PRACTICE PROBLEMS FOR PUTNAM

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1. (Putnam 1985, A1) Determine, with proof, the number of ordered triples
   \((A_1, A_2, A_3)\) of sets which have the property that
   (a) \(A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}\), and
   (b) \(A_1 \cap A_2 \cap A_3 = \emptyset\).

   Express your answer in the form \(2^{a_3}3^{b_5}5^{c_7}d\), where \(a, b, c, d\) are nonnegative integers.

2. (Putnam 1986, A1) Find, with explanation, the maximum value of \(f(x) = x^3 - 3x\) on the set of all real numbers \(x\) satisfying \(x^4 + 36 \leq 13x^2\).

3. (Putnam 1988, A2) A not uncommon calculus mistake is to believe that the product rule for derivatives say that \((fg)' = f'g'\). If \(f(x) = e^{x^2}\), determine, with proof, whether there exists an open interval \((a, b)\) and a nonzero function \(g\) defined on \((a, b)\) such that this wrong product rule is true for \(x\) in \((a, b)\).

4. (Putnam 1990, A2) Is \(\sqrt{2}\) the limit of a sequence of numbers of the form
   \(\sqrt{n} - \sqrt{m}\) \((n, m = 0, 1, 2, \ldots)\).

5. (Putnam 1992, B1) Let \(S\) be the set of \(n\) distinct real numbers. Let \(A_S\) be
   the set of numbers that occur as averages of two distinct elements of \(S\). For
   a given \(n \geq 2\), what is the smallest possible number of elements in \(A_S\).

6. (Putnam 1999, A2) Let \(p(x)\) be a polynomial that is nonnegative for all real
   \(x\). Prove that for some \(k\), there are polynomials \(f_1(x), \ldots, f_k(x)\) such that
   \[p(x) = \sum_{j=1}^{k} (f_j(x))^2.\]