = \sum_{k=0}^{d} a_k \binom{n+1}{k+1}. \]

Problem 7. ** Show that
\[ 1 - \frac{1}{2} + \frac{1}{3} - \cdots - \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \]
for all $n \in \mathbb{N}$.

Problem 8. * Suppose that $x \in \mathbb{R} \setminus \{1\}$. Show that
\[ \frac{1 - x^{n+1}}{1-x} = 1 + x + x^2 + \cdots + x^n \]
for all $n \in \mathbb{N}$.

Example 2 (Putnam 1997 B2). **** Let $r$, $s$ and $t$ be integers with $0 \leq r$, $0 \leq s$ and $r + s \leq t$. Prove that
\[ \binom{0}{r} + \binom{1}{r} + \cdots + \binom{s}{r+s} = \frac{t+1}{(t+1-s)(n+r)}. \]

Proof. The idea of induction crosses one's mind. Question is, which variable shall we use for induction? It turns out that induction with respect to $s$ works. For $s = 0$ we get
\[ \binom{0}{r} = \frac{t+1}{(t+1)(n+r)}. \]
Suppose we already have proven the identity for $s = u$. Using the identity
\[ \binom{u+1}{i} = \binom{u}{i-1} + \binom{u}{i}. \]
and the induction hypothesis, we deduce
\[ \binom{u+1}{0} + \binom{u+1}{1} + \cdots + \binom{u+1}{u+1} = \binom{u+1}{r} + \binom{u+1}{r+1} + \cdots + \binom{u+1}{r+u+1} = \binom{t+1}{r} + \binom{t+1}{r+1} + \cdots + \binom{t+1}{r+u+1}. \]
\[
= \left( \frac{\binom{u}{0}}{t} + \frac{\binom{u}{1}}{t+1} + \cdots + \frac{\binom{u}{u}}{t+u-1} \right) + \left( \frac{\binom{u}{0}}{t+1} + \frac{\binom{u}{1}}{t+u} + \cdots + \frac{\binom{u}{u}}{t+u+1} \right) = \\
\frac{t+1}{(t+1-u)(t+u)} + \frac{t+1}{(t+1-u)(t+u+1)} = \\
\frac{t+1}{(t-u)(t+u-1)} \left( \frac{t-u-r}{t+1-u} + \frac{r+1}{t+1-u} \right) = \frac{t+1}{(t-u)(t+u-1)}.
\]

This shows that the identity is true for \( s = u+1 \). \qed

**Problem 9.***

1. \(((UM)^2C^5,5)\) For \( n \geq 1 \), let \( S_n \) be the set of points \((x,y)\) in the plane with integral coordinates satisfying \( x \geq 0, y \geq 0 \) and \( x + y \leq n \). Prove that \( S_n \) is not contained in the union of \( n \) straight lines.

2. How many possibilities are there for line configuration with exactly \( n \) straight lines such that exactly one point is missed?

**2. Strong induction**

**Example 3.** Every natural number \( \geq 2 \) is a product of prime numbers.

**Proof.** We will prove by induction on \( n \) that:

\[
(2) \quad \text{Every natural number \( m \) with } 2 \leq m \leq n \text{ is a product of prime numbers.}
\]

Clearly the statement (2) is true for \( n = 1 \).

Assume that (2) is true for \( n = k \). We will prove that (2) is true for \( n = k + 1 \). Suppose that \( k + 1 \) is a prime number. In that case we are clearly finished. Otherwise, we can write \( k + 1 = ab \) with \( 2 \leq a, b \leq k \). By the induction hypothesis, both \( a \) and \( b \) are products of prime numbers, therefore \( k + 1 \) is a product of prime numbers, so (2) holds for \( n = k + 1 \). \qed

The previous example shows that one has to pick carefully which statement to prove by induction. If we try to prove my induction on \( n \) that \( n \) is a product of prime numbers, then the induction step will be impossible to prove. This is why we prove by induction on \( n \) not just that \( n \) is a product of prime numbers, but that all \( m \leq n \) are products of prime numbers. Now the induction step is very easy to prove. We call this method **strong induction**.

**Problem 10.*** A chocolate bar consists of \( nm \) tablets arranged in \( n \) rows and \( m \) columns. We want to break the chocolate bar into single tablets. We proceed as follows. We first take the \( n \times m \) bar and break it into 2 pieces. Then we take one of the pieces and again break that piece into two pieces. Now there are three pieces and we continue choosing a piece and breaking it into two until we are are left with \( nm \) \( 1 \times 1 \) tablets. Show that regardless of how you do it, you always need to break exactly \( nm - 1 \) times until you are left with single tablets.
Problem 11. ** Show by induction that every nonempty subset $S \subseteq \mathbb{N}$ has a minimal element. (OK, the statement is already obvious, but try to give an induction proof.)

Problem 12. **** Suppose that $S \subseteq \mathbb{N}^d$ is a subset. An element $a = (a_1, \ldots, a_d) \in S$ is called minimal if whenever $b = (b_1, \ldots, b_d) \in S$ with $b_i \leq a_i$ for $i = 0, 1, \ldots, d$, then $b = a$. Show that there are only finitely many minimal elements in $S$. (This problem is known as Dixon’s Lemma. It can be used to prove that ideals in the polynomial ring $K[x_1, \ldots, x_n]$ over a field $K$ are finitely generated.)

3. Recursive definitions

The induction principle can also be used in so-called recursive definitions. For example, the well known Fibonacci numbers $F_0, F_1, F_2, \ldots$ are defined by:

$$F_0 = 1, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1} \text{ for all } n \in \mathbb{N}.$$ 

Using the induction principle we can show that $F_n$ is defined for all natural numbers (and we also defined it for $n = 0$). The sequence of Fibonacci numbers is:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

Problem 13. * Show that

$$F_n^2 - F_{n+1}F_{n-1} = (-1)^n$$

for all $n \in \mathbb{N}$.

Problem 14 (MMPC 29, 3). The function $g$ is defined recursively on the positive integers by $g(1) = 1$ for $n > 1$ and $g(n) = 1 + g(n - g(n - 1))$.

(a). Find $g(1), g(2), g(3), g(4)$.

(b). Describe the pattern formed by the entire sequence $g(1), g(2), g(3), \ldots$.

(c). Prove your answer in part (b).

Problem 15. **** Let the function $f : \mathbb{N} \to \mathbb{Q}$ be defined by $f(1) = 1$, $f(2n) = f(n) + 1$ and $f(2n + 1) = 1/(f(n) + 1)$ for all $n \in \mathbb{N}$. Show that for every positive rational number $q$ there is a unique natural number $n$ such that $f(n) = q$.

Example 4 (Putnam 1985 A3). **** Let $d$ be a real number. For each integer $m \geq 0$ define a sequence $\{a_m(j)\}$, $j = 0, 1, 2, \ldots$ by the condition

$$a_m(0) = \frac{d}{2^m},$$

$$a_m(j + 1) = (a_m(j))^2 + 2a_m(j), \quad j > 0.$$ 

Evaluate $\lim_{n \to \infty} a_n(n)$. 
Proof. We first try to find a formula for $a_m(j)$ and then we will compute the limit. Let us take $a_m(0) = 1$ (i.e., $d = 2^m$) and we what we get.

$$a_m(1) = 3, \ a_m(2) = 15, \ a_m(3) = 255, \ldots$$

It seems that $a_m(i) + 1$ is always a power of two. Let us therefore look at the value of $b_m(i) := a_m(i) + 1$ rather than $a_m(i)$. Then we get

$$b_m(i + 1) = a_m(i + 1) + 1 = (a_m(i))^2 + 2a_m(i) + 1 = (a_m(i) + 1)^2 = (b_m(i))^2.$$ 

It is now easy to show by induction that

$$(3) \quad b_m(i) = b_m(0)^{2^i}$$

for all $i \in \mathbb{N}$. Indeed, for $i = 0$ we have $b_m(0) = b_m(0)^2$. If (3) is true for $i = n$, then

$$b_m(n + 1) = (b_m(n))^2 = (b_m(0)^{2^n})^2 = (b_m(0))^{2^{n+1}}.$$ 

By the induction principle, (3) holds for all nonnegative integers $i$. (If it wasn’t for the fact that this problem set is about induction, we probably wouldn’t have bothered to write down this easy induction prove.)

So finally we have

$$b_n(n) = b_n(0)^{2^n} = b_n(0)^{2^n} = (1 + a_n(0))^{2^n} = (1 + d \cdot 2^n)^{2^n}.$$ 

We need the following formula from calculus:

$$\lim_{n \to \infty} \left(1 + \frac{a}{n}\right)^n = e^a.$$ 

It follows that

$$\lim_{n \to \infty} b_n(n) = e^d,$$

so

$$\lim_{n \to \infty} a_n(n) = \lim_{n \to \infty} b_n(n) - 1 = e^d - 1.$$ 

Problem 16 (Putnam 1990 A1). *** Let

$$T_0 = 2, \ T_1 = 3, \ T_3 = 6$$

and for $n \geq 3$,

$$T_n = (n + 4)T_{n-1} - 4nT_{n-2} + (4n - 8)T_{n-3}.$$ 

The first terms are

$$2, 3, 6, 14, 40, 152, 784, 5168, 40576.$$ 

Find, with proof, a formula for $T_n$ of the form $T_n = A_n + B_n$ where $\{A_n\}$ and $\{B_n\}$ are well-known sequences.
Problem 17 (Putnam 1993 A2). Let \(\{x_n\}_{n \geq 0}\) be a sequence of nonzero real numbers such that \(x_n^2 - x_{n-1}x_{n+1} = 1\) for \(n = 1, 2, 3, \ldots\). Prove that there exists a real number \(a\) such that \(x_{n+1} = ax_n - x_{n-1}\) for all \(n \geq 1\).

Problem 18 (Putnam 1999 A6). The sequence \(\{a_n\}_{n \geq 1}\) is defined by \(a_1 = 1, a_2 = 2, a_3 = 24\) and for \(n \geq 4\),

\[
a_n = \frac{6a_{n-1}a_{n-2} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}.
\]

Show that, for all \(n\), \(a_n\) is an integer multiple of \(n\).