MATH 289, FALL 2005, PROBLEM SET 1

DUE: 9/14/2005

Hand in solutions for 3 of the problems below.

Problem 1. Let \( a_1, a_2, \ldots, a_n \) represent an arbitrary arrangement of the numbers 1, 2, \ldots, \( n \). Prove that, if \( n \) is odd, the product
\[
(a_1 - 1)(a_2 - 2) \cdots (a_n - n)
\]
is an even number.

Problem 2. Prove that
\[
\sqrt{2 + \sqrt{3}} + \sqrt{2 - \sqrt{3}} = \sqrt{6}.
\]

Problem 3. Show that there are no positive integers \( x, y \) such that \( y^2 = x^2 + x + 1 \).

Problem 4. Let \( a, b \) be nonzero numbers with
\[
a + b = \frac{1}{a} + \frac{1}{b}.
\]
Prove that
\[
a^3 + b^3 = \frac{1}{a^3} + \frac{1}{b^3}.
\]

Problem 5. Let \( f(x) \) be a polynomial of degree \( n \) with real coefficients and such that \( f(x) \geq 0 \) for every real number \( x \). Show that
\[
f(x) + f'(x) + f''(x) + \cdots + f^{(n)}(x) \geq 0
\]
for all real \( x \). (\( f^{(k)}(x) \) is the \( k \)-th derivative.)

Problem 6. Let \( S_n \) be the set of all pairs \((x, y)\) with integral coordinates such that \( x \geq 0, y \geq 0 \) and \( x + y \leq n \). Show that \( S_n \) cannot be covered by the union of \( n \) straight lines.