Math 289, Problem Set 2
Due: 9/29/2004

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Hand in solutions to 4 problems from the following list of problems: Larson, 1.9.2*, 1.9.3***, 1.9.5***, 1.10.6***, 1.10.8**, 1.10.9**, 1.10.10**, 1.11.5***.

Section 1.11 of Larson considers the extremal principle. Below is some more information on the extremal principle. You may also choose problems from the section below.

The (discrete version) of the extremal principle can be formulated as follows:

**Theorem 1** (Extremal Principle). A real valued function \( f \) on a finite set \( S \) has a maximum and a minimum.

There is also a continuous version of the Extremal Principle. This is not quite as obvious. It is well known in (advanced) calculus.

**Theorem 2** (Extremal Principle, continuous version). A continuous real-valued function \( f \) on a closed interval \([a, b] \subset \mathbb{R}\) has a maximum and a minimum.

It sometimes can be very useful to assume that a certain quantity is maximal. We will see various examples of this.

The Discrete Extremal Principle

**Example 1** ((UM)\(^{2}C^{18} 2001 2\)). **** Show that the people at a party can be divided into two groups and sent to two different rooms in such a way that, for every person in either room, at least half that person's friends at the party are in the other room. (You may assume that friendship is a symmetric relation.

**Proof.** Let \( m \) the the number of all pairs \( \{P, Q\} \) of people such that \( P \) and \( Q \) are in different rooms and \( P \) and \( Q \) are friends. We may assume that \( m \) is maximal over all possible ways of dividing the people in two groups. Suppose some person \( P \) has \( a_{P} \) friends in his own room and \( b_{P} \) friends in the other room. If \( P \) would move to the other room then we have to add \( b_{P} - a_{P} \) to \( m \). By our maximality assumption on \( m \), we get that \( b_{P} \leq a_{P} \) for all \( P \) which is what we wanted to prove. \( \square \)

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THE CONTINUOUS EXTREMAL PRINCIPLE

**Theorem 3.** Suppose that \( f \) is a real-valued differentiable function on an open interval \( (a, b) \) which has a maximum (or minimum) at \( c \in (a, b) \). Then \( f'(c) = 0 \).

*Proof.* By definition

\[
f'(c) = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h}
\]

In particular

\[
f'(c) = \lim_{h \downarrow 0} \frac{f(c + h) - f(c)}{h} \leq 0
\]

because \( f(c) \) is the maximal value of \( f \). On the other hand

\[
f'(c) = \lim_{h \uparrow 0} \frac{f(c + h) - f(c)}{h} \geq 0.
\]

This shows \( f'(c) = 0 \). \( \square \)

**Example 2.** Show that

\[
x^3 + \frac{3}{x} \geq 4
\]

for \( x > 0 \).

*Proof.* Let \( f(x) = x^3 + 3/x \). Clearly if \( 0 < x < \frac{1}{2} \) and if \( x > 2 \), then \( f(x) \geq 4 \). The continuous function \( f(x) \) has a minimum on the interval \( [\frac{1}{2}, 2] \), say at \( c \). If \( c = \frac{1}{2} \) or \( c = 2 \) then \( f(x) \geq f(c) \geq 4 \) for all \( x \in [\frac{1}{2}, 2] \). Assume now that \( \frac{1}{2} < c < 2 \). Then we must have

\[
f'(c) = 3c^2 - \frac{3}{c^2} = 0
\]

by Theorem 3. We easily solve this and find \( c = 1 \). Now we have

\[
f(x) \geq f(1) \geq 4
\]

for all \( x \in [\frac{1}{2}, 2] \). \( \square \)

An immediate consequence of the previous theorem and the Continuous Extremal Principle is the Mean Value Theorem:

**Theorem 4 (Mean Value Theorem).** Let \( f \) be a continuous real-valued function on the closed interval \([a, b] \subset \mathbb{R}\) which is differentiable on the open interval \((a, b)\). Then there exists a \( c \in (a, b) \) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}.
\]

*Proof.* Put

\[
g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).
\]

Note that \( g(a) = g(b) = 0 \). There exists a \( c \in [a, b] \) with \( g(c) \) maximal. If \( c = a \) or \( c = b \), then \( g(x) \leq 0 \) for \( x \in [a, b] \) and we can find a \( c \in (a, b) \) such that \( g(c) \) is
minimal. In any case there is a $c \in (a, b)$ for which $g(c)$ is maximal or minimal. From the previous theorem follows that

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0.$$ 

\[ \Box \]

**Theorem 5.** Suppose $f$ is a differentiable function on an interval $(a, b)$. Then $f$ is (weakly) increasing if and only if $f'(x) \geq 0$ for all $a < x < b$. (Weakly increasing means here that $f(x_1) \leq f(x_2)$ if $a \leq x_1 < x_2 \leq b$.)

**Proof.** Suppose that $a \leq x_1 < x_2 \leq b$. By the mean value theorem there exists a $c$ in the interval $(x_1, x_2)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

Since $f'(c) \geq 0$ and $x_2 > x_1$, we have $f(x_2) \geq f(x_1)$. \[ \Box \]

**Example 3.** * Show that $\sin(x) \leq x$ for $x \geq 0$ and $\sin(x) \geq x$ for $x \leq 0$. Also show that $\cos(x) \geq 1 - \frac{1}{2}x^2$ for all $x \in \mathbb{R}$.

**Proof.** Consider $f(x) = \sin(x) - x$. Then $f'(x) = \cos(x) - 1 \leq 0$. This means that $f(x)$ is weakly decreasing on $\mathbb{R}$. Since $f(0) = 0$, we have $f(x) \leq 0$ for all $x \geq 0$ and $f(x) \geq 0$ for all $x \leq 0$. Now consider $g(x) = \cos(x) - 1 + \frac{1}{2}x^2$. We have $g'(x) = -\sin(x) + x = -f(x)$ Therefore $g'(x) \geq 0$ for $x \geq 0$ and $g'(x) \leq 0$ for $x \leq 0$. It follows that $g$ is weakly increasing for $x \geq 0$ and $g$ is weakly decreasing for $x \leq 0$. Since $g(0) = 0$ we have $g(x) \geq 0$ for all $x \in \mathbb{R}$. \[ \Box \]

**Problems**

**Problem 1.** *** There are $n$ people standing in a field, each carrying a gun. Every person shoots the person nearest to him (all people shoot at the same time, all distances are distinct). Show that at least one person survives if $n$ is odd.

**Problem 2.** ** Let $f : \mathbb{Z} \to \mathbb{Z}$ be an integer-valued function on $\mathbb{Z}$ with

$$2f(n) < f(n - 1) + f(n + 1)$$

for all $n \in \mathbb{Z}$. Prove that $f$ has arbitrary large values.

**Problem 3.** **** Let $S$ be a measurable subset of $\mathbb{R}^2$ with area $A$. Show that one can choose a set $T \subset S$ of at least $A/\pi$ points in $S$ such that all pairs of distinct points in $T$ have distance at least 1. (Hint: Suppose that $S$ is maximal. Consider balls of radius 1 around each point. What can you say now?)

The following result is actually useful in coding theory and is known as the *Gilbert-Varshamov bound*. 
Problem 4. 

Let $S = \{0,1\}^n$ (this means that $S$ is the set of all sequences with just zeroes and ones of length $n$). For $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ we define the Hamming distance by

$$d(a, b) = \#\{i \mid a_i \neq b_i\}$$

i.e., the number of indices for which $a_i \neq b_i$. Prove that for every positive integer $k$ there exists a subset $T$ of $S$ with at least

$$\frac{\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{k-1}}{2^n}$$

elements such that $d(a, b) \geq k$ for all distinct $a, b \in T$. (The solution to this problem is similar to the solution of Problem 3.)

Problem 5. 

Suppose that $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ are real numbers such that $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$. Let $c_1, c_2, \ldots, c_n$ be a permutation of $b_1, b_2, \ldots, b_n$. Show that

$$a_1b_n + a_2b_{n-1} + \cdots + a_nb_1 \leq a_1c_1 + a_2c_2 + \cdots + a_nc_n \leq a_1b_1 + a_2b_2 + \cdots + a_nb_n.$$  

(Hint: Choose a permutation $c_1, \ldots, c_n$ such that $a_1c_1 + \cdots + a_nc_n$ is maximal. Prove that $c_1 \leq \cdots \leq c_n$.)