PROBLEM SET 1: INDUCTION AND BINOMIAL COEFFICIENTS

HARM DERKSEN

1. Induction

Let \( \mathbb{N} = \{1, 2, \ldots\} \) be the natural numbers (starting with 1!). One important property of \( \mathbb{N} \) is the induction principle. It is in fact rather an axiom than a theorem, i.e., we simply assume that the induction principle for natural numbers is true.

The Induction Principle. Suppose we have a statement which depends on a natural number \( n \in \mathbb{N} \). Suppose that

1. the statement is true for \( n = 1 \), and
2. if for a natural number \( m \), the statement is true for \( n = m \), then the statement is also true for \( n = m + 1 \).

Then the statement is true for all natural numbers \( n \).

Many mathematical problems involve induction proofs. Usually applying the induction principle is the easy part of the problem. However, sometimes it can be tricky to find the right induction statement. Typical applications are identities which involve an indeterminate natural number \( n \).

Example 1. Show that

\[
\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.
\]

for all \( n \in \mathbb{N} \).

Proof. We check (1) for \( n = 1 \):

\[
\sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{2}.
\]

Suppose that (1) is true for \( n = m \), i.e.,

\[
\sum_{i=1}^{m} i = \frac{m(m+1)}{2}.
\]
then
\[
\sum_{i=1}^{m+1} i = \sum_{i=1}^{m} i + (m + 1) = \frac{m(m+1)}{2} + (m + 1) = \frac{(m+1)((m+1)+1)}{2}
\]
so \(1\) is also true for \(n = m + 1\). Using the induction principle, \(1\) holds for all natural numbers \(n \in \mathbb{N}\). \(\Box\)

2. Binomials Coefficients

We define the binomial coefficient \(\binom{n}{k}\) by
\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]
if \(n\) and \(k\) are nonnegative integers, and \(0 \leq k \leq n\). If \(n\) is nonnegative and \(k\) is an integer with \(k < 0\) or \(k > n\) then we define \(\binom{n}{k} = 0\) for convenience.

Recall that \(n!\) is defined by \(0! = 1\) and \(n! = n \cdot (n-1)!\) for all natural numbers \(n\). Notice that this is a definition using induction, an \textit{inductive definition}.

There is also a combinatorial interpretation of the binomial coefficient \(\binom{n}{k}\). Suppose that are \(n\) marbles in a jar. Then there are \(\binom{n}{k}\) possibilities of choosing \(k\) marbles from the jar. Indeed, mark the marbles with \(1, 2, 3, \ldots, n\). We want to count the number of subsets of \(\{1, 2, \ldots, n\}\) which have cardinality \(k\). There are
\[
n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}
\]
k-tuples \((a_1, a_2, \ldots, a_k)\) of integers with \(a_1, a_2, \ldots, a_k\) distinct and \(1 \leq a_i \leq n\) for all \(i\). Indeed, there are \(n\) possibilities for \(a_1\) and since \(a_2\) has to be distinct from \(a_1\), there are \(n-1\) possibilities for \(a_2\), etc. Finally for \(a_k\) there are only \(n - (k-1)\) possibilities since \(a_k\) has to be distinct from \(a_1, a_2, \ldots, a_{k-1}\).

Now each subset \(\{a_1, a_2, \ldots, a_k\}\) appears \(k!\) times since there are \(k!\) ways of ordering the integers \(a_1, \ldots, a_k\). So we conclude that there are
\[
\frac{n!}{k!(n-k)!} = \binom{n}{k}
\]
ways of choosing \(k\) marbles from a jar of \(n\) marbles.

A third interpretation of \(\binom{n}{k}\) is the following identity
\[
(1 + x)^n = \binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{n}x^n.
\]

If we expand the expression
\[
(1 + x)^n = (1 + x)(1 + x)\cdots(1 + x)
\]
then we can choose for each factor \((1 + x)\) either the 1-term or the \(x\)-term. The coefficient of \(x^k\) in the expansion is exactly the number of ways we can choose \(k\) times \(x\) and \(n-k\) times 1. This number of ways is \(\binom{n}{k}\) as we have seen.
Example 2. Prove the identity

\[ \sum_{k=0}^{n} \binom{n}{k} = 2^n \]

Proof. \( \binom{n}{k} \) is the number of ways to choose \( k \) elements from a set of \( n \) elements. Then

\[ \sum_{k=0}^{n} \binom{n}{k} \]

is the number of ways to choose any subset of \( \{1, 2, \ldots, n\} \) (since any subset has at least 0 and at most \( n \) elements). The number of subsets of \( \{1, 2, \ldots, n\} \) is \( 2^n \), since each element can be either in or out the subset. \( \square \)

Proof. We use the identity

\[ (1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k. \]

If we plug in \( x = 1 \) we get the identity

\[ 2^n = (1 + 1)^n = \sum_{k=0}^{n} \binom{n}{k}. \]

\( \square \)

3. Problems

Problem 1. * Prove that

\[ 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \]

for all \( n \in \mathbb{N} \).

Problem 2. * Give 3 proofs of the identity

\[ \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \]

for all nonnegative integers \( n \) and all integers \( k \).

(a). Give a proof using our definition of \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \).

(b). Give a proof using the interpretation of \( \binom{n+1}{k} \) as the number of ways of choosing \( k \) balls among \( n+1 \) balls (think of \( n+1 \) balls as \( n \) blue balls and 1 red ball).

(c). Give a proof using \( (1 + x)^{n+1} = (1 + x)(1 + x)^n \).
Problem 3. ** Give a proof (or even better, two proofs) of the identity
\[ \sum_{k=0}^{n} \binom{n}{k} \binom{m}{m-k} = \binom{n+m}{k}. \]

Problem 4. * How many subsets of \{1, 2, \ldots, n\} are there with an even number of elements?

Problem 5. **** Show that the number of subsets of \{1, 2, \ldots, n\} whose number of elements is divisible by 4 is equal to \(2^{n-2} + 2^{n/2-1}\) if \(n\) is divisible by 8. What if \(n\) is not divisible by 8? (One may need complex numbers for this problem).

Problem 6. ** Prove the inequality
\[ \binom{n}{k} \leq \frac{n^n}{k^k(n-k)^{n-k}} \]
(Hint: \((k + (n - k))^n\))

Problem 7. *** Show that the number of integer \(n\)-tuples \((x_1, x_2, \ldots, x_n)\) of nonnegative integers with
\[ x_1 + x_2 + \cdots + x_n = k \]
is equal to \(\binom{n+k-1}{k}\).

Problem 8. ***** Prove the identity
\[ \sum_{k=0}^{n} \binom{n}{k} (-1)^{k+n} k^n = n! \]

Problem 9. ***** Prove the identity
\[ \binom{n}{1} - \frac{1}{2} \binom{n}{2} + \frac{1}{3} \binom{n}{3} - \cdots + (-1)^{n+1} \frac{1}{n} \binom{n}{n} = 1 + \frac{1}{2} + \cdots + \frac{1}{n}. \]