PROBLEM SET 7:
GAMES, PUZZLES, RECREATIONAL MATH

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1. Weighing Problems

There are many popular problems involving weighing with either a *balance scale* or a *numerical scale*. A balance scale has two sides. With one weighing there are three possible outcomes, the left side is heavier than the right side, the left side is lighter than the right side or the left and the right side have the same weight. A weighing scale which gives you the weight of an object in pounds (or another unit) we will call a numerical scale (to make a clear distinction).

Example 1. Suppose that we have 25 coins that look identical. The coins are all the same except that one coin is counterfeit and heavier than the others. How can one determine, in three weighings on a balance scale, which coin is counterfeit.

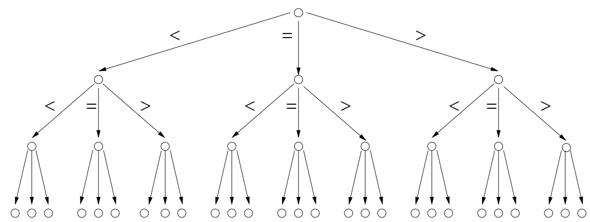
Discussion. Divide the 25 coins up in three groups, say A, B and C such that A and B have 8 coins each and C has 7 coins. Put the A coins on the left side of the scale and the B coins on the right side of the scale. If A is heavier then A contains the counterfeit coin. If B is heavier then B contains the counterfeit coin. If both sides balance, then C has the counterfeit coin.

So we found a set of 7 or 8 coins of which we know that one is counterfeit. We may take (for convenience) 1 or 2 coins from the remaining genuine coins so that we have 9 coins. We have to find out, with two remaining weighings which is the counterfeit coin among these 9 coins. Split the 9 coins up in three groups, D,E and F, such that each group has exactly 3 coins. Put the D coins on the left and E coins on the right of the scale. Again we find out which of the groups D,E,F cointains the counterfeit coin.

We now have three coins left. Put one coin on the left and one coin on the right of the scale. The coin that is the heaviest is the counterfeit. If both coins weigh the same, then the third coin is the counterfeit one.

Example 2. Suppose that we have 30 coins that look identical. The coins are all the same except that one coin is counterfeit and heavier than the others. Is it always possible to determine, in three weighings on a balance scale, which coin is counterfeit.

Discussion. The answer is no. Each weighing has three possible outcomes. One could graph the possible events as a tree:



From the 3 weighing events there can only be at most $3 \times 3 \times 3 = 27$ possible outcomes. By the pigeonhole principle there are two numbers i and j with $1 \leq i < j \leq 30$ such that the results of the weighings will be exactly the same if either the i-th coin or j-th coin is the cointerfeit coin. ☺

Example 3. Suppose you have 10 barrel of coins. Each barrel contains all real coins or it contains all fake coins. The real coins weigh 10 grams, and the counterfeit coins weigh 11 grams. There is exactly one barrel with counterfeit coins. Determine, with only one weighing on a numerical scale, which barrel cointains the cointerfeit coins. (One may assume that the barrels cointain "enough" coins.)

Proof. Take one coin from the first barrel, two coins from the second barrel, three coins from the third barrel, etc. Put these $1 + 2 + \cdots + 10 = 55$ coins on the scale. If barrel k contains the counterfeit coins, then the weight will be

$$1 \cdot 10 + 2 \cdot 10 + \cdots + 10 \cdot 10 + (11 - 10) \cdot k = 550 + k.$$

grams. This means that we can tell immediately from the weight which barrel contains the counterfeit coins. If the weight is for example 557, then this means that barrel 7 cointains the counterfeit coins. ☺

Problem 1. * Suppose that we have 3^n coins that look identical. The coins are all the same except that one coin is counterfeit and heavier than the others. How can one determine, in n weighings on a balance scale, which of the coins is counterfeit?

Problem 2. **** Suppose that we have 12 coins that look identical. The coins are all the same except that one coin is counterfeit and does not have the same weight as the real coins. We do not now if the counterfeit coin is heavier or lighter than the real coins. How can one determine, in 3 weighings on a balance scale, which of the coins is counterfeit and whether the counterfeit coin is heavier or lighter?

Problem 3. * Suppose that we have 14 coins that look identical. The coins are all the same except that one coin is counterfeit and does not have the same weight as the real coins. We do not know if the counterfeit coin is heavier or lighter than the real coins. Show that it is not always possible to determine, in 3 weighings on a balance scale, which coins is counterfeit and whether it is heavier or lighter at the same time.

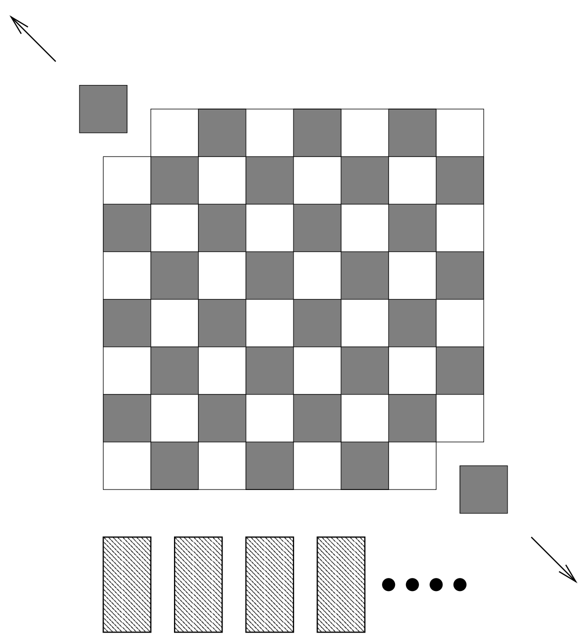
Problem 4. **** Suppose that we have a balance scale and exactly 5 weights, weighing exactly x_1, x_2, x_3, x_4 and x_5 grams. For any positive integer $n \leq 100$ one would like to be able to determine using the scale whether a given object weighs less than, more than or exactly n grams. How should one choose the weights x_1, x_2, x_3, x_4, x_5 such that this is always possible?

Problem 5. *** Suppose you have 10 barrel of coins. Each barrel contains all real coins or it contains all fake coins. The real coins weigh 10 grams, and the counterfeit coins weigh 11 grams. This time, there may be several barrels with cointerfeit coins (or even all or none of them). Determine, with only one weighing on a numerical scale, exactly which of barrels cointain the cointerfeit coins. (One may assume that the barrels cointain "enough" coins.)

2. Tilings

Vaguely speaking, an *invariant* is a quantity that remains the same after certain operations. For many problems it is useful to identify such *invariants.*

Example 4. ** We cut out two opposite corner fields of a chessboard. Is it possible to put 31 domino tiles (of size 2×1) on the remaining 62 fields of the chessboard?



Proof. The answer is no. The two corner which were cut out have the same color. Without loss of generality we may assume that they both were black. Let I be the number white fields that have been covered by domino tiles minus the number of black fields that have been covered by domino tiles. If there are no domino tiles on the chessboard, then $I = 32-30 = 2$. Every time we put another domino tile on the chessboard, then this domino tile will cover exactly one black field and one white field. This means that the quantity I doesn't change. If the whole chessboard without the two corners would be covered with domino tiles, then $I = 0$ which is impossible because I is constant and equal to 2. □

Example 5. Suppose that p, q and n are positive integers such that n is not divisible by p or by q. Prove that an $n \times n$ floor cannot be tiled by $p \times p$ or by $q \times q$ tiles.

Discussion. The idea of the proof is the following: Number the rows and columns with $0, 1, 2, \ldots, n-1$. Let us write numbers $a_{i,j}$ on square i, j such that: (1) The sum of all the numbers on a $p \times p$ or $q \times q$ square is always 0 and (2) the sum of all the numbers $a_{i,j}$, $1 \leq i, j \leq n$ is nonzero. This then would clearly prove that the $n \times n$ floor cannot be tiled with $p \times p$ and $q \times q$ tiles.

To ensure that the sum of all the numbers under a $p \times p$ square is 0, we could force that

$$a_{i,j} + a_{i+1,j} + \cdots + a_{i+p-1,j} = 0$$

for all i, j. To ensure that the sum of all the numbers under a $q \times q$ square is 0, we could force that

$$a_{i,j} + a_{i,j+1} + \cdots + a_{i,j+q-1} = 0$$

for all i, j. The easiest way to choose the $a_{i,j}$ in this fashion is to use complex numbers (but one could avoid this).

For any integer k, let $\zeta_k = e^{2\pi i/k}$ be the k-th primitive root of unity. Observe that

$$1 + \zeta + \zeta^2 + \cdots + \zeta^{l-1} = (1 - \zeta^l)/(1 - \zeta)$$

is equal to 0 if and only if l is divisible by k. Fill the $n \times n = n^2$ fields with complex numbers as follows. Number the rows and columns by $0, 1, 2, \ldots, n-1$. Put the complex number $\zeta_p^i \zeta_q^j$ on the field in row i and column j. The sum of all numbers over all fields is

$$\sum_{0 \leq i,j < n} \zeta_p^i \zeta_q^j = (1 + \zeta_p + \cdots + \zeta_p^{n-1})(1 + \zeta_q + \cdots + \zeta_q^{n-1})$$

is **nonzero** since p and q do not divide n. On the other hand, if we place a $p \times p$ tile (with one corner at (k, l)), then the sum of all complex numbers under the tile is

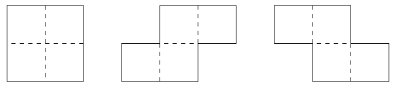
$$\sum_{i=k}^{k+p-1} \sum_{j=l}^{l+p-1} \zeta_p^i \zeta_q^j = \zeta^{2k} (\sum_{i=0}^{p-1} \zeta_p^i)(\sum_{i=0}^{p-1} \zeta_q^i) = 0$$

since

$$\sum_{i=0}^{p-1} \zeta_p^i = 0.$$

Similarly all the numbers under a $q \times q$ tile sum up to 0. This shows that it is not possible to tile the $n \times n$ floor with $p \times p$ and $q \times q$ tiles.

Problem 6. *** Show that 12×11 rectangular floor cannot be covered using only tiles of the following shapes:



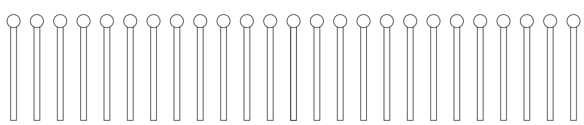
Problem 7. ***** Suppose that p, q and r are distinct prime numbers and $N > 2pqr$. Show that an $N \times N$ floor can be tiled with $p \times p$, $q \times q$ and $r \times r$ tiles. (*Hint: Write* $N = apq + bpr + cqr$ *for certain nonnegative integers* a, b, c. *Use this to divide the* $N \times N$ *floor in regions which are easy to tile.*)

Problem 8. ***** A rectangle R is divided into smaller rectangles. Each of the smaller rectangles has at least one side, whose length is an integer. Show that R itself has at least one side which is an integer.

Problem 9. ** You are at the coordinates $(1, 0, 0)$ in $\mathbb{R}^3$ where we use the usual xyz coordinate axis. A three dimensional knight jump is if you move ± 1 along one coordinate axis, ± 2 along a second coordinate axis and ± 3 along the third coordinate axis. For example one could jump from $(1, 0, 0)$ to $(1, 0, 0) + (2, -1, 3) = (3, -1, 3)$. Then one could jump to $(3, -1, 3) + (-3, 2, -1) = (0, 1, 2)$ and from there to $(0, 1, 2) + (1, 2, -3) = (1, 3, -1)$. Show that it is impossible to land at $(0, 0, 0)$ after finitely many three dimensional knight jumps.

3. Games

Problem 10. ** There are 25 matches on the table. Two players take turns. Each turn they have to take away 1,2 or 3 matches. The person taking the last match loses. Show that the second player always can win this game. (Try it first with 5,9 and 13 matches instead.)



(You can play it online at
`http://www.geocities.com/TimesSquare/4934/javamatc.html`,
`http://www.safe4kids.org/kids/smatches.htm`)

Problem 11 (IMO)**.** ***** To each vertex of a regular pentagon an integer is assigned in such a way that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively and $y < 0$ then the following operation is allowed: the numbers x, y, z are replaced by $x+y, -y, z+y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.

Problem 12 (after a well-known puzzle)**.** **** In a 4×4 square, we put the numbers $2, 1, 3, 4, 5, 6, \ldots, 15$ (see below). The last square is black. In each move, we may exchange the black square with one of its neighbors (neighbor means sharing an edge). Is it possible to get $1, 2, 3, \ldots, 15$ after finitely many moves (see second picture).

2	1	3	4
5	6	7	8
9	10	11	12
13	14	15	■

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	■

4. Extra Problems

Problem 13. **** Define a sequence $x_1, x_2, x_3, \ldots$ by $x_1 = 1$, $x_2 = 5$ and

$$x_{n+1} = \frac{x_n}{2} + x_{n-1} - \frac{x_{n-1}^2}{2x_n}$$

for $n \geq 2$. What is $\lim_{n\to\infty} x_n$?

Problem 14. *** We start with the numbers

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10$$

Then we replace two numbers, say x and y, by $xy/(x+y)$. We repeat this until there is only one number left. Show that, regardless how you do it, this number is always equal to $2520/7381$. (For example, we could replace 3 and 6 by $3 \cdot 6/(3+6) = 2$ to get the sequence

$$1, 2, 4, 5, 7, 8, 9, 10, 2.$$

Then we can replace 9 and 10 by $9 \cdot 10/(9+10) = 90/19$ to get the sequence

$$1, 2, 4, 5, 7, 8, 2, 90/19.$$

etc.)