PROBLEM SET 2: THE PIGEON HOLE PRINCIPLE

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For Thursday, September 27, choose 4 problems from the first section. Recommended are problems 2, 3, 7, 11. You may choose easier or harder problems. For October 4, choose 4 problems from the second section (diomphantine approximation). Recommended are problems 17, 18, 21, 25. If you find the problems in section 2 too hard, you can also choose some from the first section, as long as you haven’t done them before and they were not discussed in class.

The pigeon hole principle can be formulated in the following way:

**Truth 1** (Pigeon Hole Principle). Suppose that we have more than \(nr\) marbles divided over \(r\) jars. Then one jar contains at least \(n + 1\) marbles.

**Proof.** Let us prove this triviality. Suppose that all jars contain at most \(n\) marbles. Then there would be at most \(rn\) marbles, contradicting the assumptions. \(\Box\)

One may reformulate the principle as one likes (for example with pidgeons and pigeon holes). The Pigeon Hole Principle can hardly be called a theorem (since it is so trivial), but it does have a surprisingly wide range of applications. The strength of the Pigeon Hole Principle is that it proves existence without being constructive (like the Theorem above does not specify which jar contains \(n + 1\) marbles).

In Putnam problems, the Pigeon Hole Principle does not appear very often. The often appear in combinatorial problems with phrases like “Show that there exist …”. In other cases, the pigeon hole principle may be very concealed. We will see that it can be used for diomphantine approximation.

1. Some Combinatorial Problems with the Pigeon Hole Principle

**Problem 1.** * Show that \((a - b)(b - c)(c - a)\) is always even if \(a, b, c\) are integers.

**Problem 2.** * What is the maximum number of rooks that you can put on an \(8 \times 8\) chess-board such that no two rooks can hit each other.

**Problem 3.** ** What is the maximum number of bishops that you can put on an \(8 \times 8\) chess-board such that no two bishops can hit each other.

**Problem 4.** **** What is the maximum number of queens that you can put on an \(8 \times 8\) chess-board such that no two queens can hit each other.
Problem 5. ** Suppose that $a_1, a_2, \ldots, a_{30}$ are integers. Show that
\[
\prod_{1 \leq i < j \leq 30} (a_j - a_i)
\]
is divisible by $2001 = 3 \cdot 23 \cdot 29$.

Problem 6. ** Let $f$ be a polynomial with real coefficients of degree $\leq d$. Suppose that
\[
\#\{f(1), f(2), \ldots, f(n)\} < \frac{n}{d}
\]
show that $f$ is constant.

Problem 7. ** Prove that for every integer $n \geq 2$ there exists an integer $m$ such that $k^3 - k + m$ is not divisible by $n$ for all integers $k$.

Problem 8. ** If $A$ is a set of 20 distinct integers chosen from the set \{1, 4, 7, 10, 13, \ldots, 100\}, prove that $A$ must contain two distinct integers whose sum is 104.

Example 1. *** Suppose that $a_1, \ldots, a_n$ are integers. Show that we can find a nonempty subset $I \subseteq \{1, 2, \ldots, n\}$ such that
\[
\sum_{i \in I} a_i
\]
is divisible by $n$.

Proof. Define $b_0 = 0$ and
\[
b_i = a_1 + a_2 + \cdots + a_i
\]
for $i = 1, 2, \ldots, n$. We have $n + 1$ integers $b_0, b_1, \ldots, b_n$ and $n$ classes modulo $n$. By the Pigeon Hole Principle, two of the $b_i$'s must be the same modulo $n$. So assume $j > i$ and $b_j$ and $b_i$ are congruent modulo $n$. We may assume that $j > i$. Then we have
\[
b_j - b_i = a_{i+1} + \cdots + a_j
\]
is divisible by $n$. $\square$

Problem 9. *** Show that, given a 7-digit number, you can cross out some digits at the beginning and at the end such that the remaining number is divisible by 7. For example, if we take the number 1234589, then we can cross out 1 at the beginning and 89 at the end to get the number $2345 = 7 \times 335$.

Problem 10. **** Let $a_1, a_2, \ldots, a_{10}$ be distinct integers from \{1, 2, \ldots, 99\}. Show that \{a_1, a_2, \ldots, a_{10}\} contains two disjoint non-empty subsets with the sum of the numbers from the first equal to the sum of the elements from the second subset.
Problem 11 (\textit{UM)}^2C^{13} 1996 5). \*** The Duma (the Russian parliament) has 1994 members. Assume that each member has slapped the face of exactly one other member. Prove that one can form within the Duma a 665-member committee in which no member has slapped the face of another.

Problem 12 (Putnam 2000 B1). \*** Let \(a_j, b_j, c_j\) be integers \(1 \leq j \leq N\). Assume for each \(j\), at least one of \(a_j, b_j, c_j\) is odd. Show that there exist integers \(r, s, t\) such that \(ra_j + sb_j + tc_j\) is odd for at least \(4N/7\) values of \(j\), \(1 \leq j \leq N\).

Example 2 ((\textit{UM)}^2C^{18} 2001, Dutch Mathematical Olympiad). \*** A subset \(S\) of natural numbers is called square-free if for all distinct \(a, b \in S\) we have that the product \(ab\) is not a square. What is the maximal cardinality of a square-free subset \(S \subseteq \{1, 2, \ldots, 30\}\)?

\textit{Proof.} Every integer \(n\) can be uniquely written in the form \(n = m^2r\) with \(m\) and \(r\) integers and \(r\) square-free. Let \(T\) be the set of square-free natural numbers \(\leq 30\), i.e.,

\[T := \{1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30\}.

For every \(r \in T\), define

\[S_r = \{m^2r \mid m \in \mathbb{N}\}.

Suppose that \(a \in S_r\) and \(b \in S_t\), say \(a = m^2r\) and \(b = n^2t\), then

\[ab = (mn)^2rt\]

is a square if and only if \(r = t\).

From this we easily see that \(T\) is square-free. The set \(T\) has 19 elements.

On the other hand if \(S\) is a set with \(\geq 20\) elements, then by the Pidgeon Hole Principle there exist distinct \(a, b \in S\) such that they lie in the same \(S_r\) for some \(r \in T\) (because \(T\) has only 19 elements). Now \(ab\) is a square, and \(S\) cannot be square-free. \(\square\)

Problem 13 (Dutch Mathematical Olympiad). \*** A set \(S\) of natural numbers is called super square free if for every nonempty subset \(I \subseteq S\) we have that

\[\prod_{a \in I} a\]

is not a square. What is the maximal cardinality of a super square free subset \(S \subseteq \{1, 2, \ldots, 30\}\)?

Problem 14 (Putnam 1958 B2). \*** Given a set of \(n + 1\) positive integers, none of which exceeds \(2n\), show that at least one member of the set must divide another member of the set.

Problem 15 (Putnam 1989 B4). \*** Can a countably infinite set have an uncountable collection of non-empty subsets such that the intersection of any two of them is finite?
Problem 16. **** The set $M$ consists of 2001 distinct positive integers, none of which is divisible by any prime $p > 23$. Prove that there are distinct $x, y, z, t$ in $M$ such that $xyzt = u^4$ for some integer $u$.

2. DIOPHANTINE APPROXIMATION

The Pigeon Hole Principle can be used to prove that good approximations of irrational numbers exist.

Example 3. **** Suppose that $\alpha$ is an irrational real number. Show that there are infinitely many integers $q$ such that

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{q^2}.$$

Proof. Let $N$ be an integer. Let us define the set $S$ by

$$S := \{q\alpha - \lfloor q\alpha \rfloor \mid 0 \leq q \leq N\}.$$ 

We partition the interval $[0, 1]$ into $N$ intervals $[\frac{i}{N}, \frac{i+1}{N}]$, $i = 0, 1, \ldots, N-1$. Since the cardinality of $S$ is $N+1$, there must be two distinct elements $a_1, a_2 \in S$ lying in the same interval, so we get

$$|a_1 - a_2| \leq \frac{1}{N}.$$

We can write $a_1 = b_1\alpha - \lfloor b_1\alpha \rfloor$ and $a_2 = b_2\alpha - \lfloor b_2\alpha \rfloor$. We may assume that $b_2 > b_1$. We now have

(1) $$|a_1 - a_2| = |q_N\alpha - p_N| \leq \frac{1}{N}$$

where $q_N = b_1 - b_2$ and $p_N = \lfloor b_1\alpha \rfloor - \lfloor b_2\alpha \rfloor$. Dividing by $q_N$ gives us

$$\left|\alpha - \frac{p_N}{q_N}\right| \leq \frac{1}{Nq_N} \leq \frac{1}{q_N^2}.$$ 

Note that $q_1, q_2, \ldots$ contains infinitely many distinct natural numbers because from (1) follows that

$$\lim_{N \to \infty} |q_N\alpha - p_N| = 0.$$ 

and $|q_N\alpha - p_N| > 0$ for all $N$ since $\alpha$ is irrational. \square

Problem 17. *** Prove Theorem 1 below (compare to the previous example).

Theorem 1. *** A number $\alpha \in \mathbb{R}$ is irrational if and only if there exist integers $q_1, q_2, \ldots$ and $p_1, p_2, \ldots$ such that

$$\lim_{n \to \infty} q_n\alpha - p_n = 0$$

and $q_n\alpha - p_n \neq 0$ for all $n$. 

Problem 18. ** Use Theorem 1 to prove

\[ e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \]

is irrational.

Problem 19. *** Find integers \( q_1, q_2, \ldots \text{ and } p_1, p_2, \ldots \) such that

\[ \lim_{n \to \infty} q_n \sqrt{2} - p_n = 0. \]

Use this to show that \( \sqrt{2} \) is irrational.

Problem 20 (PUTNAM 1949 b1). *** Each rational number \( p/q \) (\( p, q \) relatively prime positive integers) of the open interval \((0, 1)\) is covered by a closed interval of length \( 1/2q^2 \), whose center is at \( p/q \). Prove that \( \sqrt{2}/2 \) is not covered by any of the above closed intervals.

Problem 21. **** Let \( x_1, x_2, \ldots, x_n \in \mathbb{R} \) with \( |x_i| \leq 1 \) for \( i = 1, 2, \ldots, n \). Show that there exist \( a_1, a_2, \ldots, a_n \in \{-1, 0, 1\} \), not all equal to 0, such that

\[ |a_1 x_1 + a_2 x_2 + \cdots + a_n x_n| \leq \frac{n}{2^n - 1} \]

Problem 22 (IMO 1987 3). ***** Let \( x_1, x_2, \ldots, x_n \) be real numbers satisfying \( x_1^2 + x_2^2 + \cdots + x_n^2 = 1 \). Prove that for every integer \( k \geq 2 \) there are integers \( a_1, a_2, \ldots, a_n \), not all 0, such that \( |a_i| \leq k - 1 \) for all \( i \) and

\[ |a_1 x_1 + a_2 x_2 + \cdots + a_n x_n| \leq \frac{(k - 1)\sqrt{n}}{k^n - 1}. \]

Problem 23 (Putnam 1989 A5). ***** Let \( m \) be a positive integer and let \( G \) be a regular \((2m + 1)\)-gon inscribed in the unit circle. Show that there is a positive constant \( A \), independent of \( m \), with the following property. For any point \( p \) inside \( G \) there are two distinct vertices \( v_1 \) and \( v_2 \) of \( G \) such that

\[ ||p - v_1| - |p - v_2|| < \frac{1}{m} - \frac{A}{m^2}. \]

Here \( |s - t| \) denotes the distance between the points \( s \) and \( t \).

Problem 24. ***** Show that

\[ \sum_{p} \frac{1}{2^p} \]

where \( p \) runs over all prime numbers, is an irrational number.

Problem 25. *** Does there exist \( x \geq 1 \) with \( \cos(x) + \cos(\sqrt{2}x) > 1.9999? \)