PROBLEM SET 4: THE EXTREMAL PRINCIPLE

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The (discrete version) of the extremal principle can be formulated as follows:

**Truth 1.** *(Extremal Principle)* A real valued function $f$ on a finite set $S$ has a maximum and a minimum.

There is also a continuous version of the Extremal Principle. This is not quite as obvious. It is well known in (advanced) calculus.

**Theorem 1.** *(Extremal Principle, continuous version)* A continuous real-valued function $f$ on a closed interval $[a, b] \subseteq \mathbb{R}$ has a maximum and a minimum.

It sometimes can be very useful to assume that a certain quantity is maximal. We will see various examples of this.

1. THE DISCRETE EXTREMAL PRINCIPLE

**Example 1** ((UM)^2C^18 2001 2). **** Show that the people at a party can be divided into two groups and sent to two different rooms in such a way that, for every person in either room, at least half that person's friends at the party are in the other room. (You may assume that friendship is a symmetric relation.

**Proof.** Let $m$ be the number of all pairs $\{P, Q\}$ of people such that $P$ and $Q$ are in different rooms and $P$ and $Q$ are friends. We may assume that $m$ is maximal over all possible ways of dividing the people in two groups. Suppose some person $P$ has $a_P$ friends in his own room and $b_P$ friends in the other room. If $P$ would move to the other room then we have to add $b_P - a_P$ to $m$. By our maximality assumption on $m$, we get that $b_P \leq a_P$ for all $P$ which is what we wanted to prove.

**Problem 1.** * There are $n$ people standing in a field, each carrying a gun. Every person shoots the person nearest to him (all people shoot at the same time, all distances are distinct). Show that at least one person survives if $n$ is odd.

**Problem 2.** * Let $f : \mathbb{Z} \to \mathbb{Z}$ be an integer-valued function on $\mathbb{Z}$ with $2f(n) < f(n - 1) + f(n + 1)$ for all $n \in \mathbb{Z}$. Prove that $f$ has arbitrary large values.

**Problem 3.** ** Let $S$ be a measurable subset of $\mathbb{R}^2$ with area $A$. Show that one can choose a set $T \subset S$ of at least $A/\pi$ points in $S$ such that all pairs of distinct points in $T$ have distance at least 1.

The following result is actually useful in coding theory.
Problem 4. *** Let $S = \{0, 1\}^n$. For $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ we define the Hamming distance by

$$d(a, b) = \#\{i \mid a_i \neq b_i\}$$

i.e., the number of indices for which $a_i \neq b_i$. Prove that for every positive integer $k$ there exists a subset $T$ of $S$ such that $d(a, b) \geq k$ for all distinct $a, b \in T$ and

$$\#T \geq \frac{2^n}{\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{k-1}}.$$

(The solution to this problem is similar to the solution of Problem 3.)

Problem 5. *** Suppose that $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ are real numbers such that $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$. Let $c_1, c_2, \ldots, c_n$ be a permutation of $b_1, b_2, \ldots, b_n$. Show that

$$a_1b_n + a_2b_{n-1} + \cdots + a_nb_1 \leq a_1c_1 + a_2c_2 + \cdots + a_nc_n \leq a_1b_1 + a_2b_2 + \cdots + a_nb_n.$$

Problem 6. ***

(a). Suppose that $x_1, x_2, \ldots, x_n$ are positive real numbers. Use Problem 5 to prove

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_{n-1}}{x_n} + \frac{x_n}{x_1} \geq n.$$

(b). Suppose that $y_1, \ldots, y_n$ are nonnegative real numbers. Prove that

$$\frac{y_1 + y_2 + \cdots + y_n}{n} \geq \left( y_1 y_2 \cdots y_n \right)^\frac{1}{n}.$$

(The arithmetic average is greater or equal than the geometric average. We will give many more proofs of this in the future. Hint: Reduce to the case that $y_1y_2\cdots y_n = 1$, then use (a).)

Problem 7 (IMO 1997 3). ***** Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying the conditions

$$|x_1 + x_2 + \cdots + x_n| = 1$$

and

$$|x_i| \leq \frac{n + 1}{2}, \quad i = 1, 2, \ldots, n.$$ 

Show that there exists a permutation $y_1, y_2, \ldots, y_n$ of $x_1, x_2, \ldots, x_n$ such that

$$|y_1 + 2y_2 + \cdots + ny_n| \leq \frac{n + 1}{2}.$$

Problem 8 (Putnam 1995 A-4). ***** Suppose we have a necklace of $n$ beads. Each bead is labeled with an integer and the sum of all these labels is $n - 1$. 

Prove that we can cut the necklace to form a string whose consecutive labels
\( x_1, x_2, \ldots, x_n \) satisfy
\[
\sum_{i=1}^{k} x_i \leq k - 1, \quad \text{for } k = 1, 2, \ldots, n.
\]

**Problem 9** (Putnam 1996 B3). \(*\ast\ast\ast\ast\) Given that \( \{x_1, x_2, \ldots, x_n\} = \{1, 2, \ldots, n\} \), find, with proof, the largest possible value, as a function on \( n \) (with \( n \geq 2 \)) of
\[
x_1 x_2 + x_2 x_3 + \cdots + x_{n-1} x_n + x_n x_1.
\]

**Problem 10.** \(*\ast\ast\ast\ast\ast\) Suppose that there are \( n \) lines in the Euclidean plane \( \mathbb{R}^2 \) such that
(a). Every two lines intersect;
(b). Through any intersection point of two lines there goes at least one other line.
Prove that all lines go through one point. (**Hint:** Choose a line \( \ell \) and an intersection point \( P \), not on \( \ell \), such that the distance of \( P \) to \( \ell \) is minimal. Deduce a contradiction.)

**Problem 11.** \(*\ast\ast\ast\ast\) Suppose that in the plane \( \mathbb{R}^2 \), there are \( n \) blue points and \( n \) red points (all of them distinct). Show that you can label the blue points with \( B_1, B_2, \ldots, B_n \) and the red points with \( R_1, R_2, \ldots, R_n \) such that the line segments \( B_i R_i \) do not intersect each other. (**Hint:** Choose a labeling with \( \sum_{i=1}^{n} |B_i R_i| \) minimal, where \( |B_i R_i| \) is the distance from \( B_i \) to \( R_i \).)

2. The Continuous Extremal Principle

**Theorem 2.** Suppose that \( f \) is a real-valued differentiable function on an open interval \( (a, b) \) which has a maximum (or minimum) at \( c \in (a, b) \). Then \( f'(c) = 0 \).

**Proof.** By definition
\[
f'(c) = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h}
\]
In particular
\[
f'(c) = \lim_{h \downarrow 0} \frac{f(c + h) - f(c)}{h} \leq 0
\]
because \( f(c) \) is the maximal value of \( f \). On the other hand
\[
f'(c) = \lim_{h \uparrow 0} \frac{f(c + h) - f(c)}{h} \geq 0.
\]
This shows \( f'(c) = 0 \). \( \square \)

**Example 2.** Show that
\[
x^3 + \frac{3}{x} \geq 4
\]
for \( x > 0 \).
Proof. Let $f(x) = x^3 + 3/x$. Clearly if $0 < x < \frac{1}{2}$ and if $x > 2$, then $f(x) \geq 4$. The continuous function $f(x)$ has a minimum on the interval $[\frac{1}{2}, 2]$, say at $c$. If $c = \frac{1}{2}$ or $c = 2$ then $f(x) \geq f(c) \geq 4$ for all $x \in [\frac{1}{2}, 2]$. Assume now that $\frac{1}{2} < c < 2$. Then we must have

$$f'(c) = 3c^2 - \frac{3}{c^2} = 0$$

by Theorem 2. We easily solve this and find $c = 1$. Now we have

$$f(x) \geq f(1) \geq 4$$

for all $x \in [\frac{1}{2}, 2]$. \qed

**Problem 12.** * Find the maximum value of \[ x^{\frac{1}{2}} \]

for $x > 0$.

An immediate consequence of the previous theorem and the Continuous Extremal Principle is the Mean Value Theorem:

**Theorem 3** (Mean Value Theorem). *Let $f$ be a continuous real-valued function on the closed interval $[a, b] \subset \mathbb{R}$ which is differentiable on the open interval $(a, b)$. Then there exists a $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.\]

Proof. Put

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Note that $g(a) = g(b) = 0$. There exists a $c \in [a, b]$ with $g(c)$ maximal. If $c = a$ or $c = b$, then $g(x) \leq 0$ for $x \in [a, b]$ and we can find a $c \in (a, b)$ such that $g(c)$ is minimal. In any case there is a $c \in (a, b)$ for which $g(c)$ is maximal or minimal. From the previous theorem follows that

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0.$$

\qed

**Problem 13.** **Prove the following theorem.

**Theorem 4.** *Suppose $f$ is a differentiable function on an interval $(a, b)$. Then $f$ is (weakly) increasing if and only if $f'(x) \geq 0$ for all $a < x < b$.*

**Example 3.** *Show that $\sin(x) \leq x$ for $x \geq 0$ and $\sin(x) \geq x$ for $x \leq 0$. Also show that $\cos(x) \geq 1 - \frac{1}{2}x^2$ for all $x \in \mathbb{R}$. 


Proof. Consider $f(x) = \sin(x) - x$. Then $f'(x) = \cos(x) - 1 \leq 0$. This means that $f(x)$ is weakly decreasing on $\mathbb{R}$. Since $f(0) = 0$, we have $f(x) \leq 0$ for all $x \geq 0$ and $f(x) \geq 0$ for all $x \leq 0$. Now consider $g(x) = \cos(x) - 1 + \frac{1}{2}x^2$. we have $g'(x) = -\sin(x) + x = -f(x)$. Therefore $g'(x) \geq 0$ for $x \geq 0$ and $g'(x) \leq 0$ for $x \leq 0$. It follows that $g$ is weakly increasing for $x \geq 0$ and $g$ is weakly decreasing for $x \leq 0$. Since $g(0) = 0$ we have $g(x) \geq 0$ for all $x \in \mathbb{R}$.

Problem 14. * Show that 

$$e^x \geq x + 1$$

for all $x \in \mathbb{R}$.

Problem 15. **** Let $f$ be a continuous real-valued function on the closed interval $[a, a+2h] \subset \mathbb{R}$ which has a second derivative on the open interval $(a, a+2h)$. Then there exists a $c \in (a, a+2h)$ such that

$$f''(c) = \frac{f(a) - 2f(a + h) + f(a + 2h)}{h^2}.$$

In particular, it follows from the previous problem that if $f$ is a function on $\mathbb{R}$ with a continuous second derivative and $f''(x) \geq 0$ for all $x \in \mathbb{R}$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(a) + f(b)}{2}$$

for all $a, b \in \mathbb{R}$. Such a function is often called *convex*. A *concave* function on $\mathbb{R}$ is a function satisfying $f''(x) \leq 0$ for all $x \in \mathbb{R}$. For a concave function we have

$$f\left(\frac{a+b}{2}\right) \geq \frac{f(a) + f(b)}{2}.$$